Reflections on relations among reflections

Oleg Viro

June 12, 2014

Involutions

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$$(xy)^{-1} = yx$$

Plane Isometries

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Examples of plane isometries:

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Lemma. A plane isometry is recovered from its restriction to any three non-collinear points.

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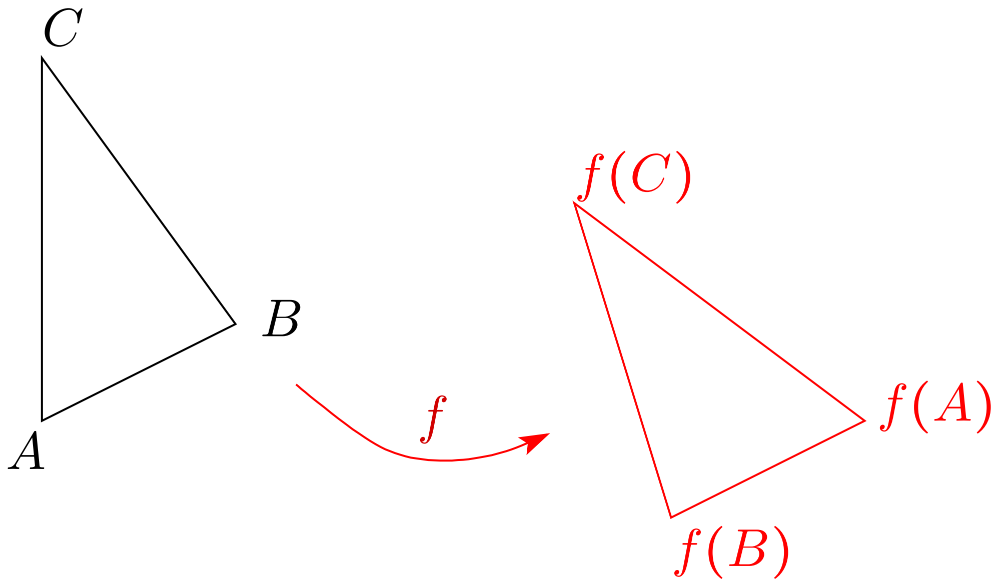

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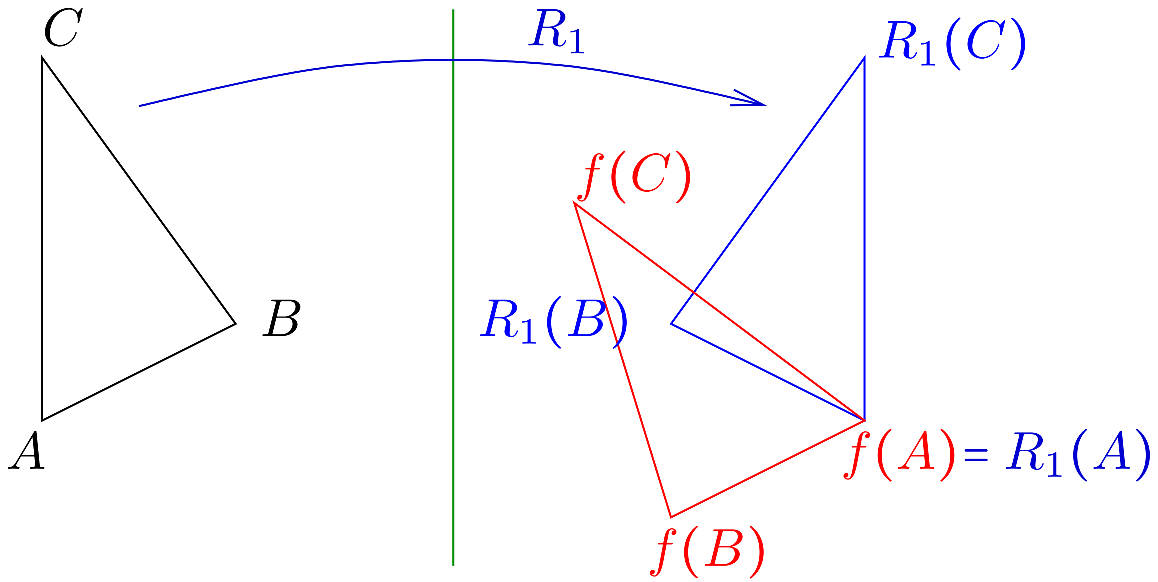

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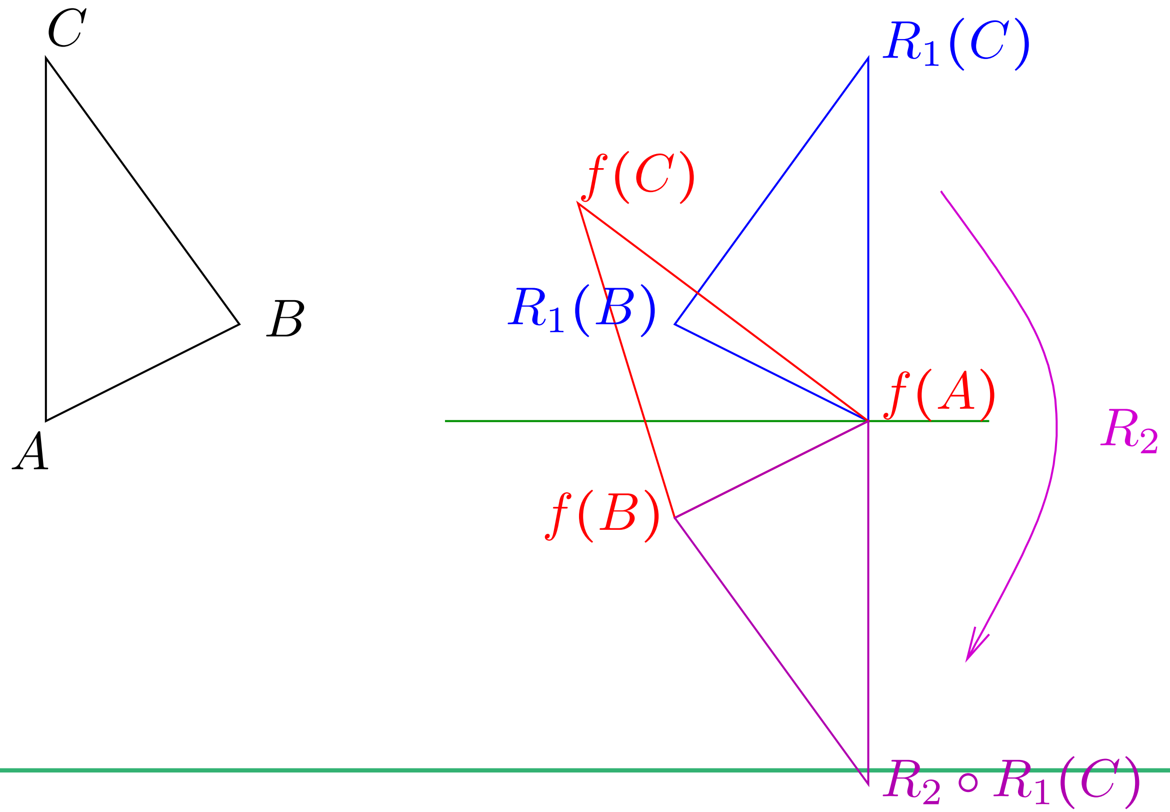

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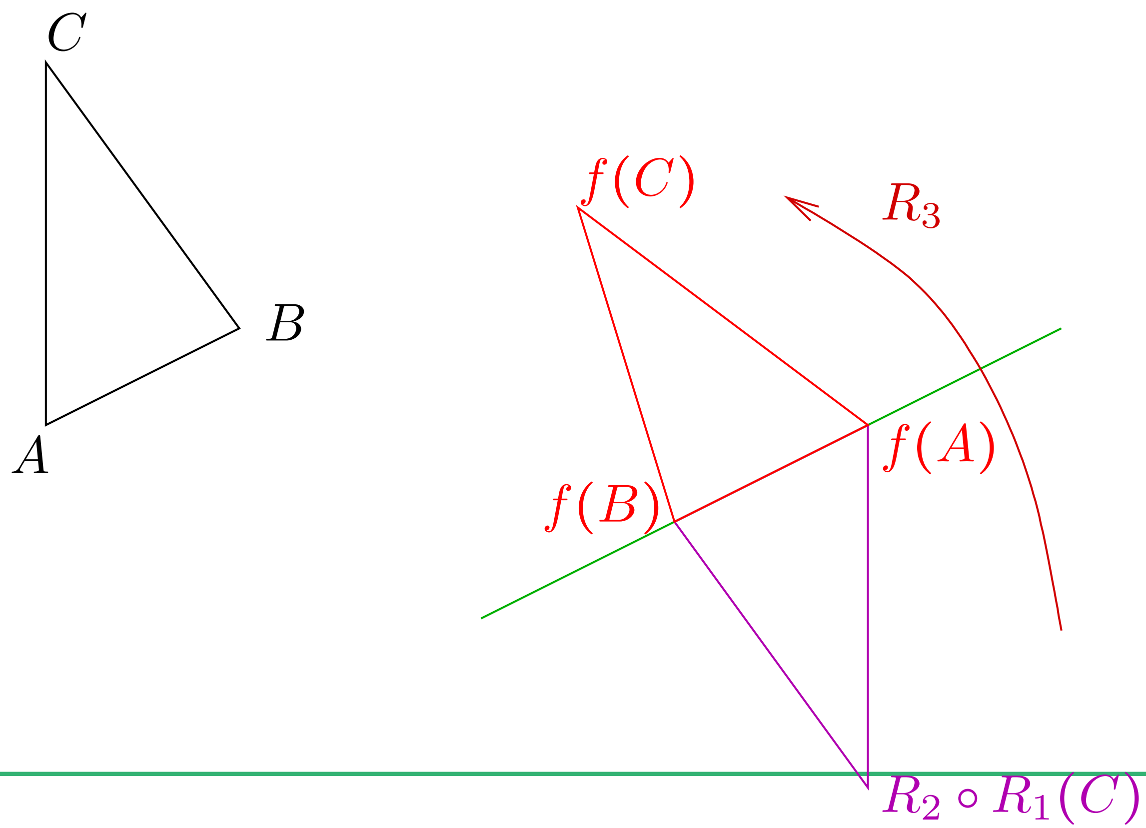
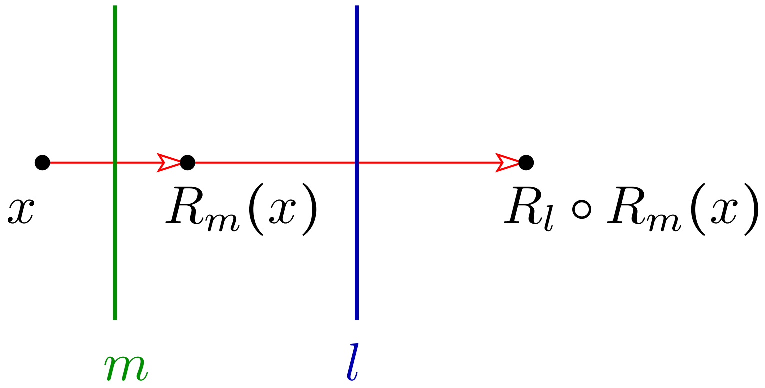


Compositions of two reflections

translation



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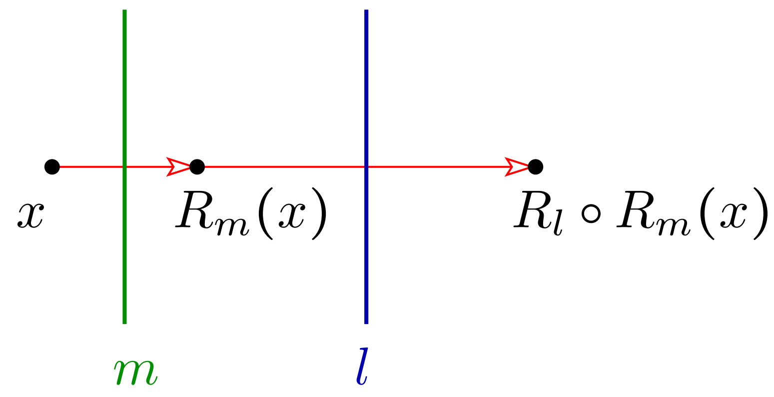

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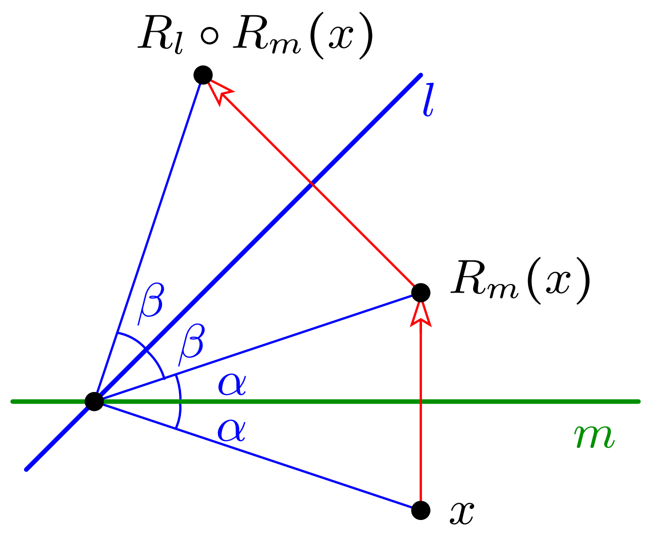

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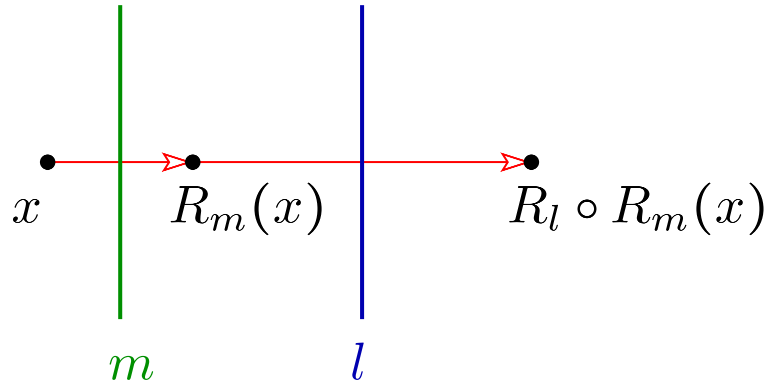

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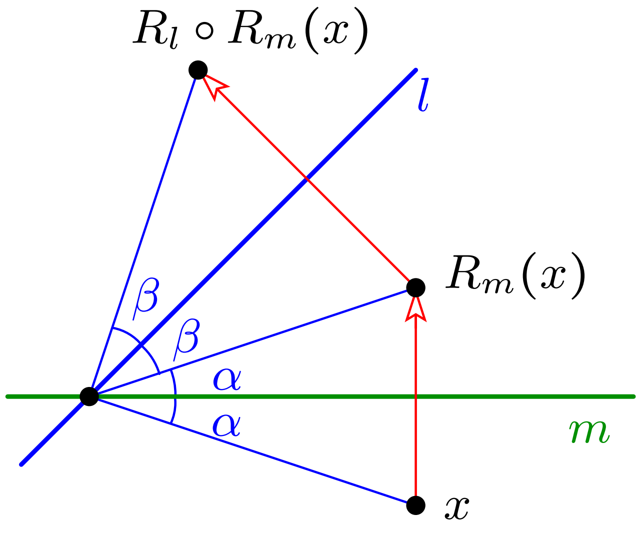

Presentations are not unique:

Compositions of two reflections

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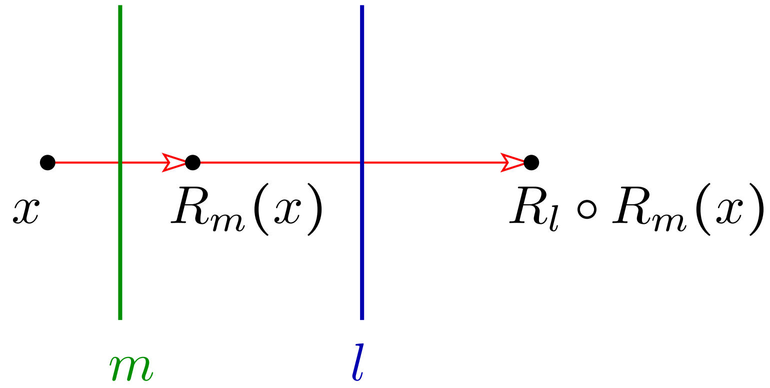

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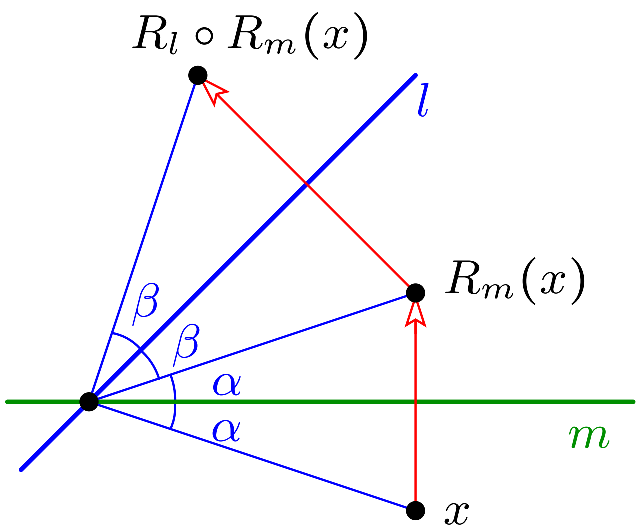

Presentations are not unique: $R_m \circ R_l = R_{m'} \circ R_{l'}$

Relations

Theorem. Any relation among reflections in lines follow from $R_l^2 = 1$ and $R_m \circ R_l = R_{m'} \circ R_{l'}$.

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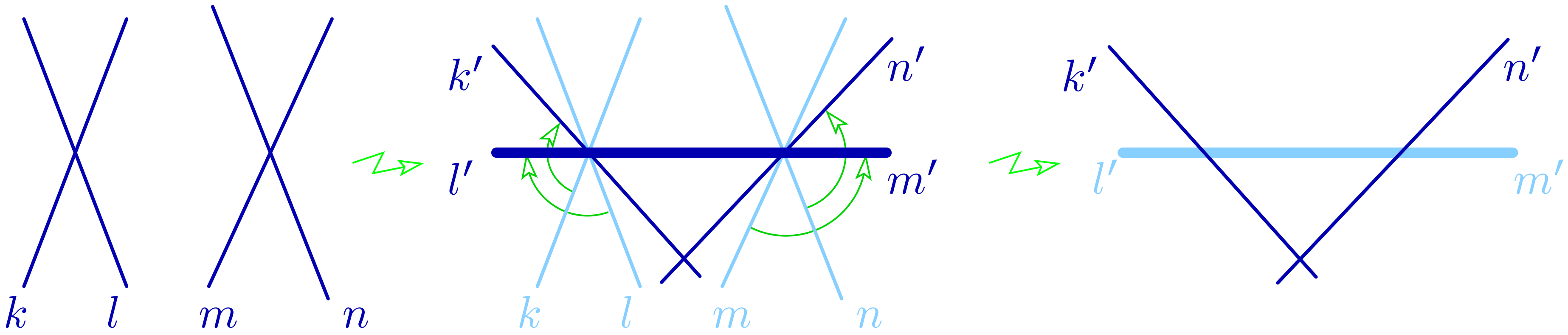

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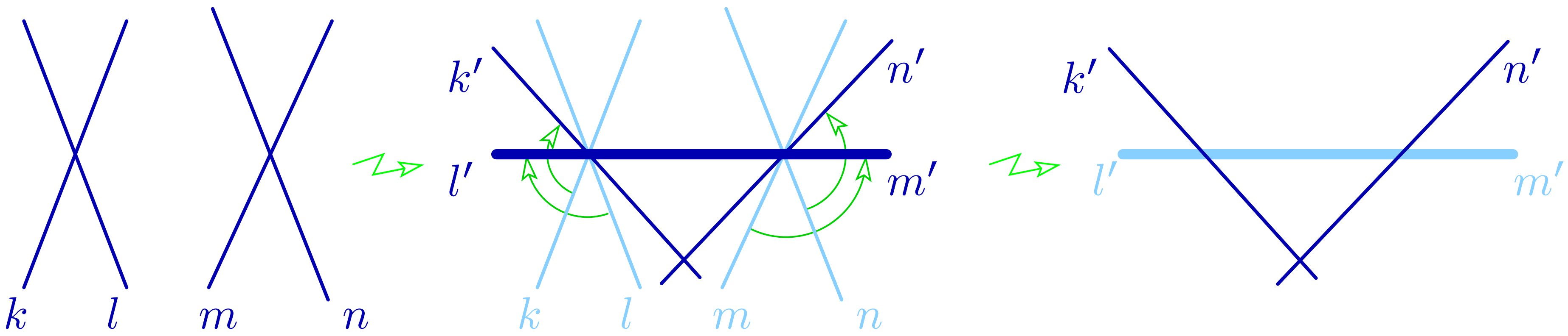

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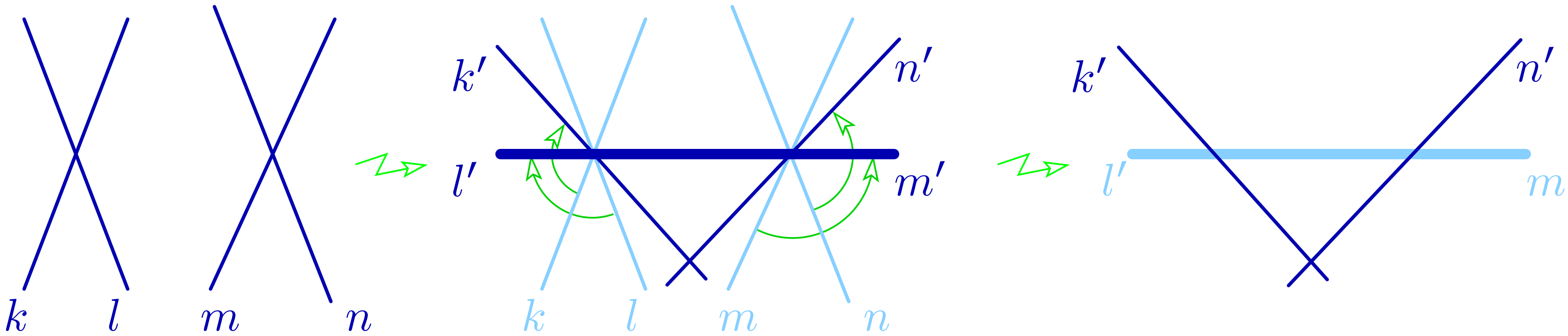

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A composition of odd number of reflections reverses orientation and cannot be id .

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One more relation...

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Group G generated by the set of involutions S.

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Three lines are concurrent or parallel

iff the composition of the reflections is a reflection.

Reflections in lines

The composition of 2 reflections in planes

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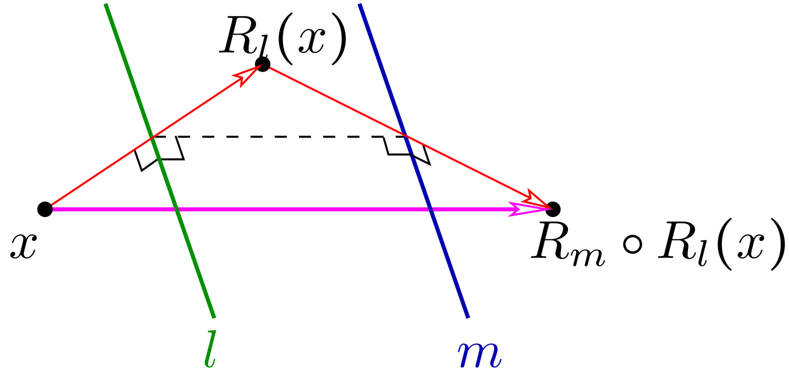

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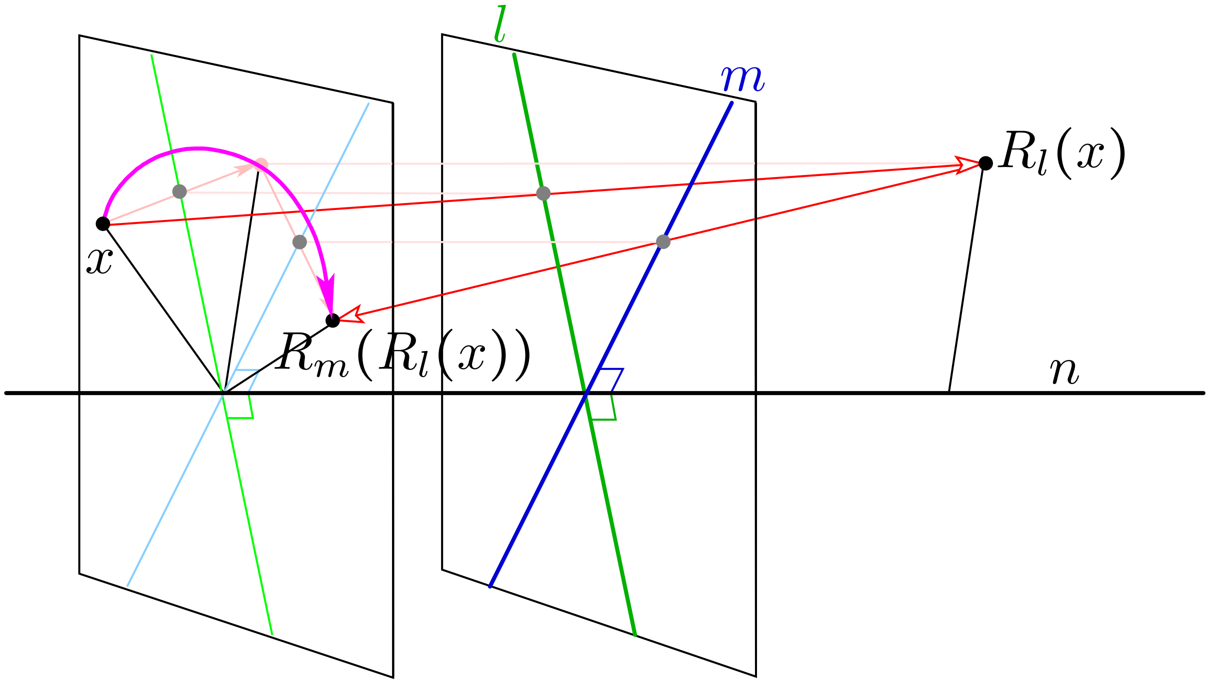

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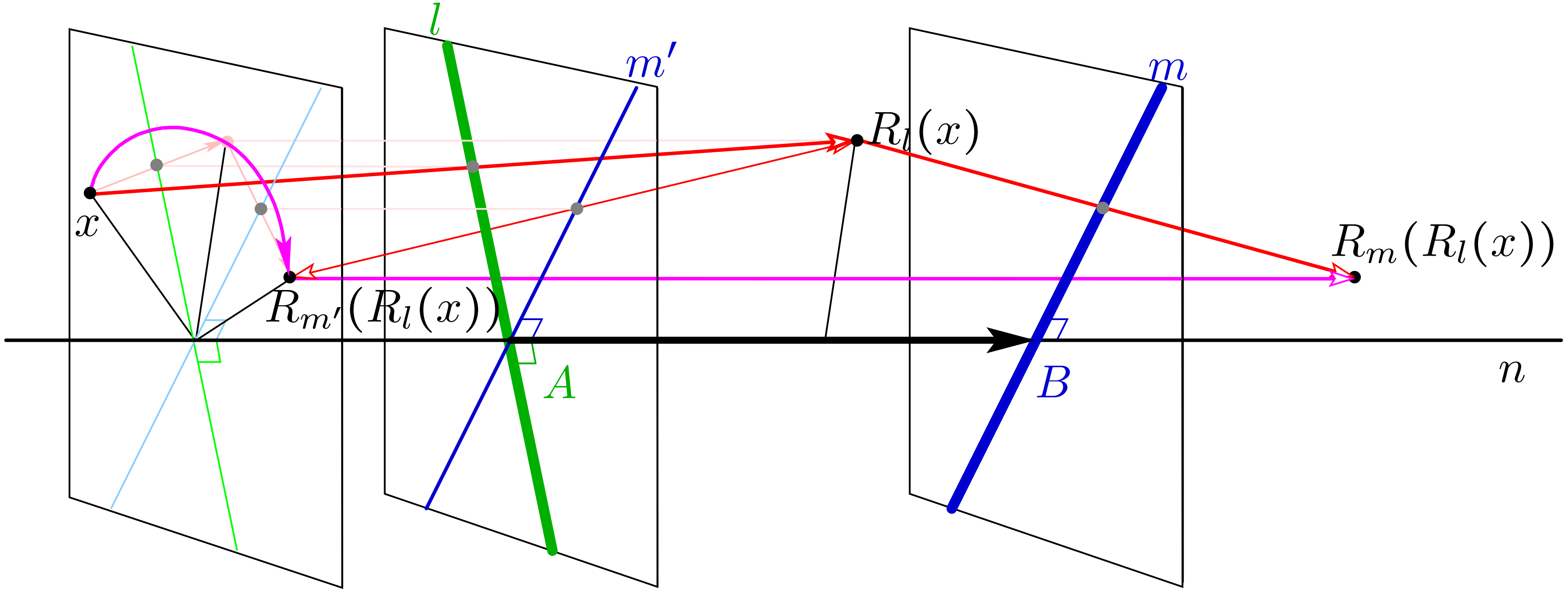

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Theorem. Any isometry of the 3-space preserving orientation

is a composition of reflections in lines.

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Vectors on 2-sphere and rotations.

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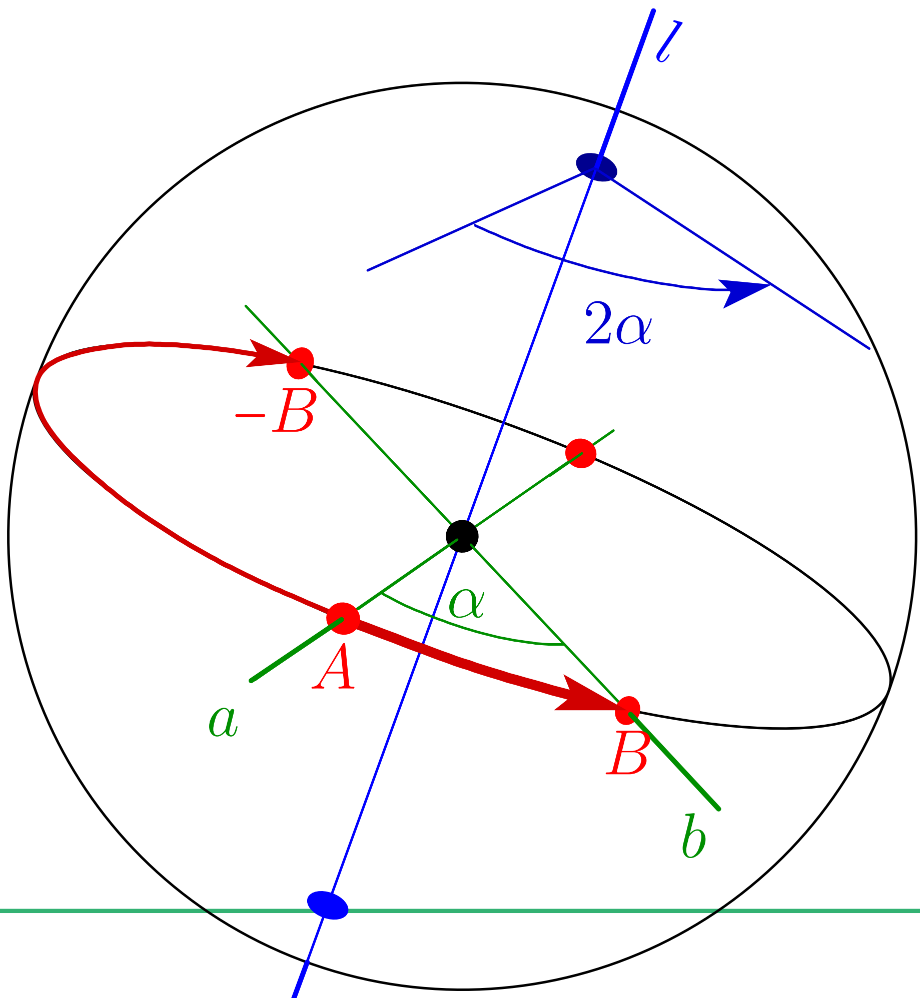
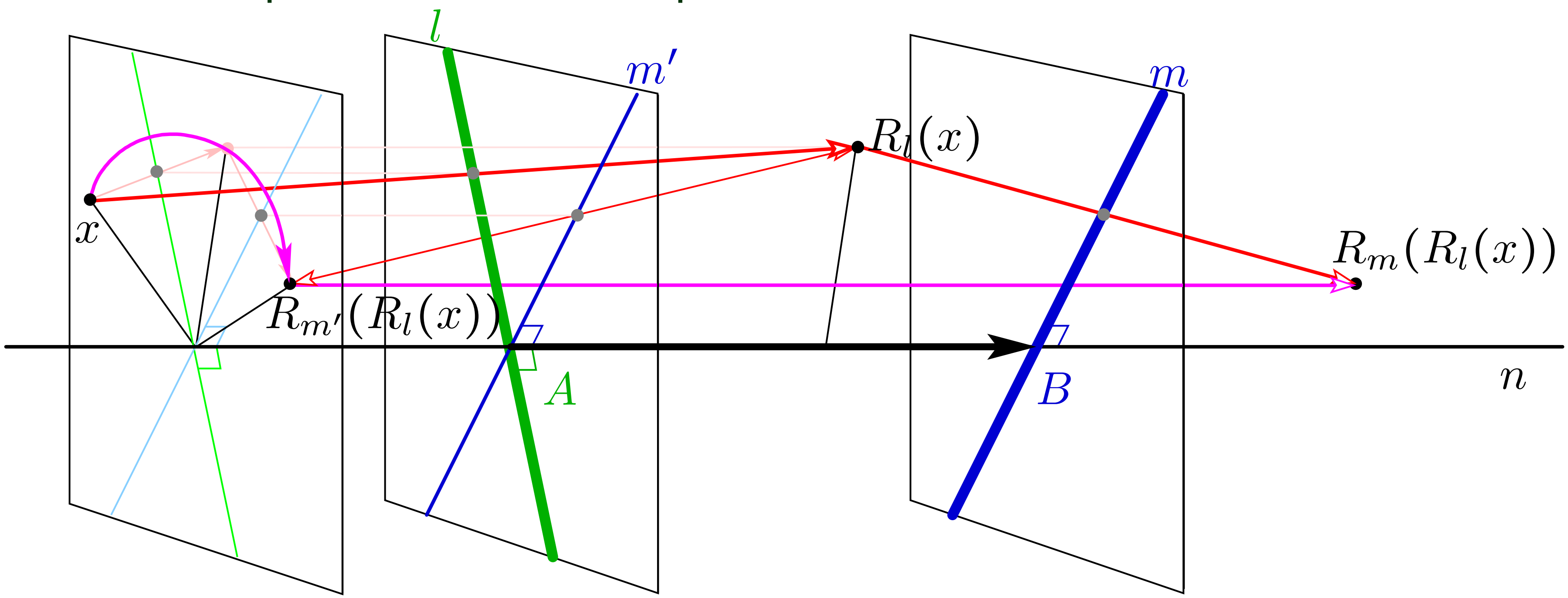


Compose screw displacements

A screw displacement is a composition of reflections in scew lines:



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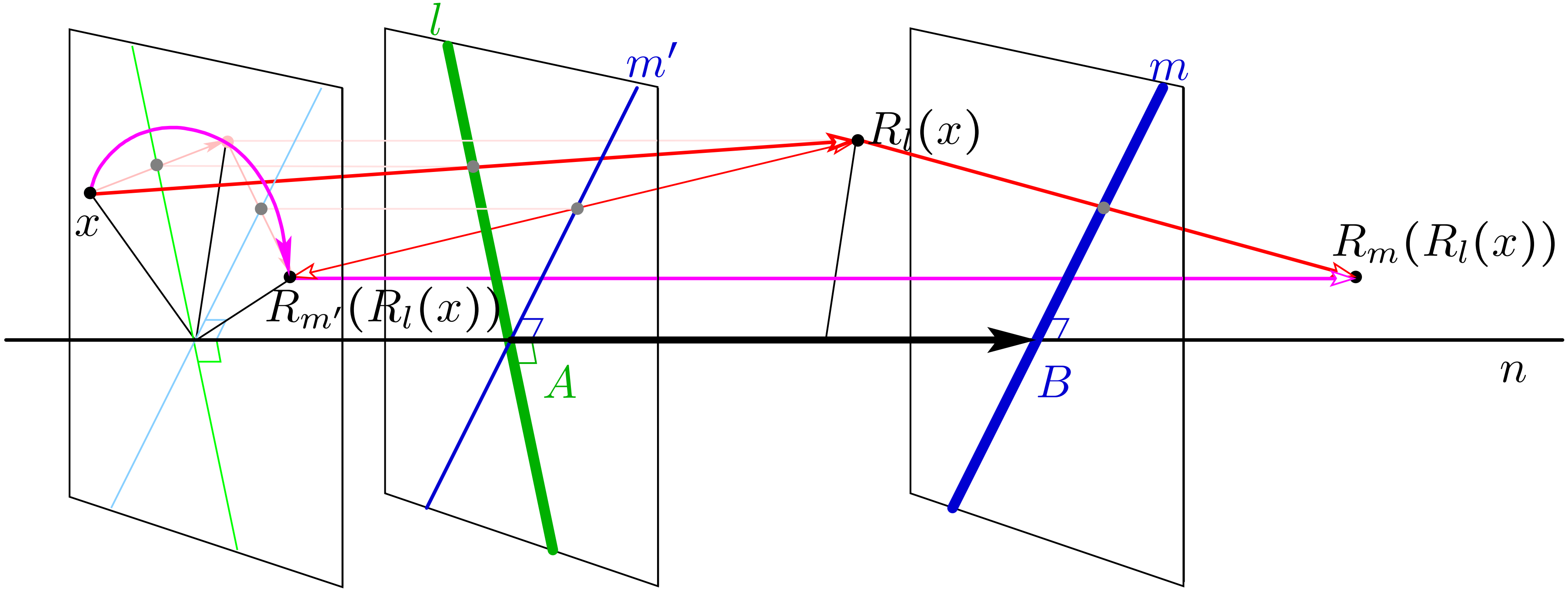

Defined up to screw displacements with the same axes.

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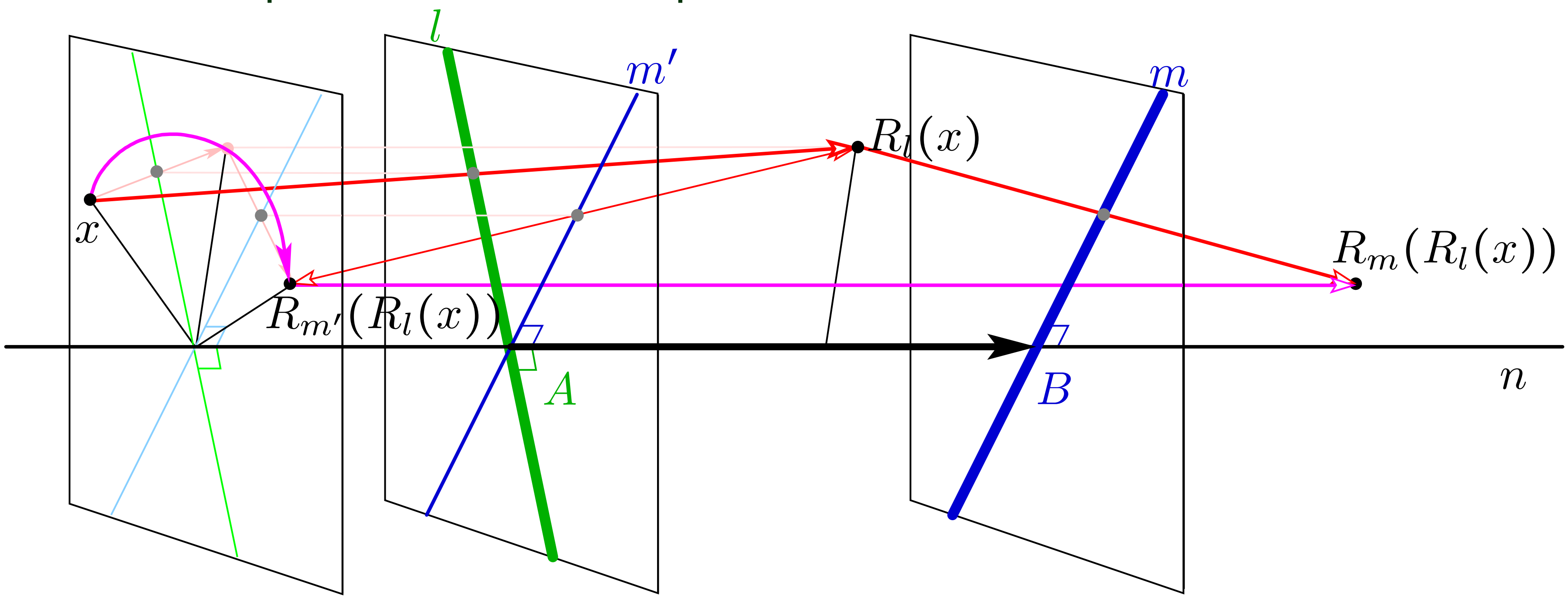

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