Variations on Arnold's Strangeness

Oleg Viro

October 31, 2009

Arnold's strangeness
- Genericity of plane curves.
- Perestrojkas
- Strangeness
- The direction of change
- How it works

Formulas for strangeness

Algebraic curves

Arnold's strangeness

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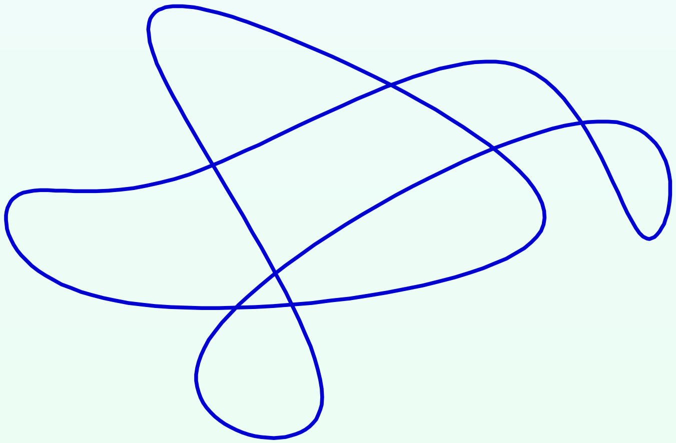

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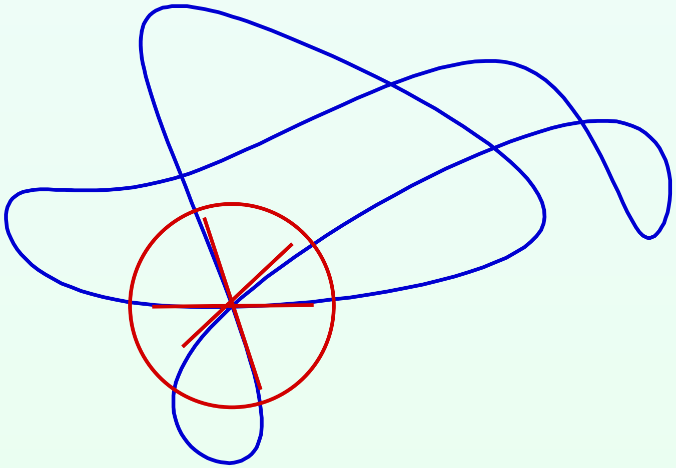

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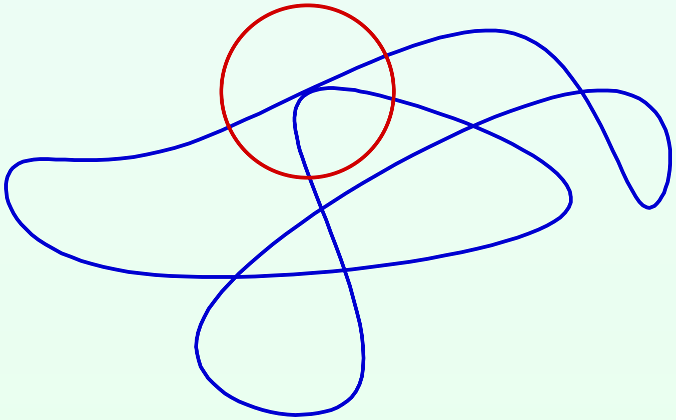

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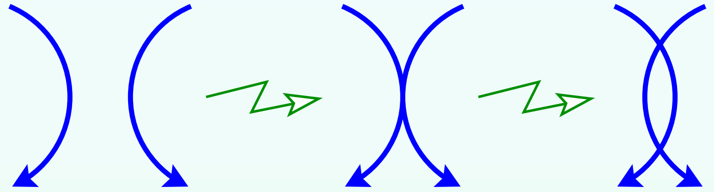

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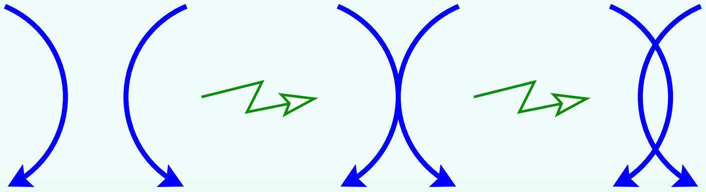

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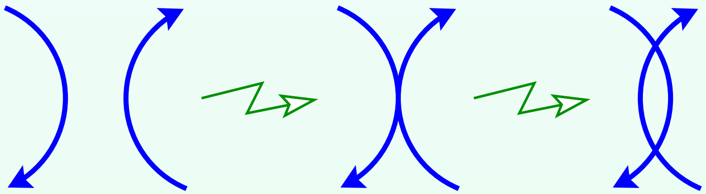

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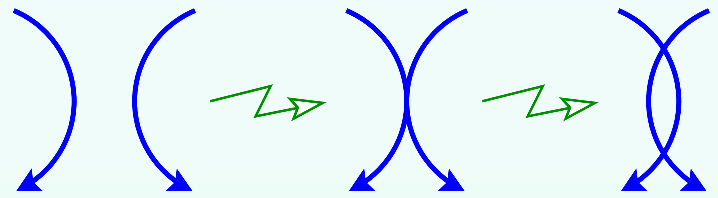

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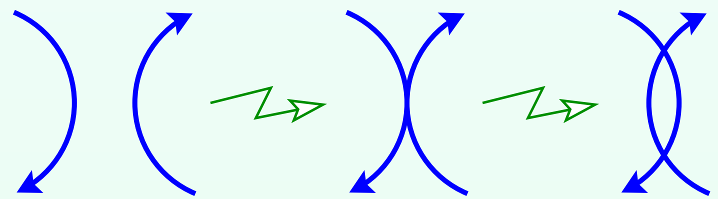

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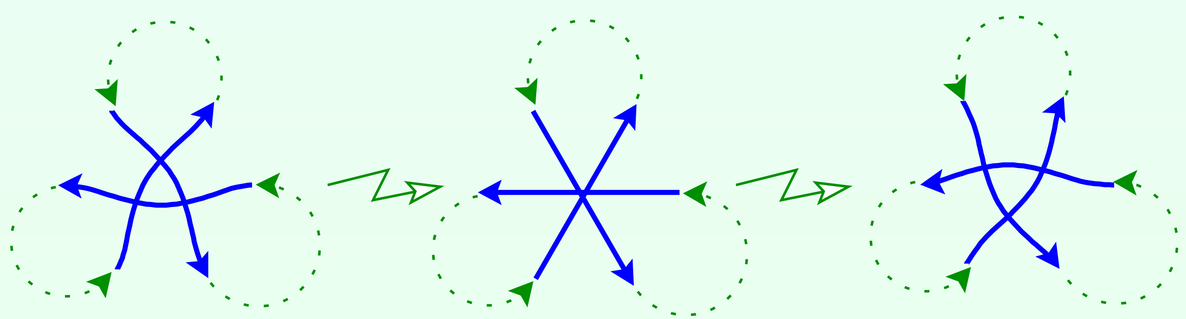

Triple point perestrojka.

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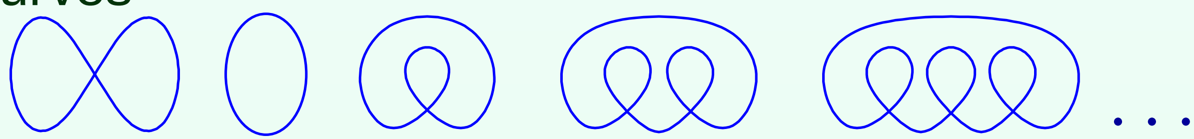
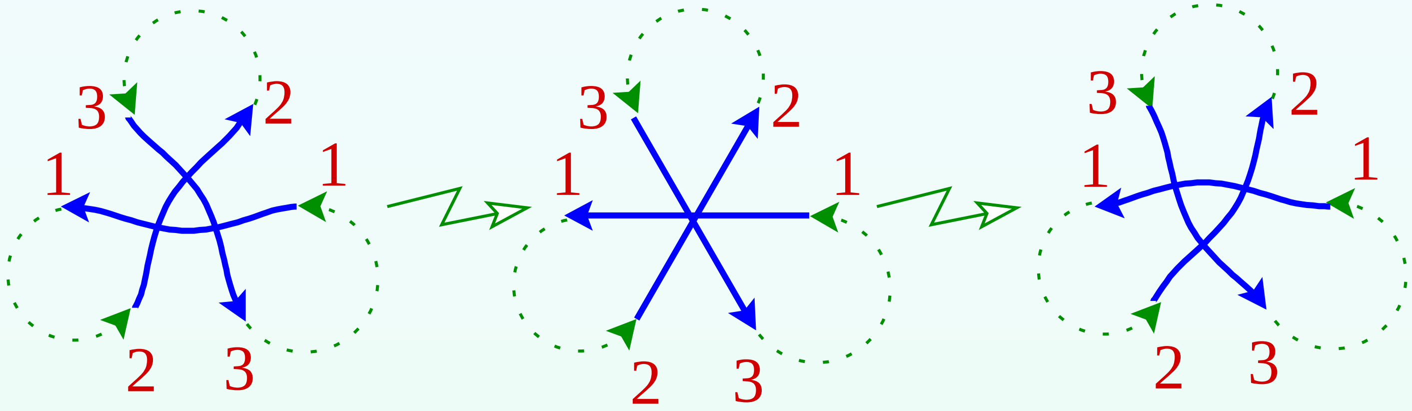


K_0 K_1 K_2 K_3 K_4

the strangeness is

$$St(K_0) = 0, \;\; St(K_{i+1}) = i \;\; (i = 0, 1, \dots).$$

The direction of change



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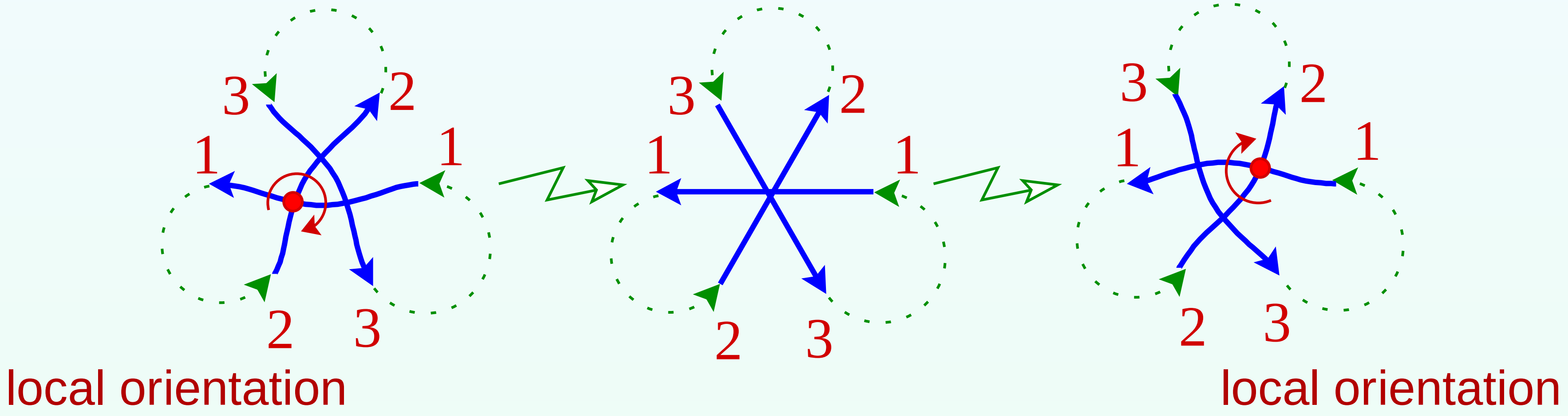

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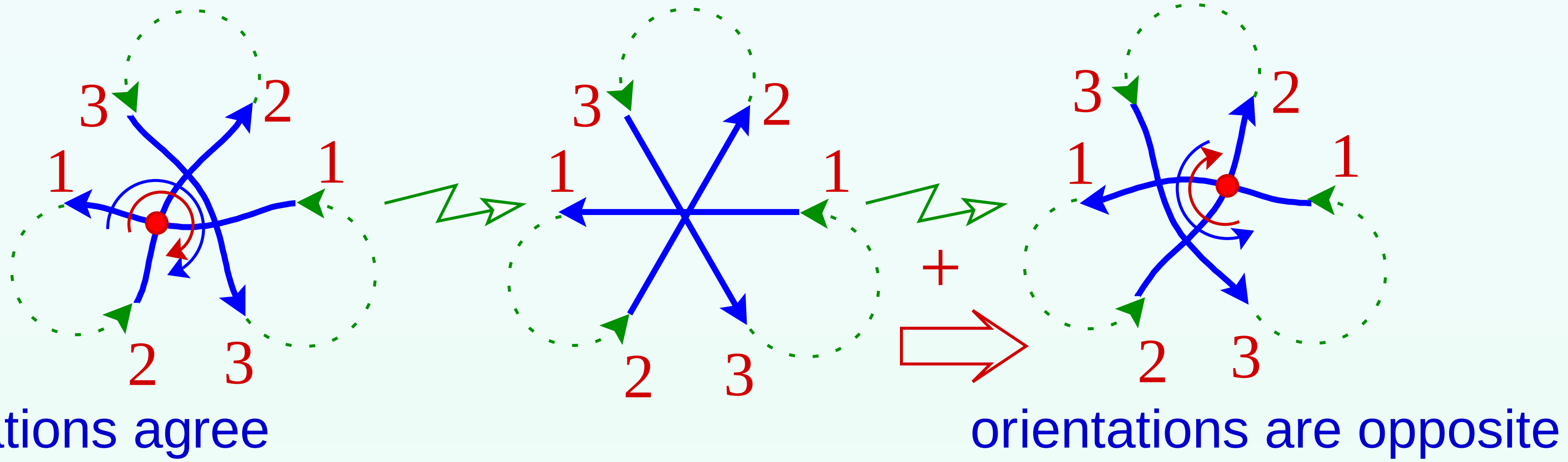

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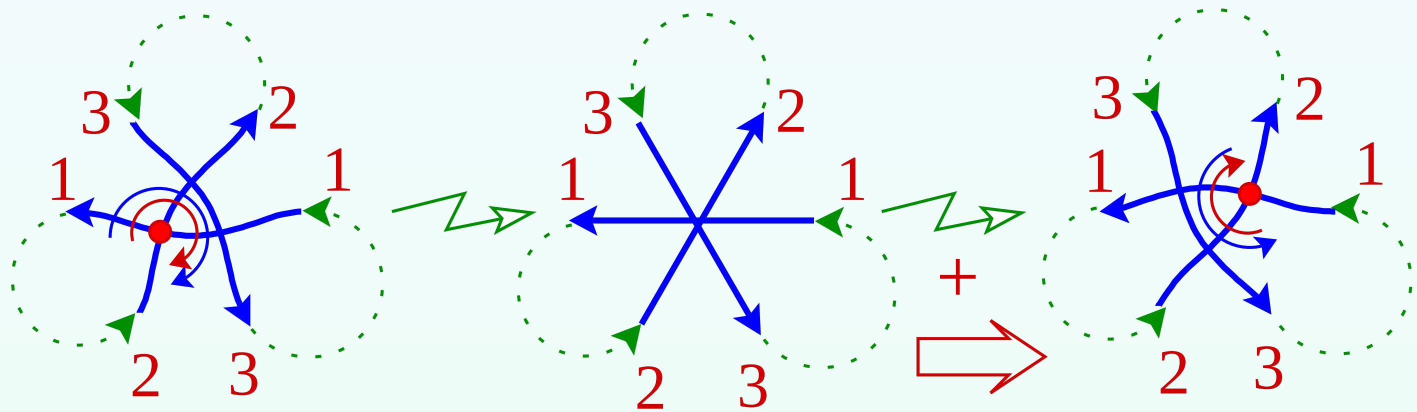

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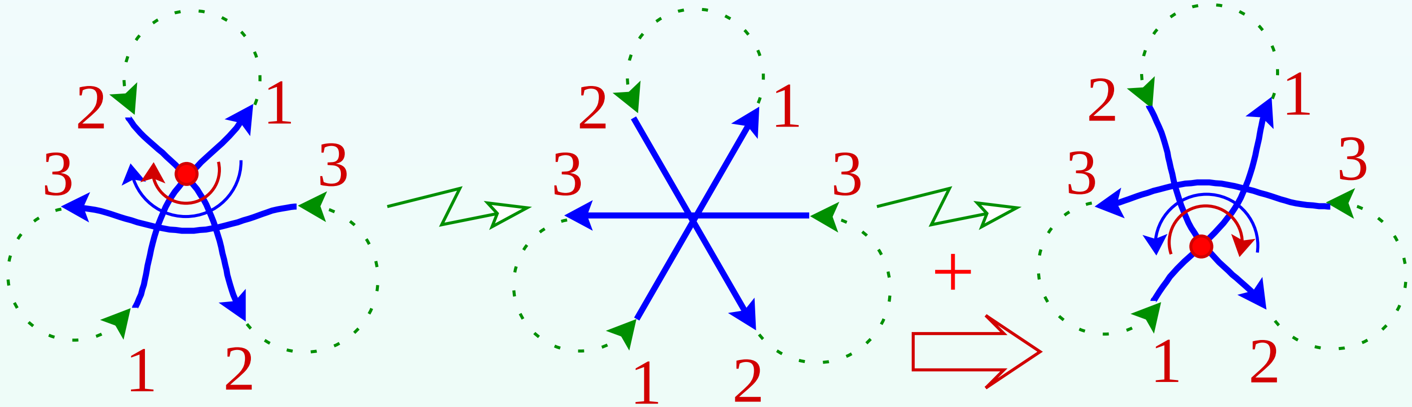

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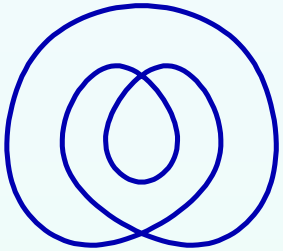
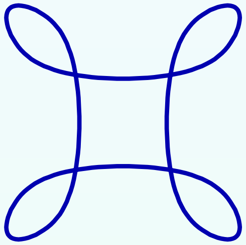
Curves  and  have Whitney number 3.

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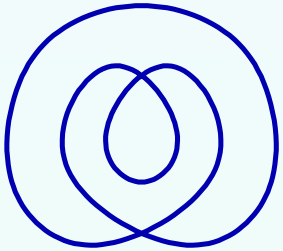
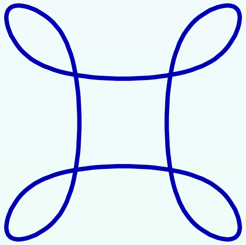
Curves  and  have Whitney number 3.

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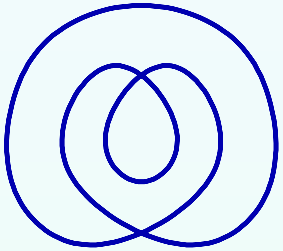
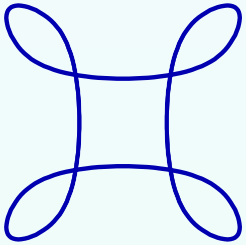
Curves  and  have Whitney number 3.

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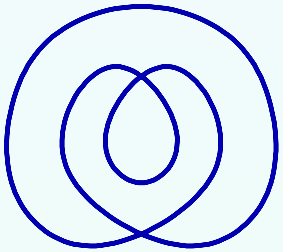
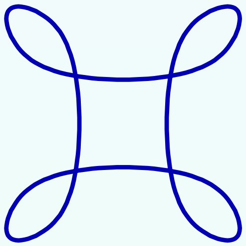
Curves  and  have Whitney number 3.

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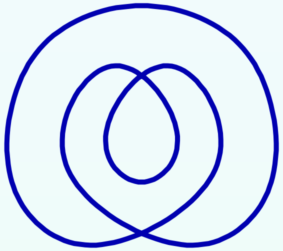
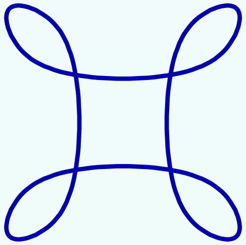
Curves  and  have Whitney number 3.

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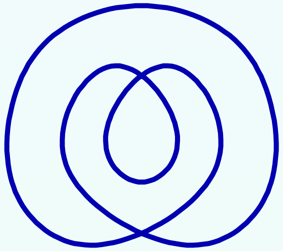
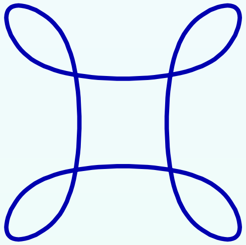
Curves  and  have Whitney number 3.

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Formula for St wanted!

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- Prepare to formulas
- Index on projective plane
- Shumakovitch's formula for St
- On the projective plane

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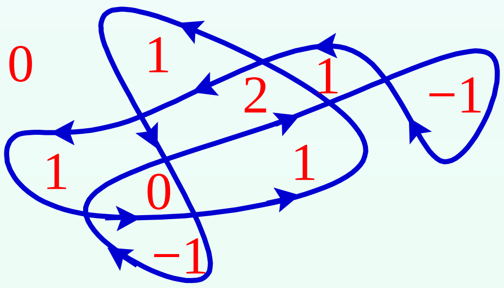

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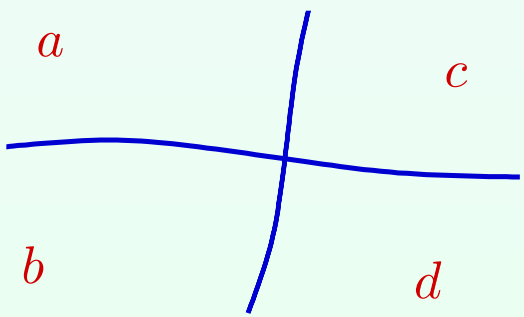

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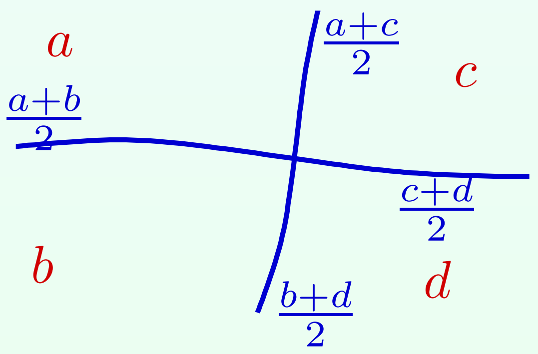

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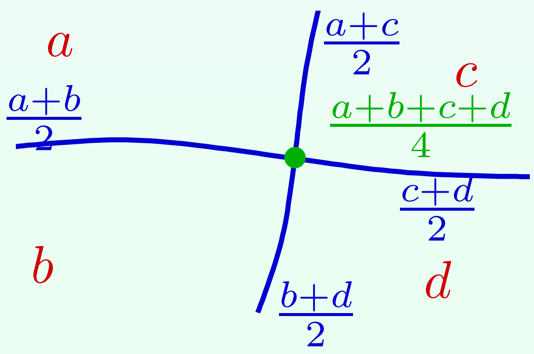

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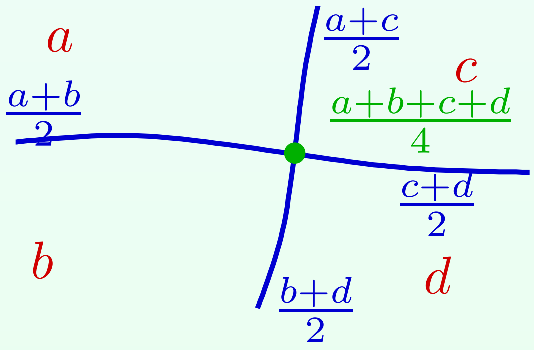

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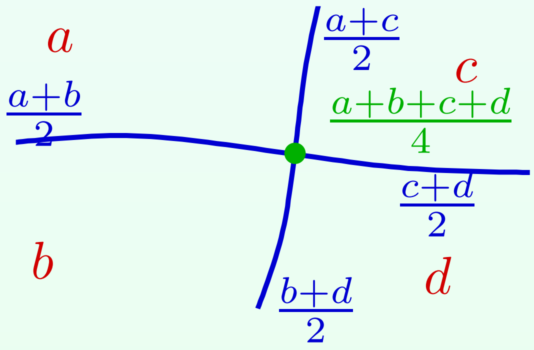

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The extension of n th power of $x \mapsto \operatorname{ind}_C(x)$,
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Hence $\operatorname{ind}^2_C(x)$ is a well-defined number independent on the local orientation.

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Assign to a double point v of C the number $s(v) = \pm 1$,
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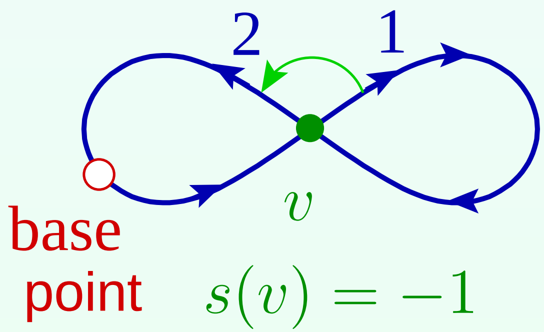

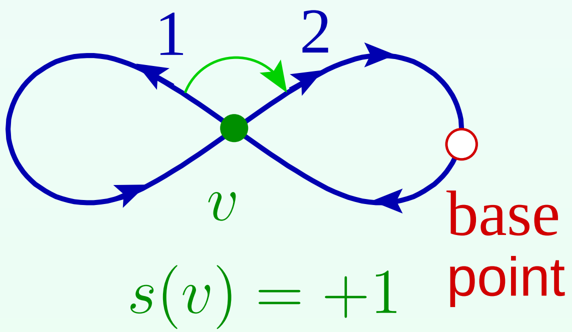

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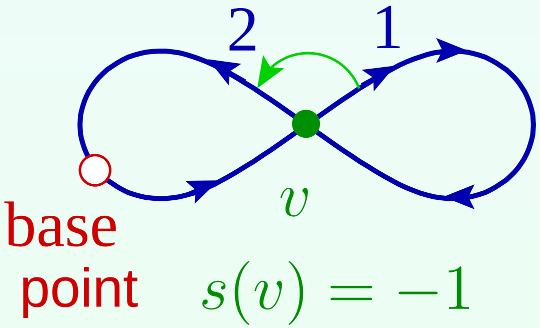

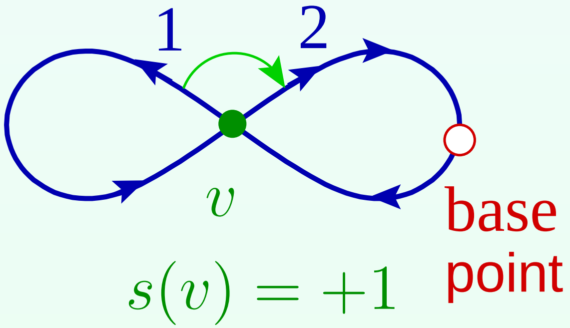

The First Shumakovitch Formula for $St(C)$:

$$St(C) = \sum_{\text{double points } v \text{ of } C} s(v)\,\mathrm{ind}_C(v) + \mathrm{ind}^2_C(f) - \frac{1}{2}.$$

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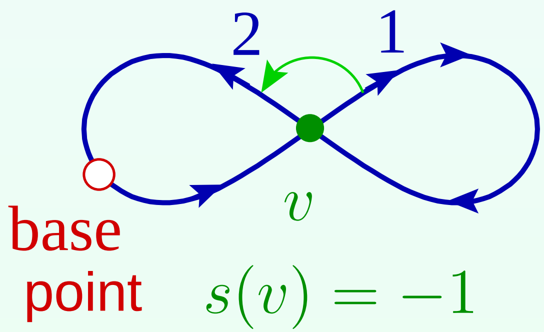

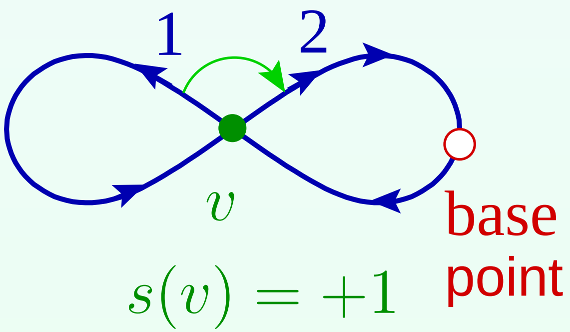

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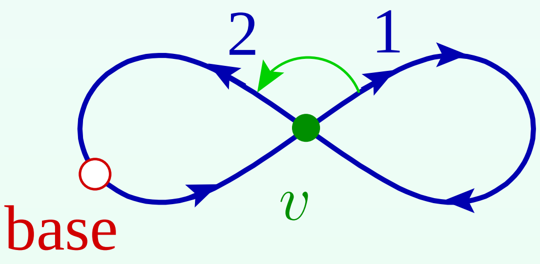

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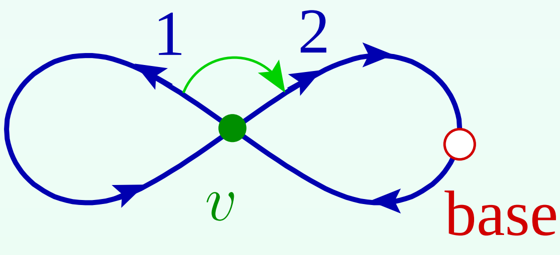

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Does the formula

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make sense for a generic curve on $\mathbb{R}P^2$?

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At first glance, no,

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Another way to understand the formula.

At each double point the ordering of branches determines a local orientation such that $s(v) = +1$ with respect to it. Take $\mathrm{ind}_C(v)$ with respect to this local orientation

and sum up over all double points.

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Thus the right hand side makes sense.

The number given by the formula has all the properties expected from $St(C)$.

Arnold's strangeness

Formulas for strangeness

Algebraic curves

- Choice of curves
- New perestrojkas
- Strangeness of $\mathbb{R}A$
- Formula for strangeness
- Cusp perestrojka

Algebraic curves

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Consider irreducible real algebraic plane projective curves A

Choice of curves

Consider irreducible real algebraic plane projective curves A of degree d

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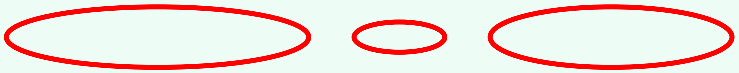

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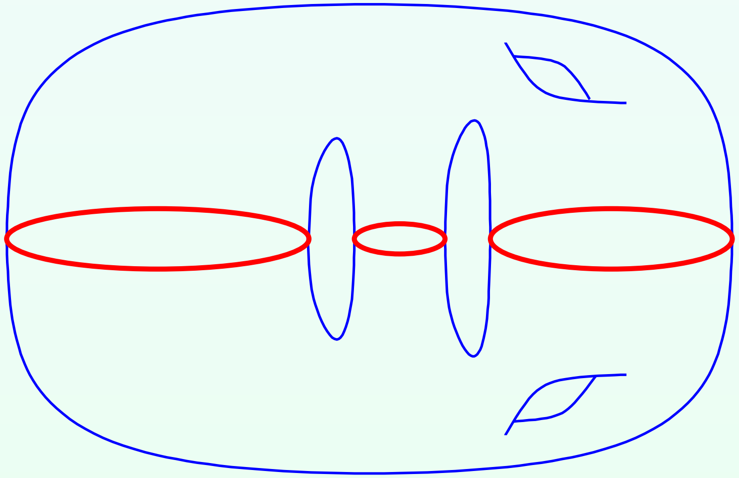

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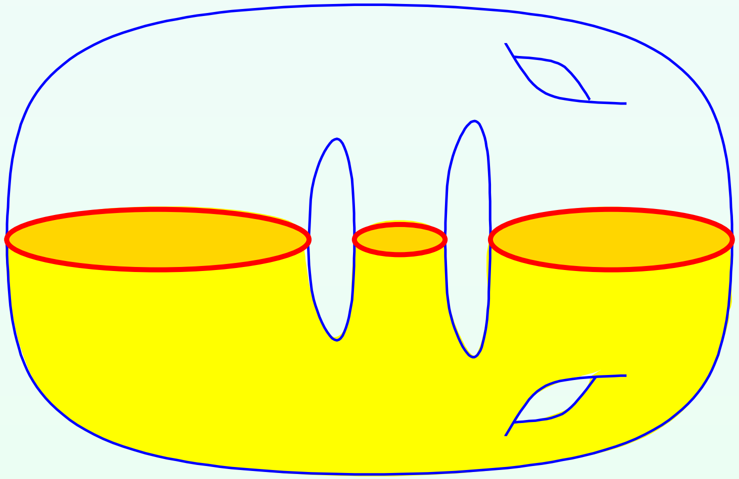

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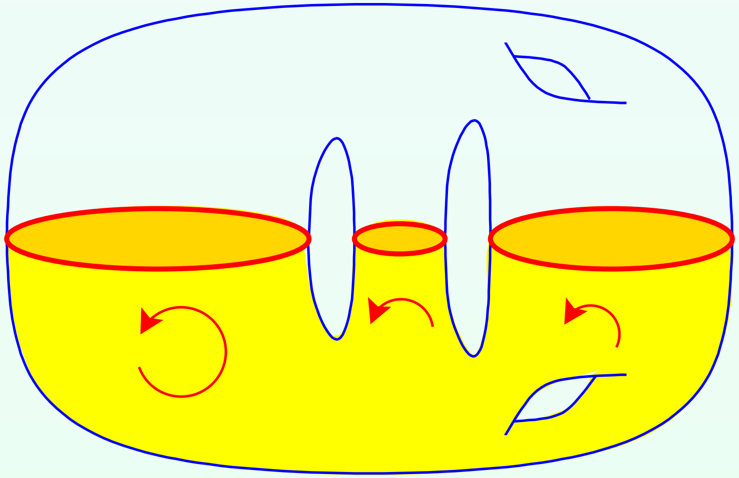

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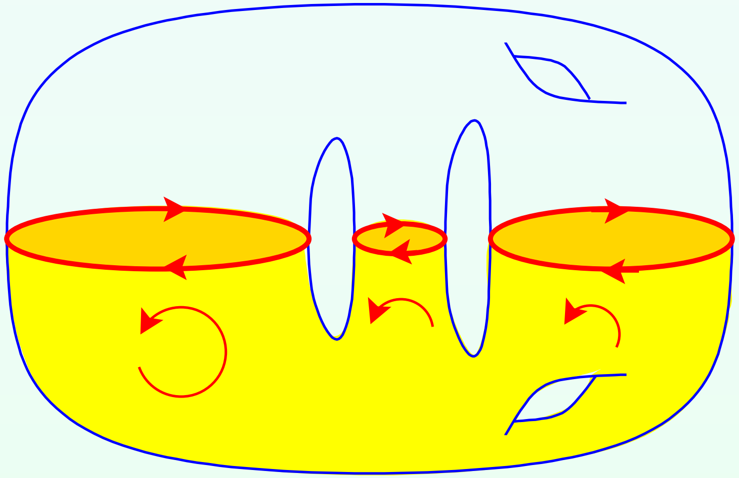

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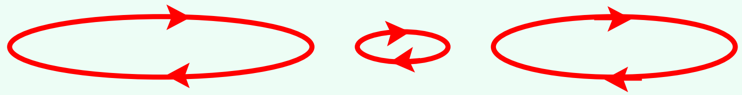

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Consider irreducible real algebraic plane projective curves A of degree d, genus g and type I , with $\mathbb{R}A$ equipped with a complex orientation.

The latter is equivalent to a choice of a half $\mathbb{C}A_+$ of $\mathbb{C}A$.

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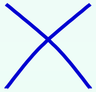
- real double points with two real branches ,

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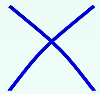
- real double points with two real branches ,
- solitary real double point with two imaginary conjugate branches,

isolated point in $\mathbb{R}A$, local normal form $x^2 + y^2 = 0$.

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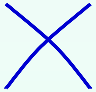
- real double points with two real branches  ,
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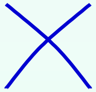
- real double points with two real branches ,
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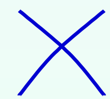
- real double points with two real branches ,
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Another way to get this local orientation: perturb the curve keeping type I and converting the solitary point into an oval.

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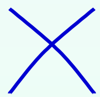
• real double points with two real branches ,

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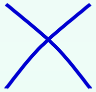
- real double points with two real branches ,
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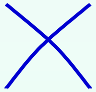
- real double points with two real branches ,
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Denote the number of the latter points by τ .

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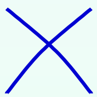
- real double points with two real branches ,
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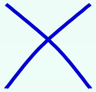
- real double points with two real branches ,
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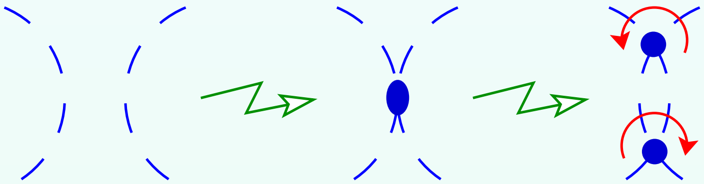

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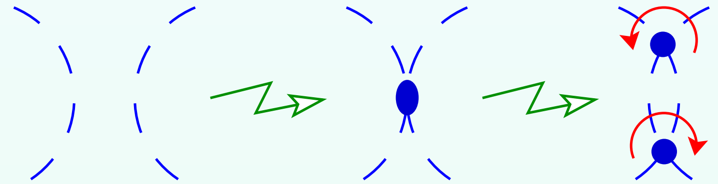

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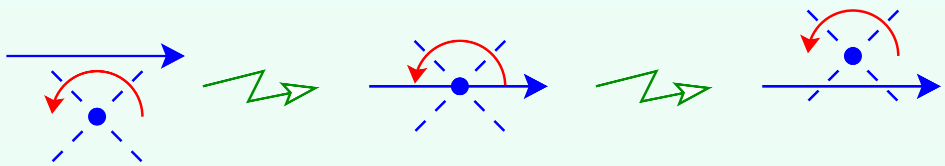

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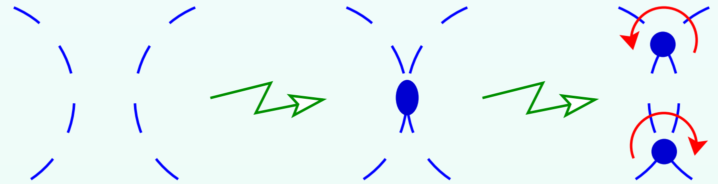

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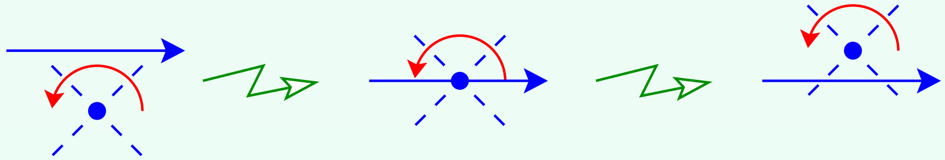

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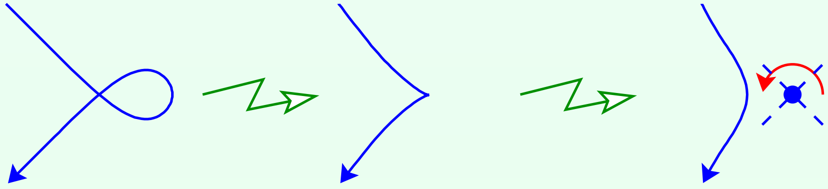

Cusp perestrojka.

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The result does not depend on the marked points,
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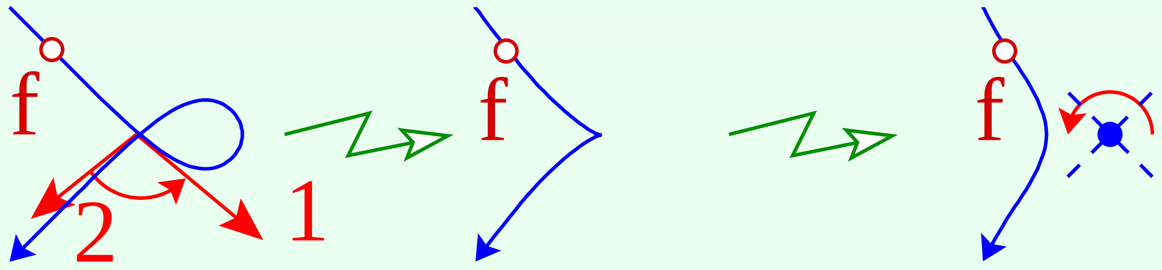

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A true Strangeness, which is an invariant of degree one, can be obtained by constructing a co-orientation of the union of the triple point strata both with all three branches real and with one real and two imaginary branches.

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Mark an ordinary point f_K on each infinite connected component K of $\mathbb{R}A$.

Let

$$St(A) = \sum_{\text{real double points } v \text{ of } A} \mathrm{rot}_v(A) + \sum_{\text{components } K \text{ of } \mathbb{R}A} \left(\mathrm{ind}^2_{\mathbb{R}A}(f_K) - \frac{1}{2}\right)$$

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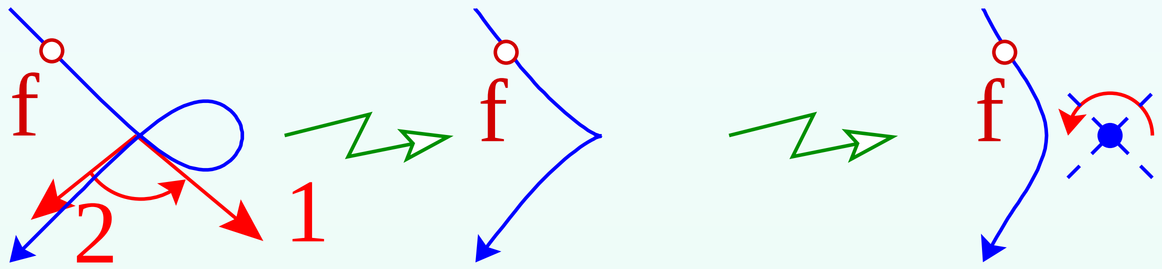
If the branches of A at v are real, this is the orientation defined by the orientations of the second branch followed by the first branch.
If the branches are imaginary, this is the orientation defined by the complex orientation of the curve.

Cusp perestrojka

This is why $St(A)$ does not change under the cusp perestrojka:



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