Link invariants a la Alexander module

Oleg Viro

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The main construction

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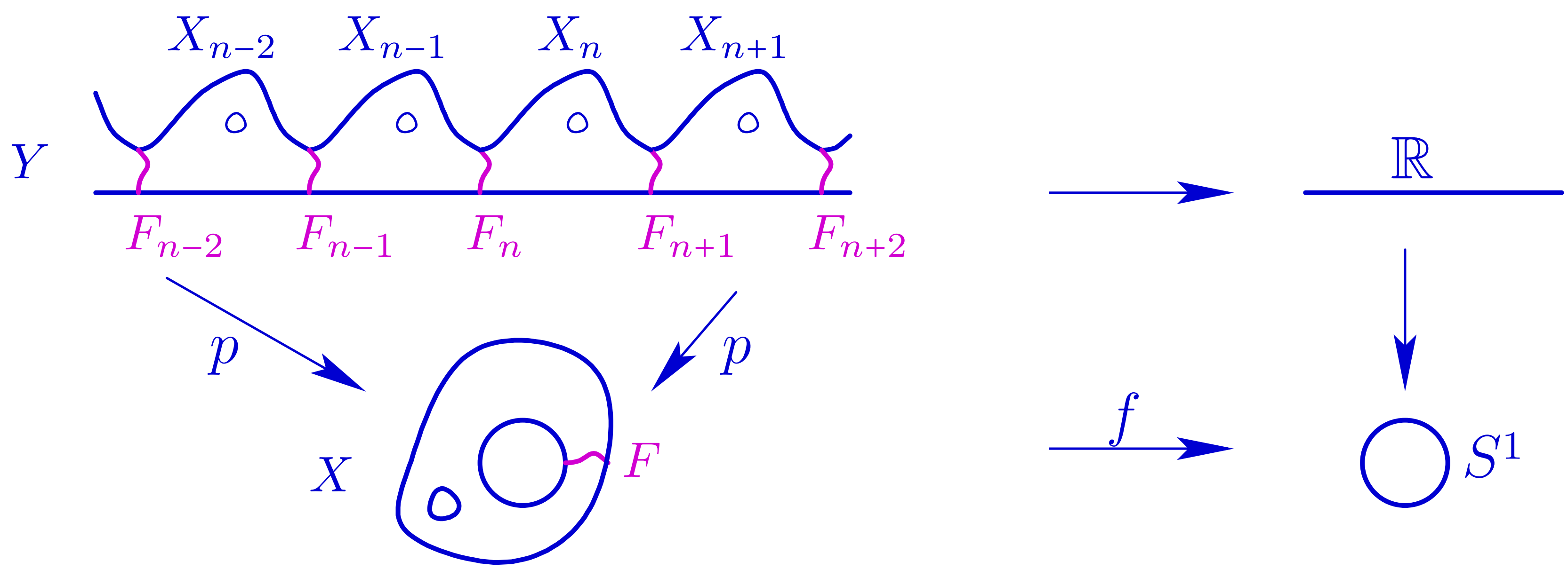

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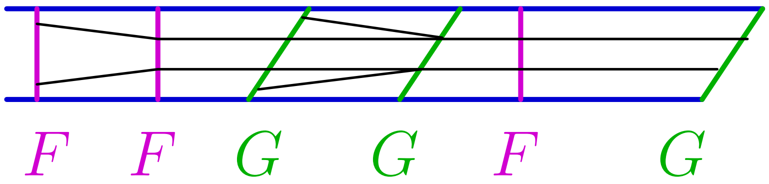

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If $X = S^3 \smallsetminus K$, $Z(F) = H_1(F;\mathbb{Q})$, then
this is Seifert's calculation of the Alexander module $H_1(Y;\mathbb{Q})$ of K.

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For 3-manifolds and various TQFT's, it was studied by Pat Gilmer in 90s.

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The invariant is a bigraded $\mathbb{Z}[\mathbb{Z}]$-module. It is trivial, unless $\chi(\Lambda) = 0$.

Theory of Skeletons

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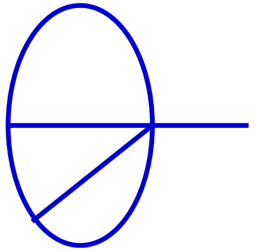

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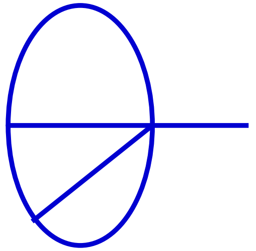

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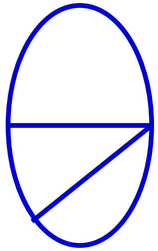

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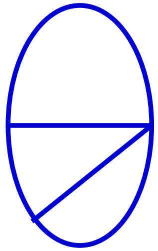

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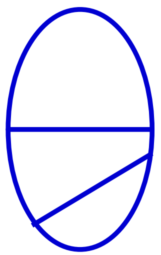

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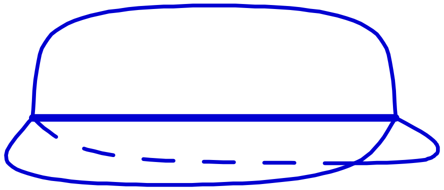

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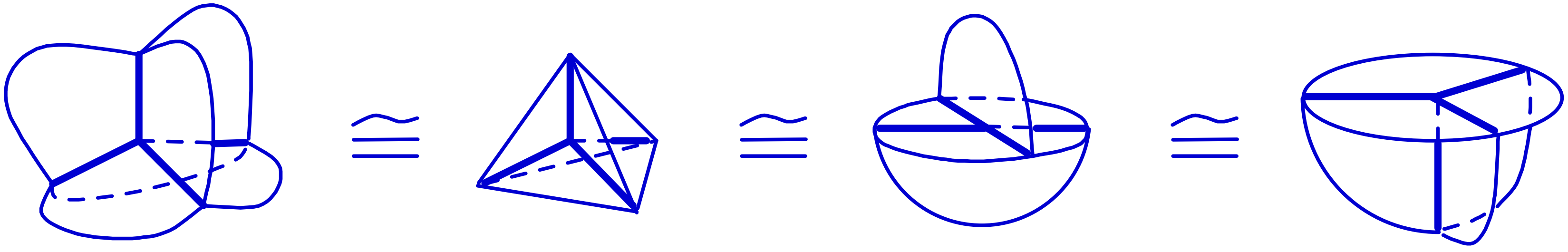

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A generic 2-polyhedron that is not equipped with gleams is considered shadowed with all gleams equal zero.

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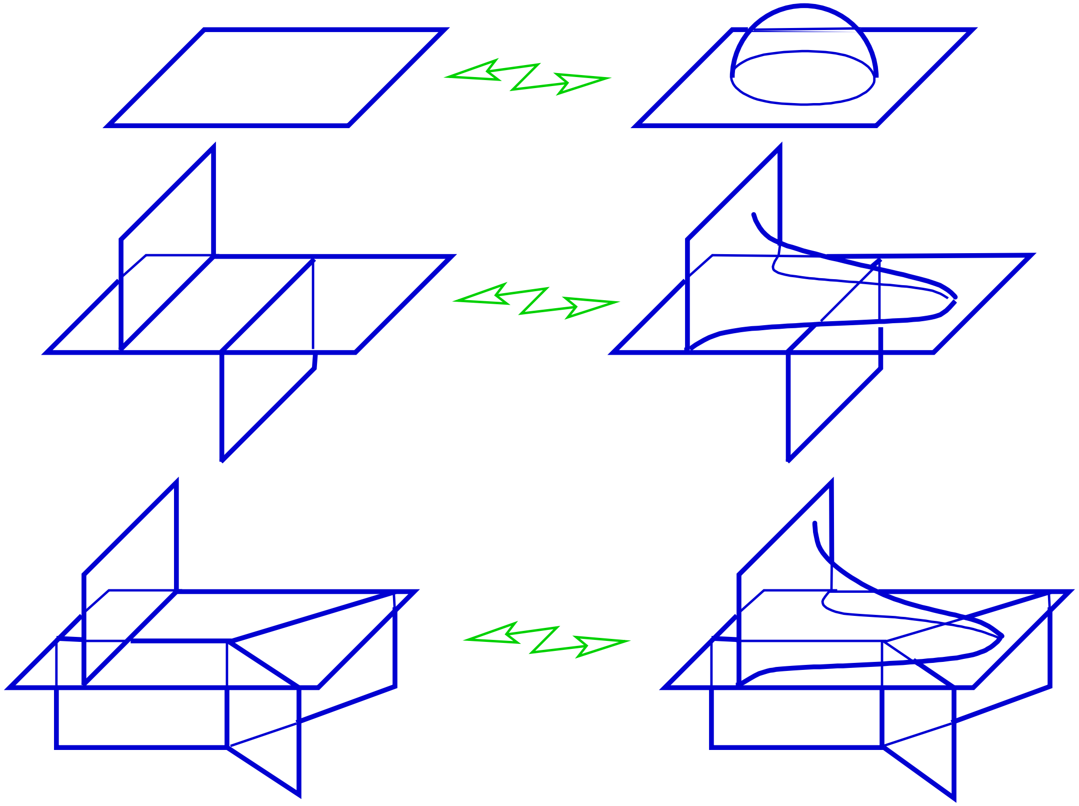

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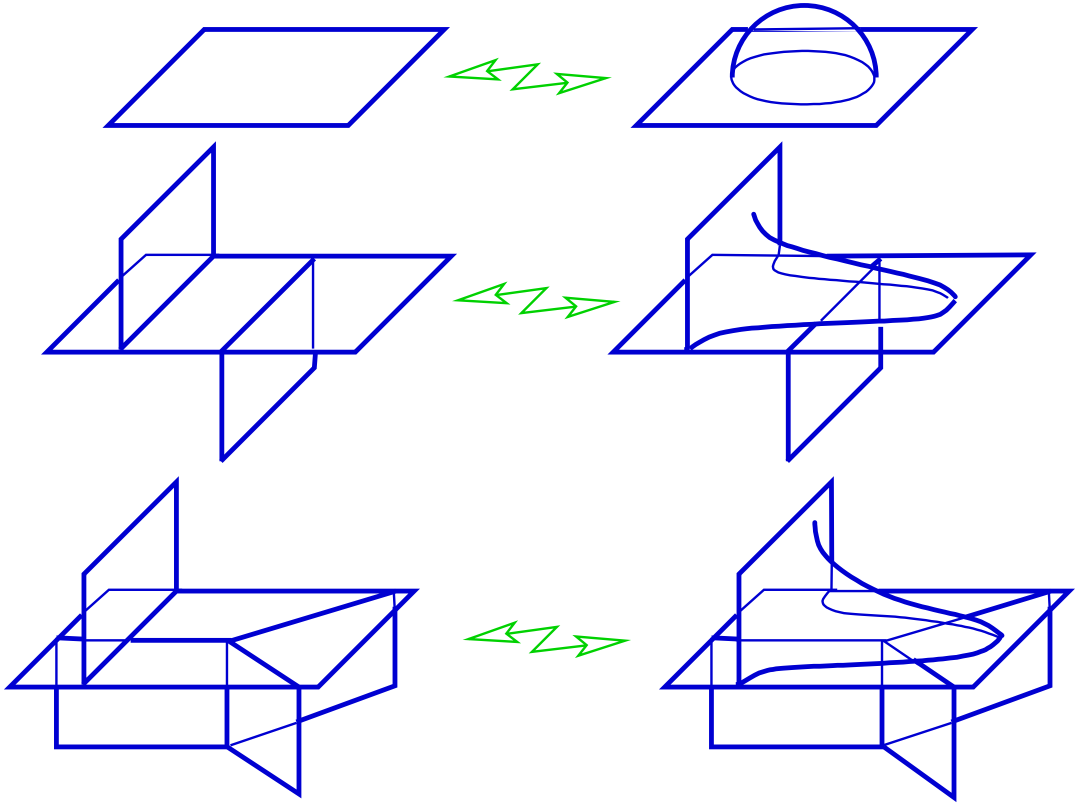
Corollary. Any quantity calculated for a generic 2-polyhedron and invariant with respect the three Matveev-Piergallini moves is a **topological invariant of a 3-manifold**.



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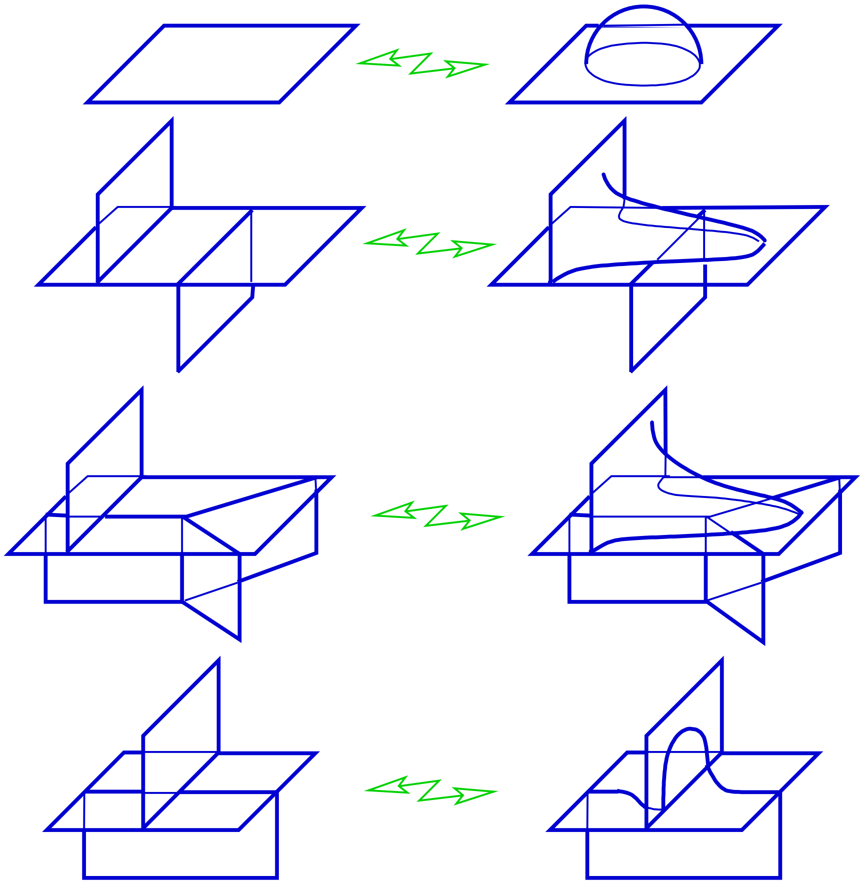

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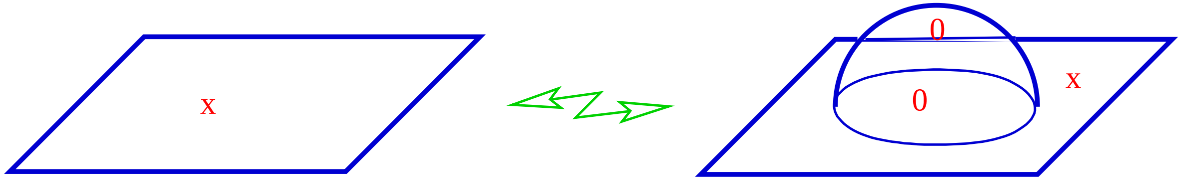

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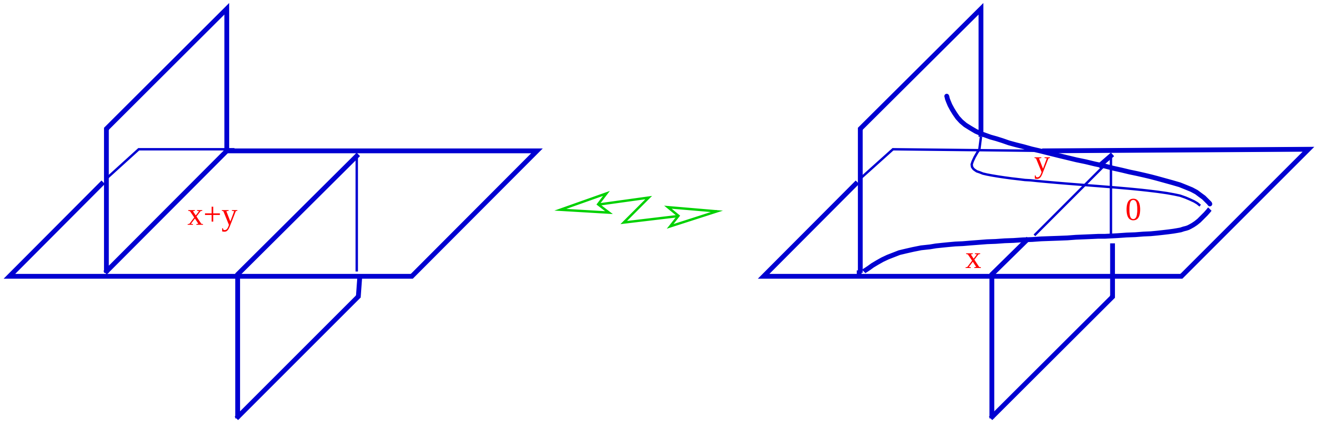

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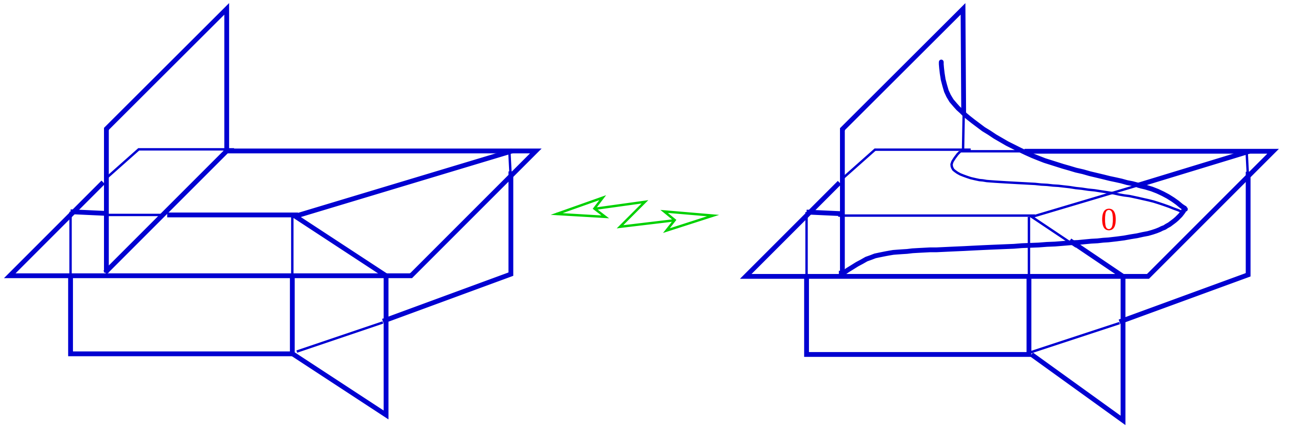

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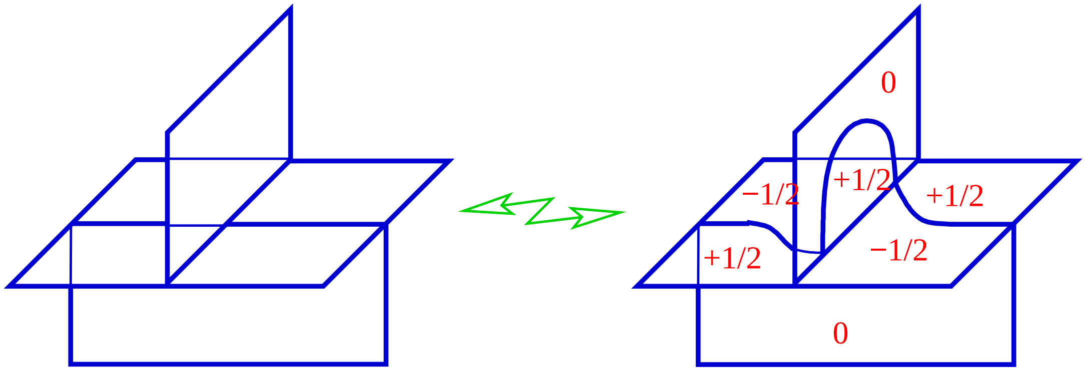

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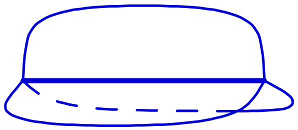
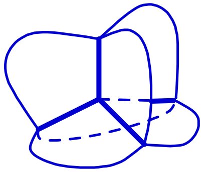
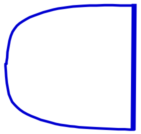
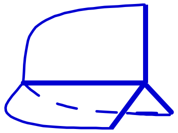
A generic 2-polyhedron with boundary has interior points with neighborhoods homeomorphic to $\mathbb{R}^2$, or , or , and boundary points with no neighborhoods of these sorts, but with neighborhoods homeomorphic to  or .

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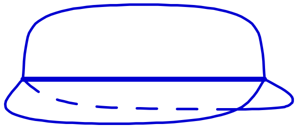
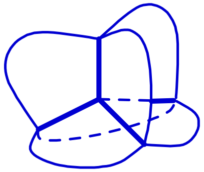
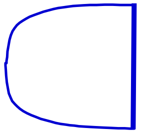
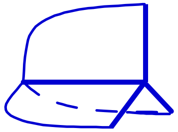
A generic 2-polyhedron with boundary has interior points with neighborhoods homeomorphic to $\mathbb{R}^2$, or , or , and boundary points with no neighborhoods of these sorts, but with neighborhoods homeomorphic to  or .

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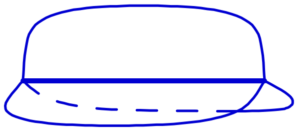
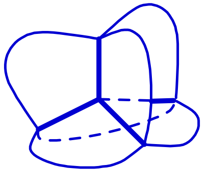
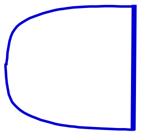
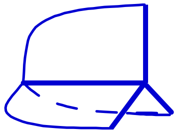
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

and boundary points with no neighborhoods of these sorts,

but with neighborhoods homeomorphic to  or .

The boundary of a generic 2-polyhedron is a generic 1-polyhedron.

A generic 2-polyhedron X whose boundary ∂X is a disjoint union of 3-valent graphs Γ_0 and Γ_1 is a cobordism between Γ_0 and Γ_1.

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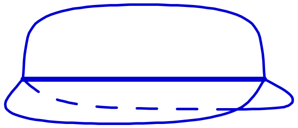
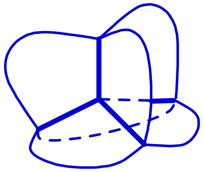
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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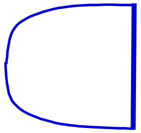
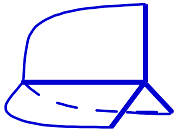
but with neighborhoods homeomorphic to  or .

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Generic shadowed 2-polyhedra with boundary are called equivalent, if they can be transformed to each other by the moves.

Recall: moves do not affect the boundary.

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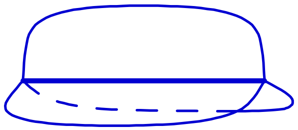
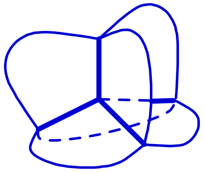
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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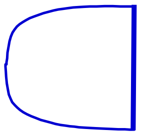
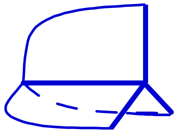
but with neighborhoods homeomorphic to  or .

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Generic shadowed 2-polyhedra with boundary are called equivalent, if they can be transformed to each other by the moves.

Recall: moves do not affect the boundary.

Any two trivalent graphs are cobordant, but there are many non-equivalent generic shadowed 3-polyhedra.

Relative 2-skeletons

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A relative generic 2-skeleton of a compact **3-manifold** W is a generic 2-polyhedron X with boundary such that:

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Choose a framing of ∂X in ∂W.

Now all 2-strata of X have self-intersections.

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Face state sums

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For a state s, let $Z(s) =$

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What w_i and t to choose?

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Not all the axioms of modular category are needed.

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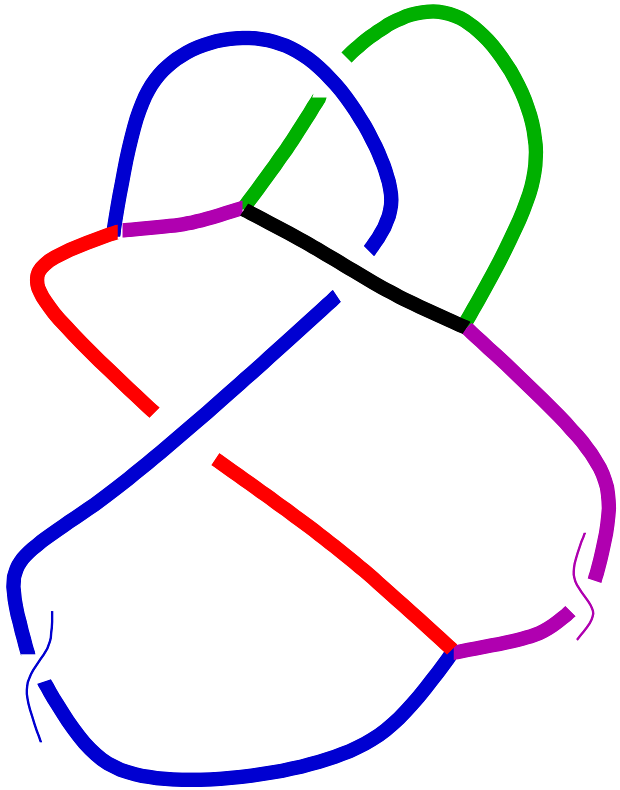

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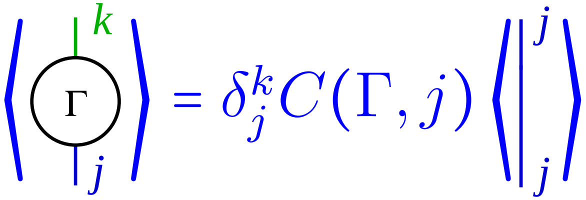

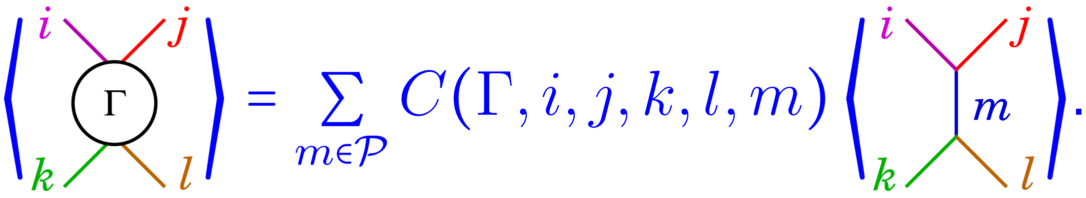

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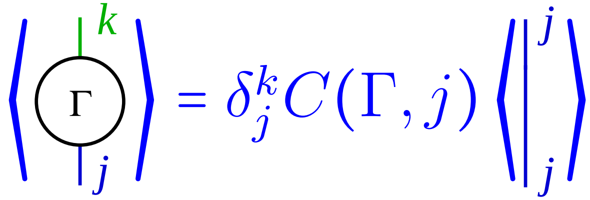
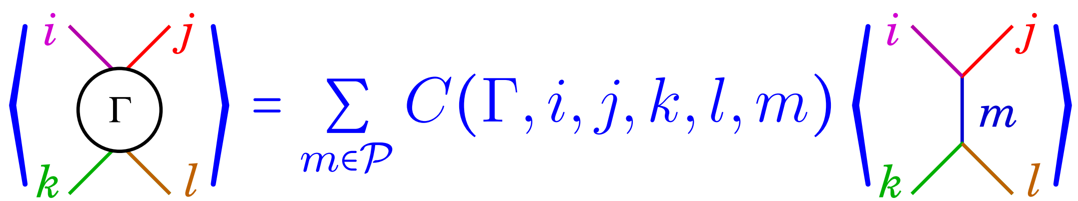
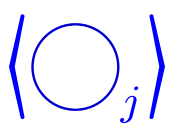
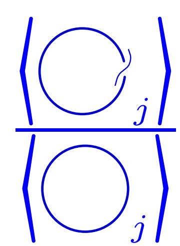
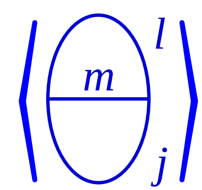




Theorem. If $w_2(j) =$ 

, $t(j) =$ 

, $w_1(j,m,l) =$ 

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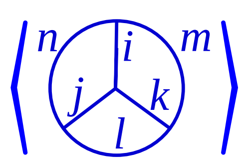
$w_0\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix} =$ 

, $w_3 = \sum_j w_2^2(j)$, then Z_X is invariant under moves and defines a TQFT.

Construction of TQFT

Correction: the state sums define a functor

$$\big(\text{trivalent graphs and their cobordisms}\big) \to \mathrm{Vect}\, k\,.$$

but only a **semifunctor** $\big(\text{manifolds, their cobordisms}\big) \to \mathrm{Vect}\, k\,.$

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In order to turn a functor

$$(\text{trivalent graphs and their cobordisms}) \to \mathrm{Vect}\, k$$

to a functor

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factorize $C(\text{1-skeleton of a manifold} M)$ by $\mathrm{Ker}\, Z_{\text{2-skeleton of } M\times I}$.

Denote $C(\text{1-skeleton of a manifold} M)/\mathrm{Ker}\, Z_{\text{2-skeleton of } M\times I}$ by $Z(M)$ and $Z_{\text{2-skeleton of a cobordism } W}$ by Z_W. **This is a TQFT!**

Old and new TQFT'es

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There many invariants of framed colored trivalent graphs for which the S-matrix is not invertible.

Upgrading the colored Jones

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The only restriction: $H_i \cap S$ is a disk for each i and in the state sum the colors of these disks coincide with the colors of the corresponding components of L.

Building a special 2-skeleton

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Choose a Seifert surface $F \subset S^3$ for L transversal to R and ∂m_i and disjoint from ∂l_i .

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Partial state sums

Instead,
we split the state sum that provides the value at q of the colored Jones into partial state sums based on $U \subset S$,
and apply the Seifert-Turaev construction to each of them and to the infinite cyclic covering $\widetilde{U} \to U$ defined by $F \cap U = F \cap R$.

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The arcs on ∂m_i contribute via w_1,
the vertices (i.e., intersections of ∂m_i with 1-strata of R) via w_0.

Modules of a link

Application of the Seifert-Turaev construction to the partial sums gives,

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The coefficients are products of values at q of Tchebyshev polynomials.

Khovanov homology for surfaces in $S^3 \times S^1$

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This can be obtained from a link $\bar{\Lambda} \subset S^4$ by a surgery along an unknotted component of $\bar{\Lambda}$ homeomorphic to S^2.

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Let the intersection $L = S^3 \times \{1\} \cap \Lambda$ be transversal, and $\widetilde{\Lambda} \subset S^3 \times \mathbb{R}$ be the preimage of Λ under $S^3 \times \mathbb{R} \to S^3 \times S^1 : (x, y) \mapsto (x, e^{2\pi i y})$.

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Luoying Weng calculated $Z_{i,j}(\Lambda)$ for many such surfaces.

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Can it detect linking/knotting of a surface consisting of

a sphere and sphere with 2 handles?

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Theorem. $Z_{i,j}(\Lambda)$ is invariant under isotopy of Λ in $S^3 \times S^1$.

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Because cobordisms needed for Khovanov homology
are surfaces in $S^3 \times I$,
while in the proof we meet
a cobordism between a link in $S^3 \times \{\mathrm{pt}\}$ and a skew copy of it.

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Let $L_{t,n} = \widetilde{\Lambda}_t \cap (S^3 \times \{n\}) \subset S^3 \times \mathbb{R}$,

and $W_{t,n} = \widetilde{\Lambda}_t \cap (S^3 \times [n, n+1])$.

Pull this new stuff back by $\widetilde{h}_t : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}$:

$$\widetilde{h}_t^{-1}(L_{t,n}) = L_n \subset \widetilde{h}_t^{-1}(S^3 \times \{n\}) ,$$

$$\widetilde{h}_t^{-1}(W_{t,n}) = \widetilde{\Lambda} \cap \widetilde{h}_t^{-1}(S^3 \times [n, n+1])$$

Table of Contents