Global complex obstructions to a real Morse modification (Klein's enigma)

Oleg Viro
a joint work with Slava Kharlamov

July 5, 2017

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(German: "sind entwicklungsfähig nicht" - is not viable)

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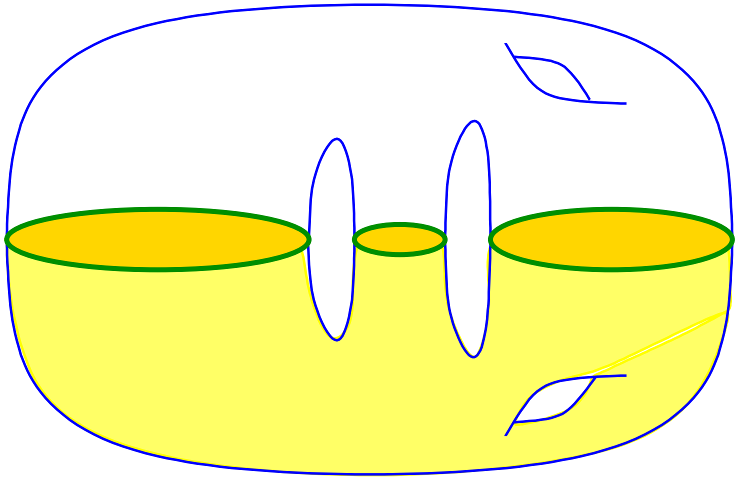

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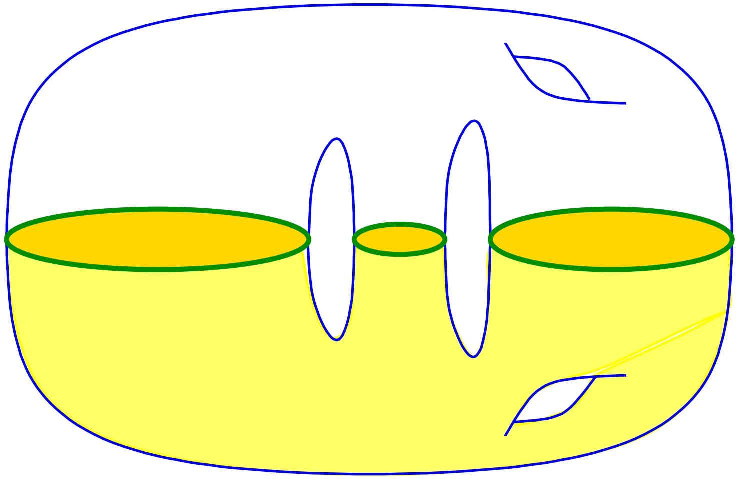

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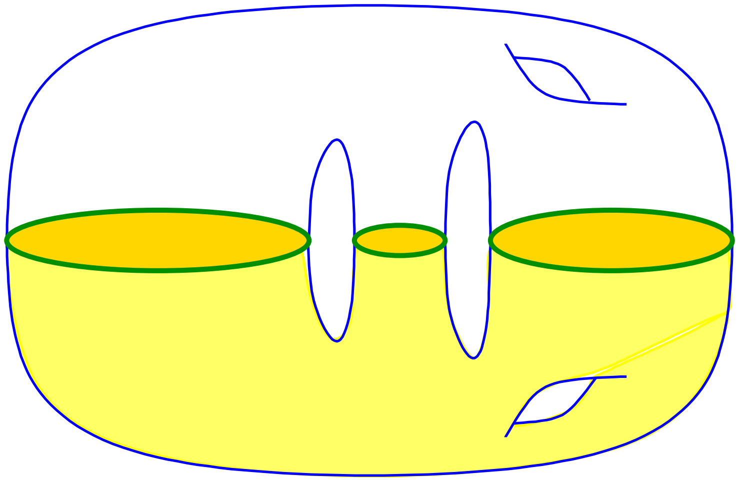

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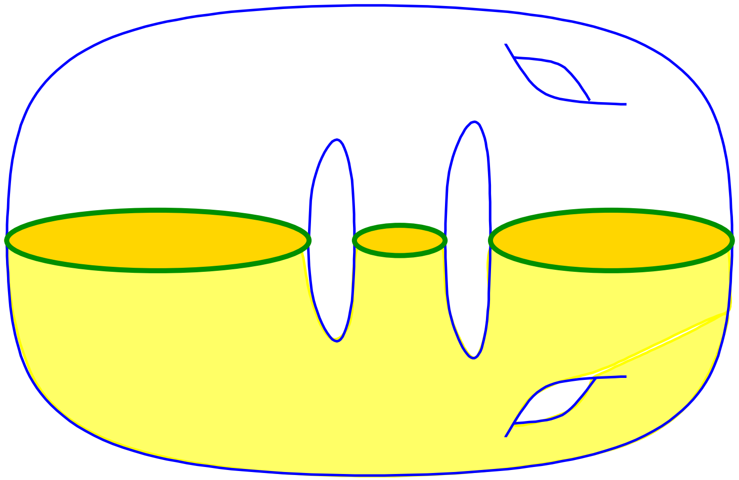

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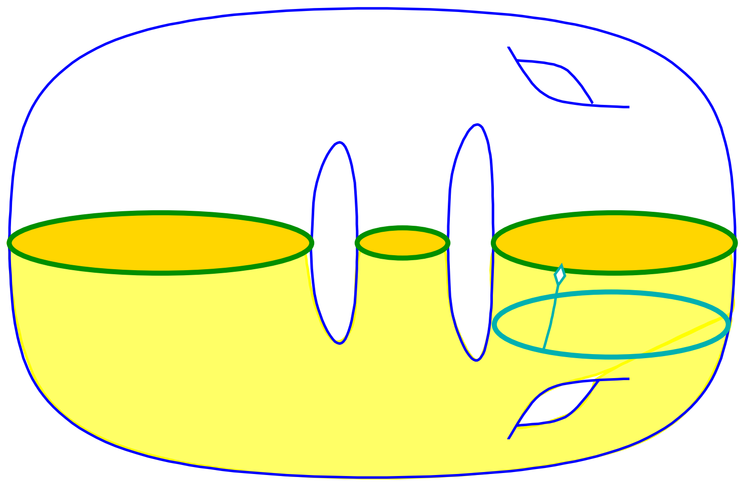

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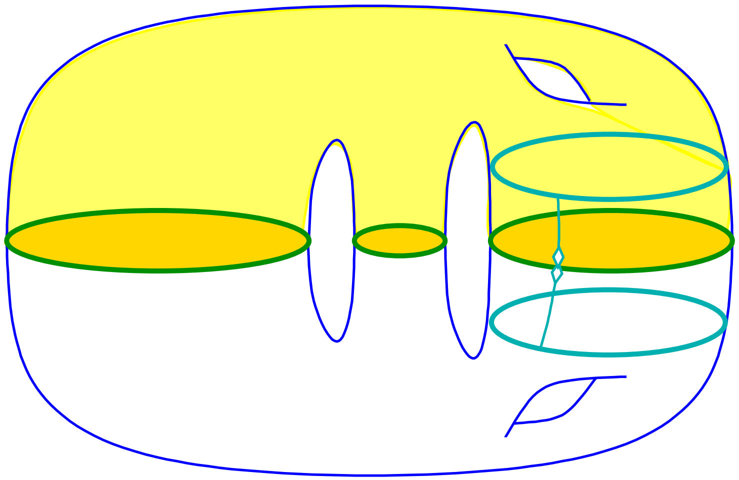

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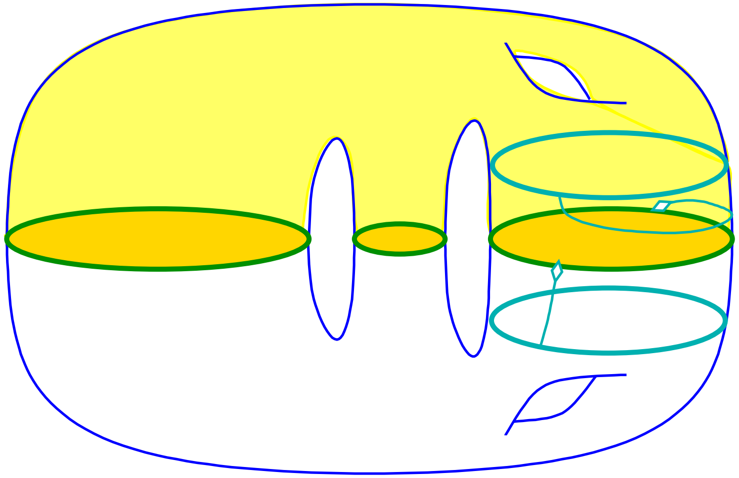

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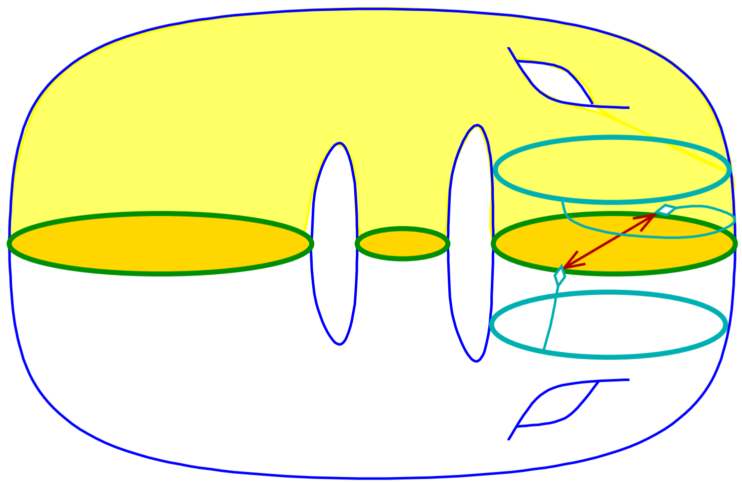

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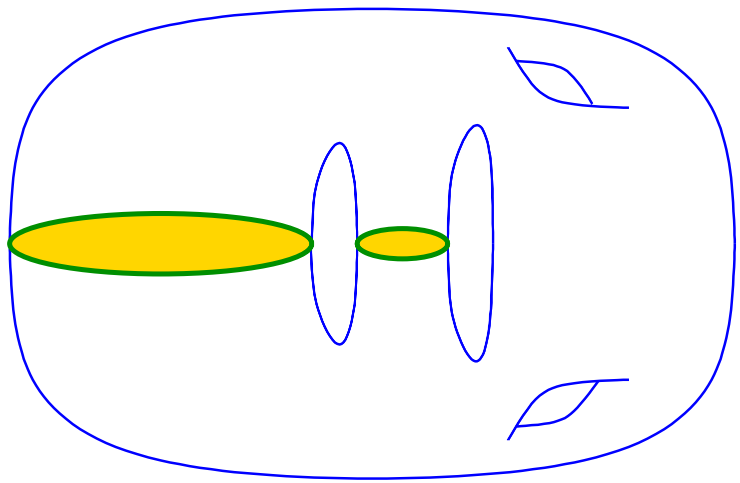

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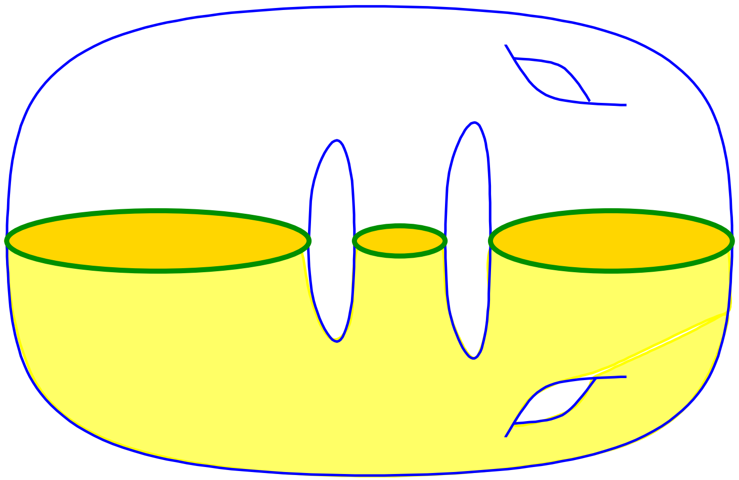

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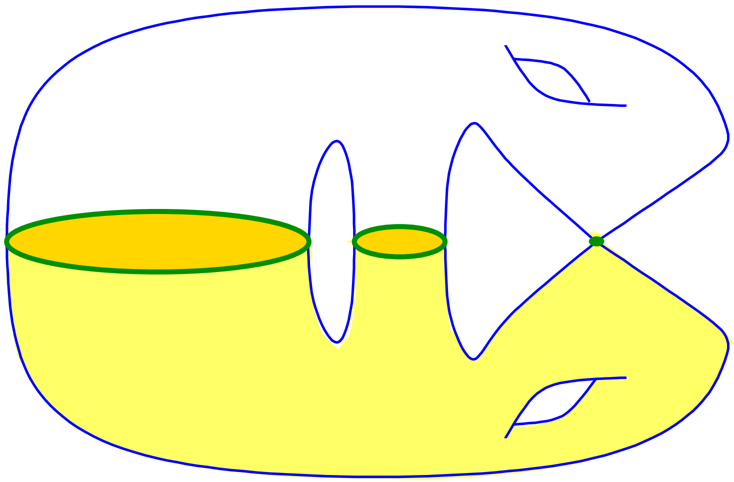

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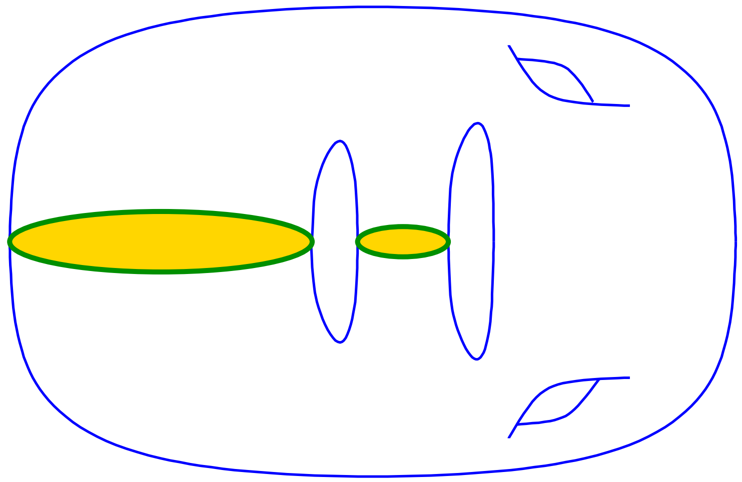

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The talk is about this theorem and its high-dimensional generalizations.

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The distribution of isotopy classes of non-singular plane projective curves between the types in low degrees.

degree	type I	type II
1	1	0
2	1	1
3	1	1
4	2	4
5	3	6
6	14	50

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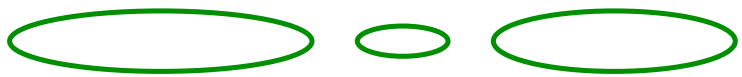

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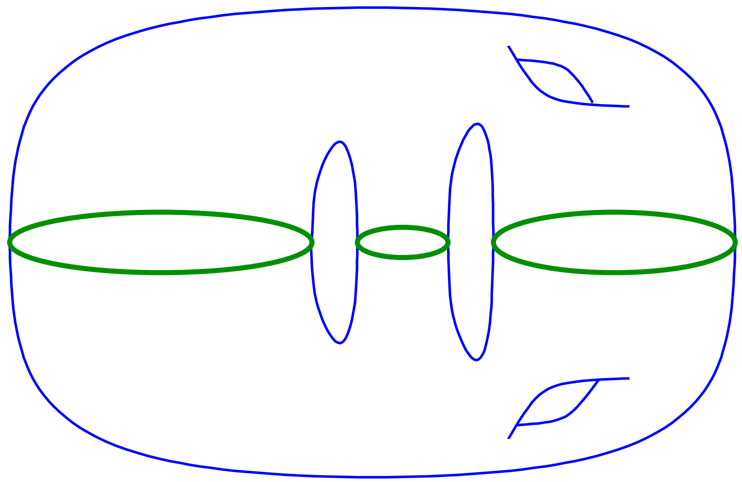

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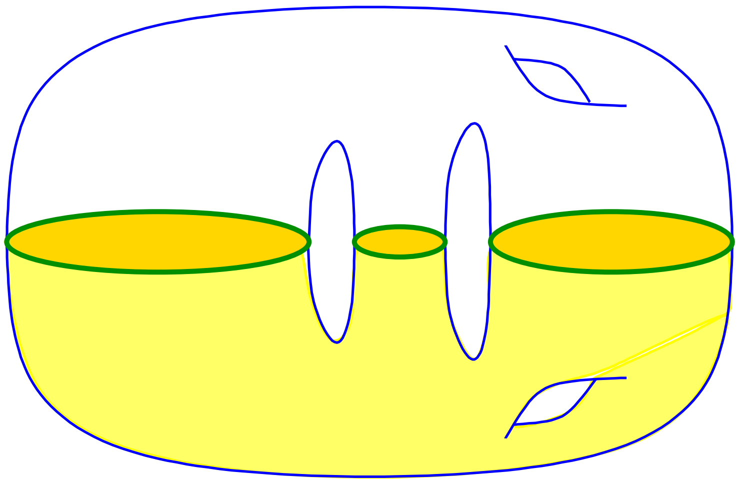

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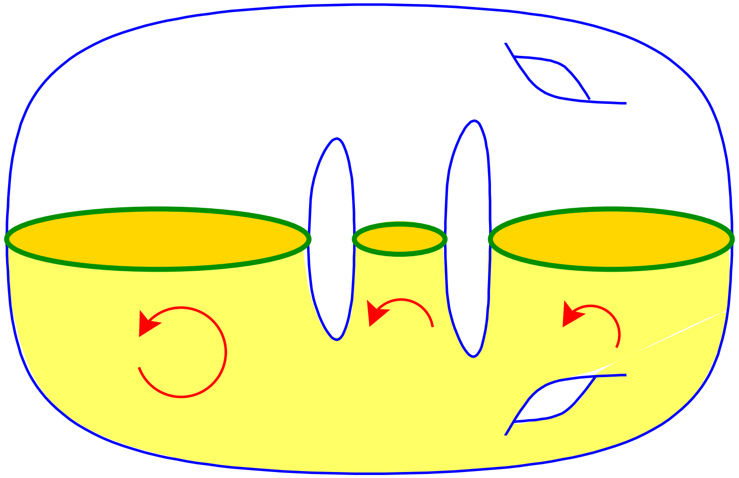

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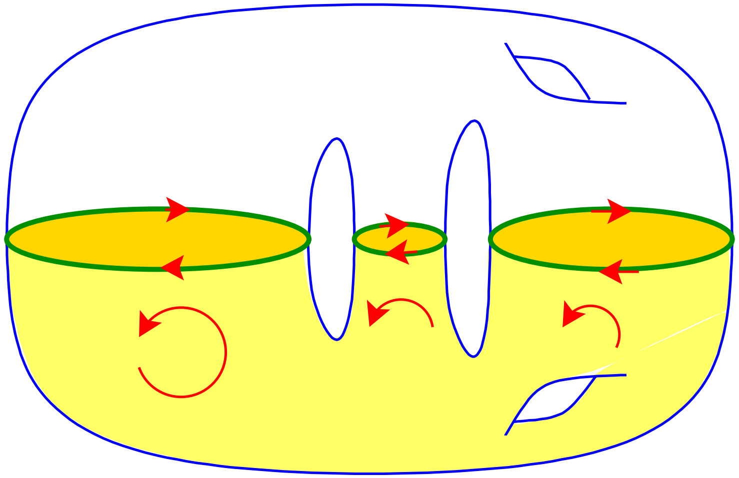

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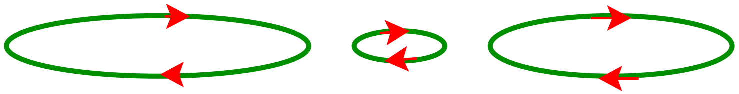

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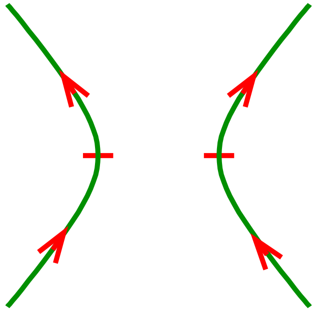

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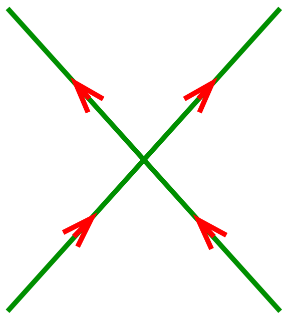

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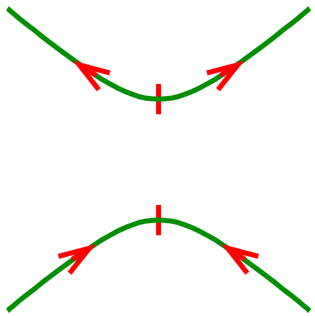

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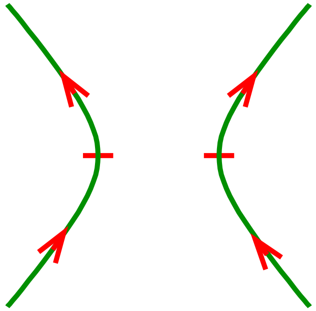

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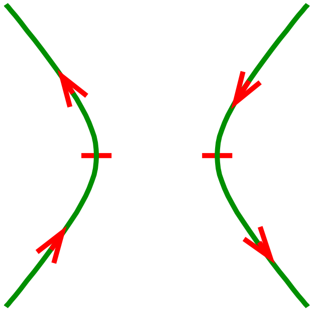

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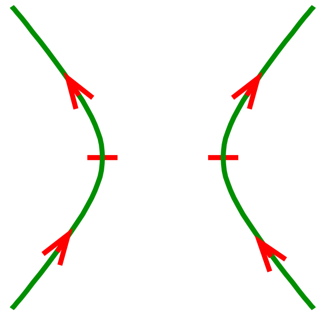

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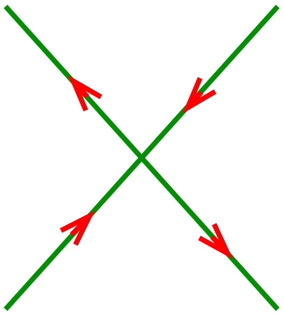

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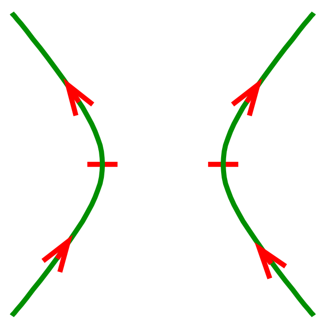

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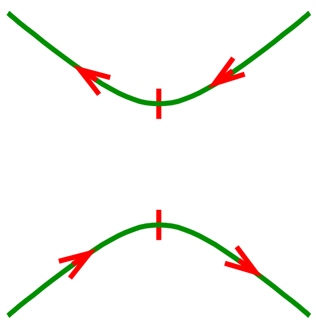

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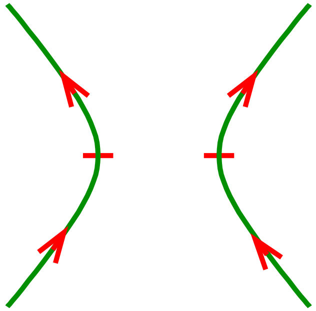
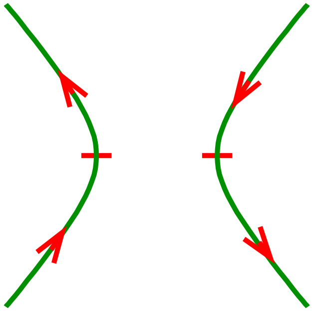
Like this: 

Not like that: 

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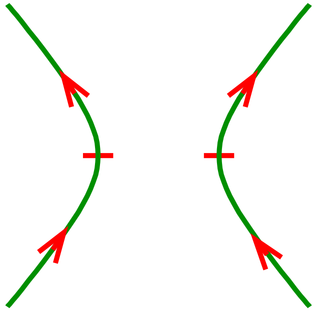
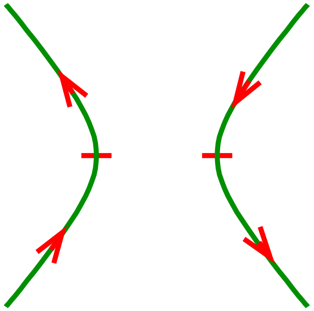
Like this:  Not like that: 

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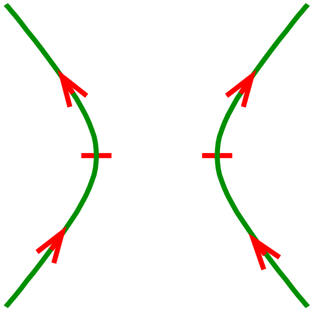
Hyperbola is of type I. 

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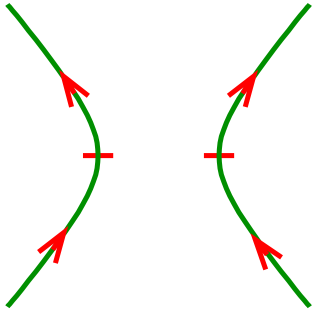
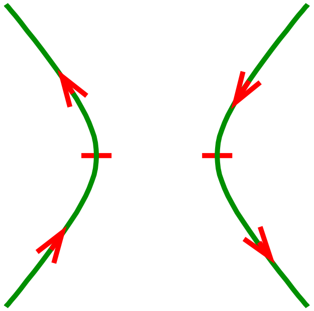
Like this:  Not like that: 

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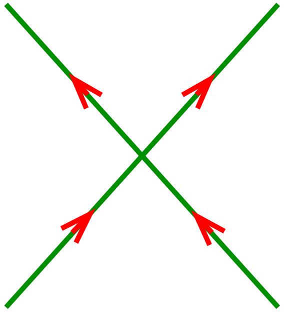
Hyperbola is of type I. 

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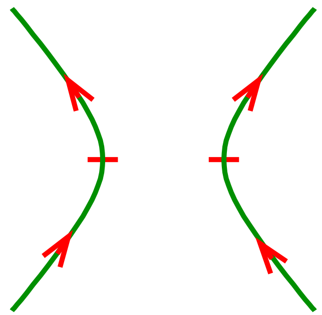

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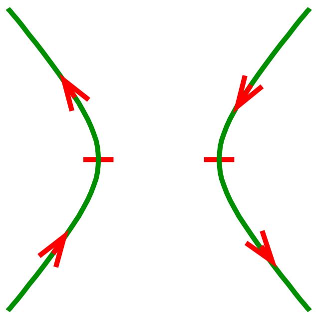

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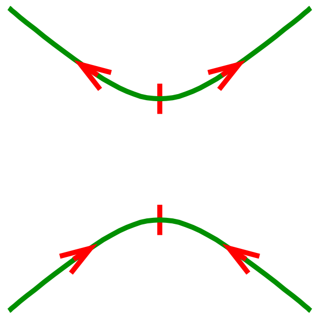

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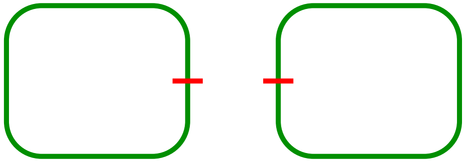

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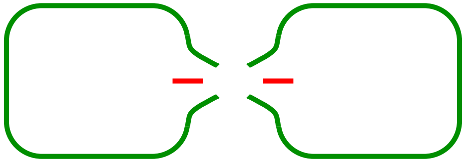

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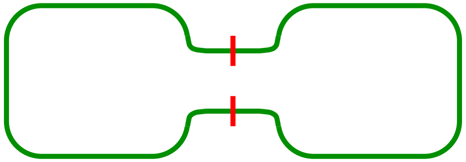

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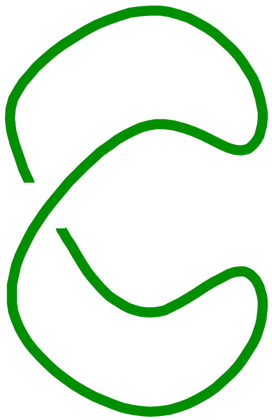
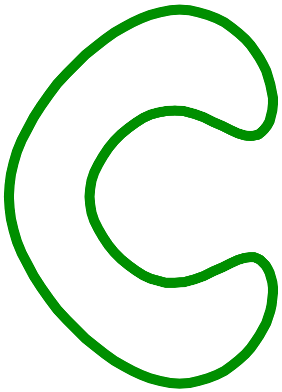
first: 

second: 

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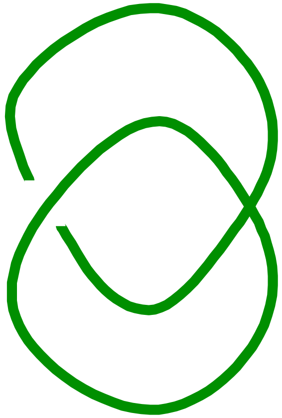
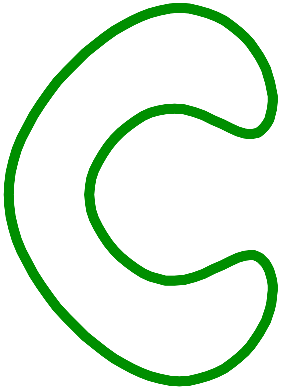
first:  second: 

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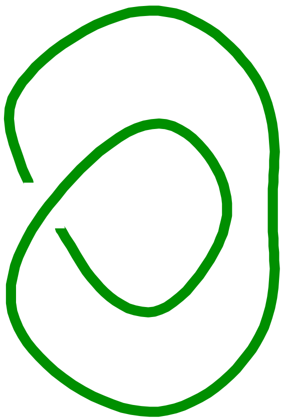
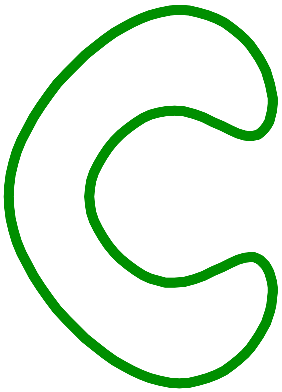
first: 

second: 

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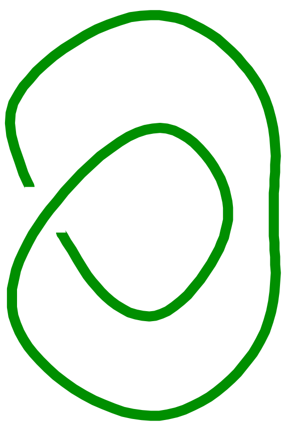
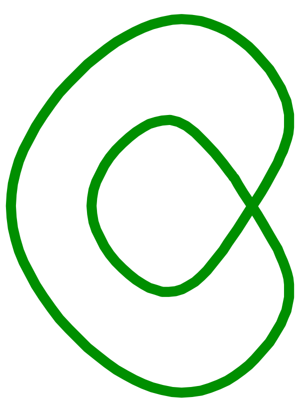
first: 

second: 

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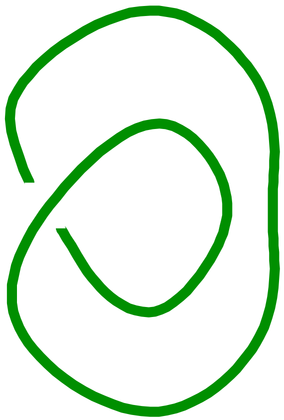
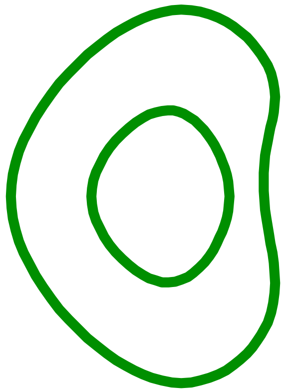
first: 

second: 

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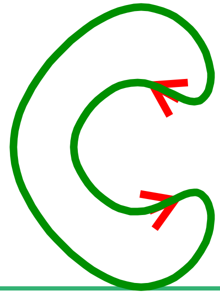

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Thus the $b_*(A_\mathbb{R})$ does not increase. $\square$

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Stiefel orientations generalize both orientations and Spin-structures.

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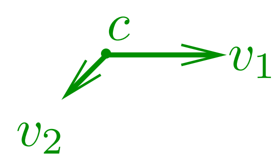

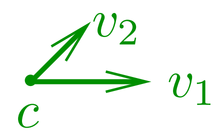

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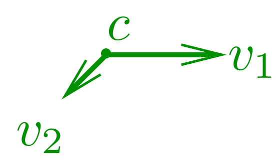

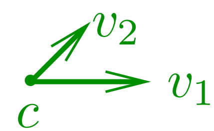

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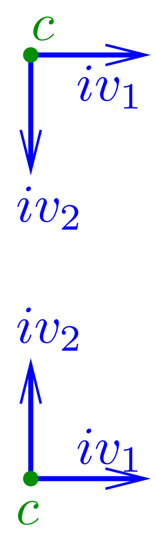

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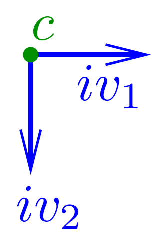

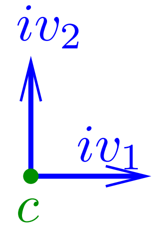

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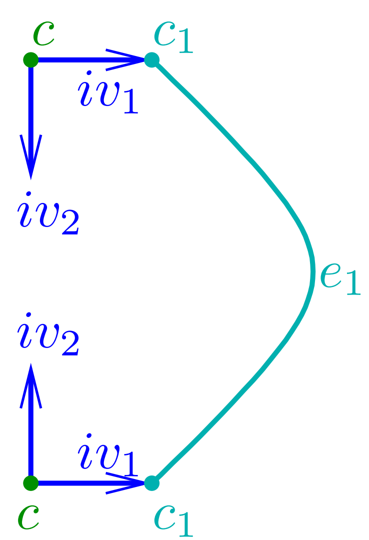

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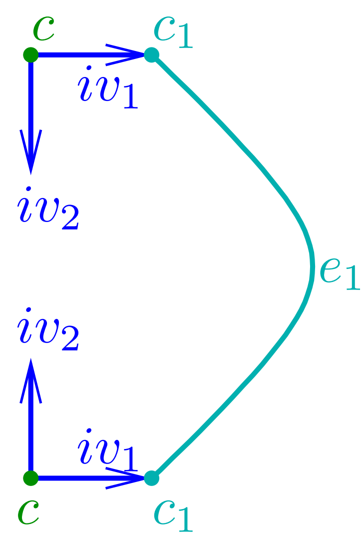

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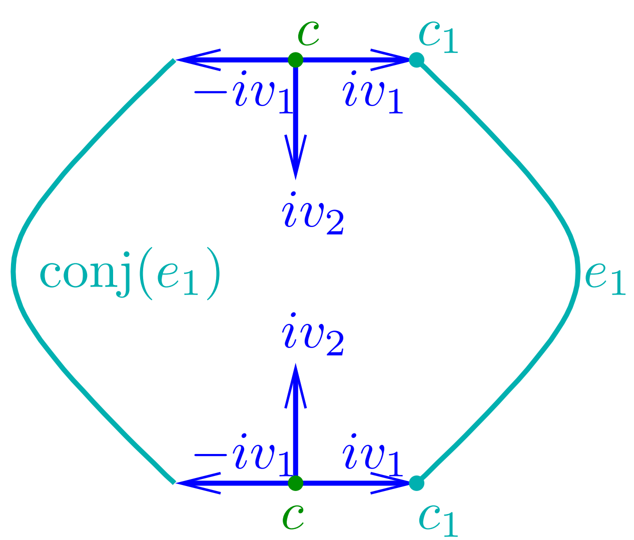

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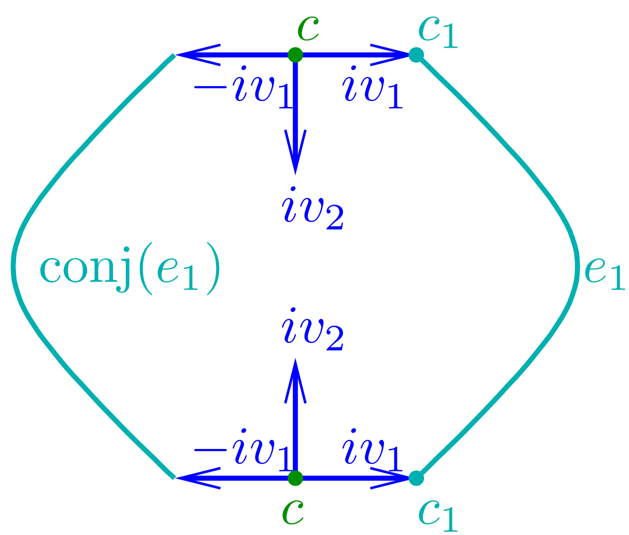

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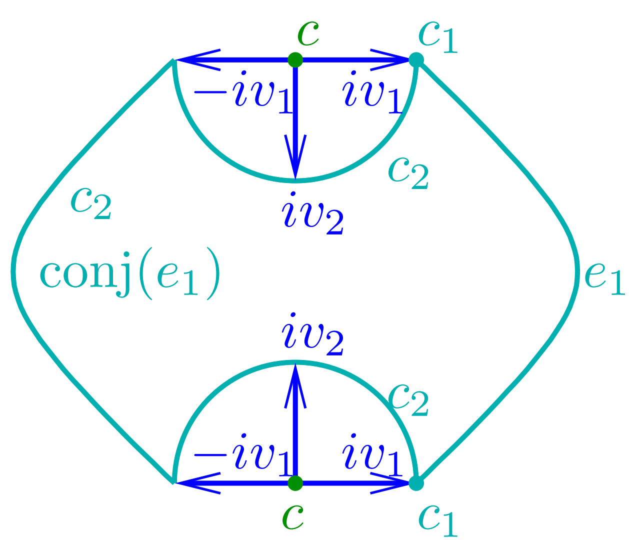

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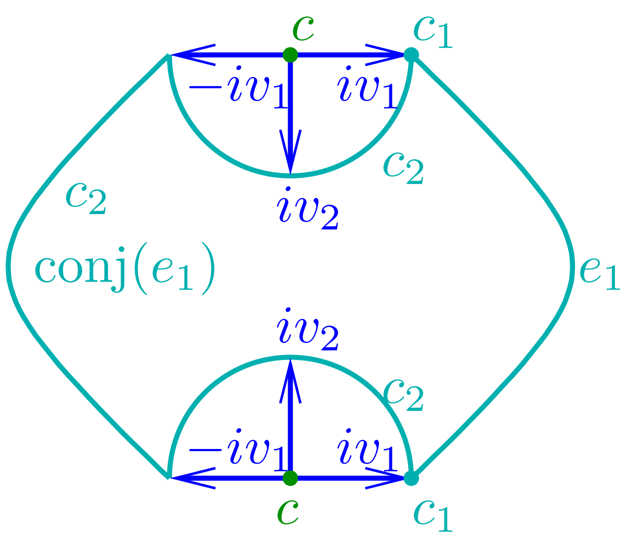

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Construct similarly cycles c_3 ,. . . , c_{n-k}.

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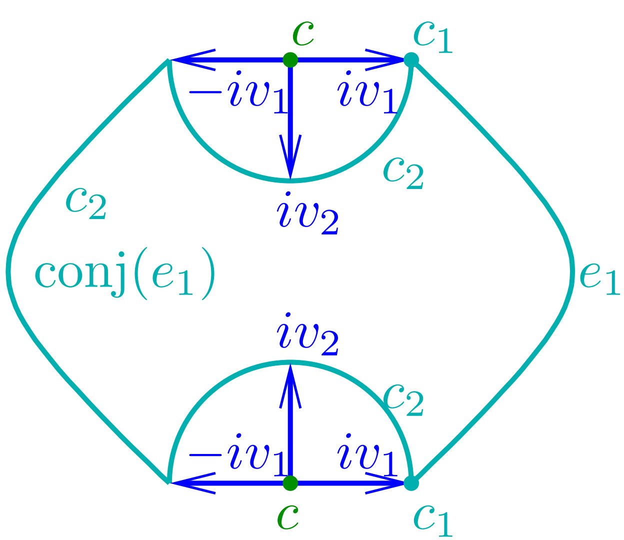

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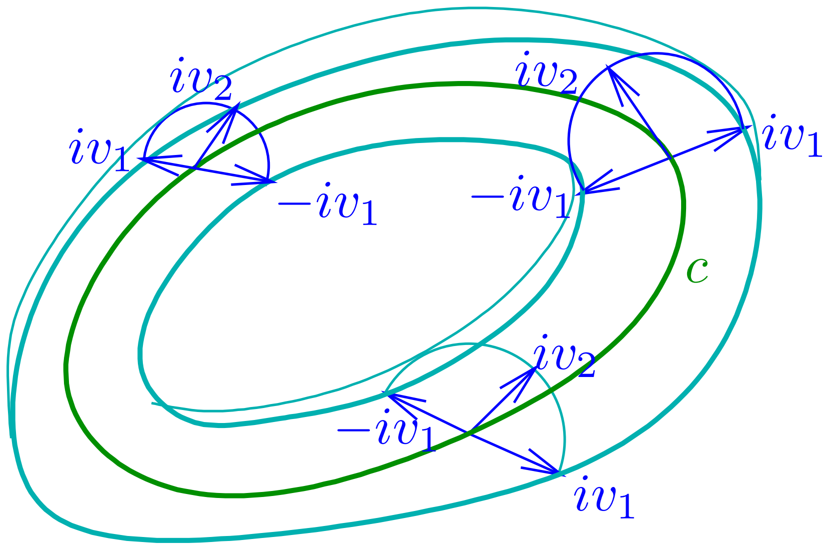

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In the case of a surface of type I with $b_1(A_{\mathbb{C}}) = 0$ the result is Stiefel k-orientations with $k = 0, 1$, i.e., a semi-orientation and Spin-structure.

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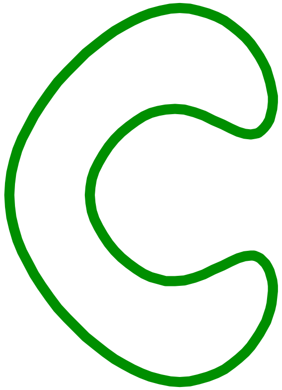

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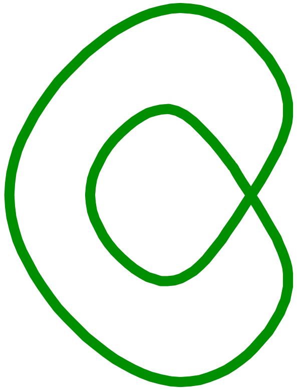

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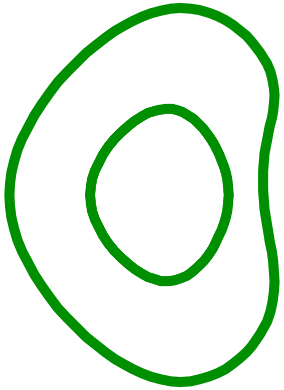

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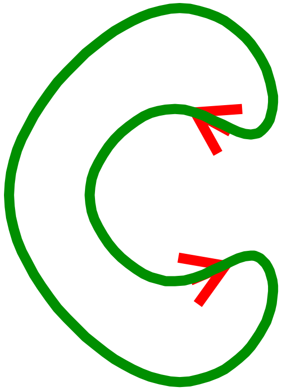

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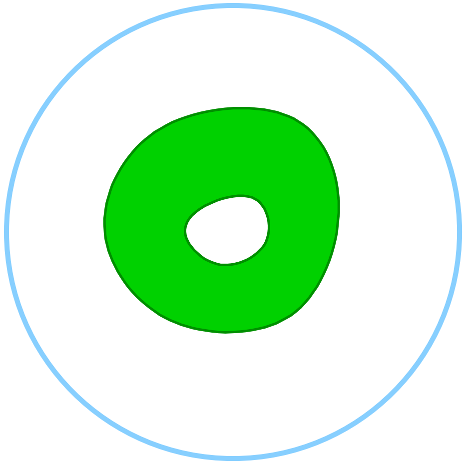

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Counter-examples to Klein in each odd dimension ≥ 3.

If n is odd, then $Q_{p,q}$ are of type I iff p and q are even.

Counter-examples to Klein in each even dimension ≥ 4.

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This is a work in progress.

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