SIGNATURES OF LINKS

OLEG VIRO

Uppsala University, Uppsala, Sweden
POMI, St. Petersburg, Russia

This paper is an English translation of my note [7] published in 1977.

Let $L = (S^{2n-1}, l)$ be an oriented link (i.e., a pair in which l is an oriented smooth closed $(2n-3)$-submanifold of the sphere S^{2n-1}) and let ζ be a complex number with $|\zeta| = 1$. Let A be an oriented smooth compact connected submanifold of D^{2n} with $\partial A = l$. Let $\mathbb{C}_\zeta$ denote the local coefficient system on $D^{2n} \smallsetminus A$ with fiber $\mathbb{C}$ defined by the homomorphism $\pi_1(D^{2n} \smallsetminus A) \to U(1)$ mapping classes of meridian loops to ζ.

Denote by $\sigma_\zeta(L)$ the signature of the Hermitian intersection form

$$H_n(D^{2n} \smallsetminus A; \mathbb{C}_\zeta) \times H_n(D^{2n} \smallsetminus A; \mathbb{C}_\zeta) \to \mathbb{C} \tag{1}$$

for even n and the signature of the Hermitian form obtained from the skew-Hermitian intersection form (1) by multiplying by $\zeta - \bar{\zeta}$. The number $\sigma_\zeta(L)$ depends only on L and ζ.

If V is a Seifert matrix of L then $\sigma_\zeta(L)$ is equal to the signature of the Hermitian matrix $(1-\zeta)V + (1-\bar{\zeta})V^\top$ for even n and $\frac{\zeta}{1+\zeta}V + \frac{\bar{\zeta}}{1+\bar{\zeta}}V^\top$ for odd n (here $^\top$ denotes transposition).

For even n the signature of the m-sheeted cyclic covering space of D^{2n} branched over A is equal to $\sum_{\zeta^m=1} \sigma_\zeta(L)$.

The following theorem generalizes the results by Murasugi [3], Tristram [5], the author [8] and Kauffman and Taylor [1].

Theorem 1. *If ζ is a root of an integer polynomial f irreducible over $\mathbb{Z}$ with $f(1)$ divisibe by a prime number p then for any integer r with $0 \le r \le \frac{n}{2}$*

$$\begin{aligned} &|\sigma_\zeta(L)| + \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{r-1-s}(S^{2n-1} \smallsetminus l; \mathbb{C}_\zeta) \\ &\quad \le \sum_{s=0}^{2r} (-1)^s \left[\dim_{\mathbb{Z}_p} H_{n-2-s}(A; \mathbb{Z}_p) + \dim_{\mathbb{Z}_p} H_{n-1-s}(A; \mathbb{Z}_p)\right] \end{aligned} \tag{2}$$

and for any oriented smooth closed submanifold Σ of S^{2n} transversally intersecting S^{2n-1} in l

$$(3)\quad |\sigma_\zeta(L)| \le \frac{1}{2}\dim_{\mathbb{Z}_p} H_{n-1}(\Sigma;\mathbb{Z}_p) + 1 - \sum_{s=0}^{2r-1}(-1)^s \dim_{\mathbb{Z}_p} H_s(\Sigma;\mathbb{Z}_p) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{Z}_p} H_s(\Sigma, l;\mathbb{Z}_p) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{s+1}(S^{2n-1}\smallsetminus l;\mathbb{C}_\zeta). \quad \square$$

The following theorem generalizes Shinohara's theorem [4] on σ_{-1} and reduces calculation of the signatures of algebraic 1-knots to a calculation of signatures of the torus knots.

Theorem 2. *Let $K_i = (S^{2n-1}, k_i)$ and $L_i = (S^{2n-1}, l_i)$ with $i = 1, 2$ be oriented knots, U_i be a tubular neighborhood of k_i. Suppose l_i lies in U_i and realizes the r-fold generator of $H_{2n-3}(U_i) = \mathbb{Z}$. If there exists a fiber-preserving diffeomorphism $h : U_1 \to U_2$ with $h(k_1) = k_2$ and $h(l_1) = l_2$, preserving linking numbers if $n = 2$, then*

$$\sigma_\zeta(L_1) - \sigma_\zeta(L_2) = \sigma_{\zeta^r}(K_1) - \sigma_{\zeta^r}(K_2).$$

Theorem 2 is deduced from the Wall theorem on non-additivity of signature [9].

References

[1] Louis H. Kauffman and L. Taylor, *Signature of links*, Trans. Amer. Math. Soc., **216** (1976), 351-365.

[2] J. Milnor, *A duality theorem for Reidemeister torsion,* Ann. Math. **76** (1962), 137–147.

[3] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc., **117** (1965), 387-422.

[4] Y. Shinohara, *On the signature of knots and links*, Trans. Amer. Math. Soc., **156** (1971), 273-285.

[5] A. G. Tristram, *Some cobordism invariants of links*, Proc. Cambridge Philos. Soc., **66** (1969), 257-264.

[6] O. Y. Viro, V. G. Turaev, *Estimates of Twisted Homology,* Tezisy VII Vsesojuznoj topologicheskoj konferencii, Minsk, 1977, p. 42 (Russian).

[7] O. Y. Viro, *Signatures of links,* Tezisy VII Vsesojuznoj topologicheskoj konferencii, Minsk, 1977, p. 43 (Russian).

[8] O. Y. Viro, *Positioning in codimension 2, and the boundary*, Uspechi Mat. Nauk, **30** (1975), no. 1, (181), 231-232.

[9] C. T. C. Wall, *Non-additivity of the signature*, Invent. Math., **7** (1969), 269-274.

Department of Mathematics, Uppsala University, Box 480, S-751 06 Uppsala, Sweden

E-mail address: oleg@math.uu.se