Pictorial Calculus for Isometries

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Abstract. In this paper a new graphical calculus for operating with isometries of low dimensional spaces is proposed. It generalizes a well-known graphical representation of vectors and translations in an affine space. Instead of arrows, we use arrows framed with affine subspaces at their end points. The head to tail addition of vectors and translations is generalized to head to tail composition rules for isometries. The material of this paper is elementary and can be used even in the framework of high-school geometry.

1. INTRODUCTION. Among the objects of plane geometry, rigid motions (aka isometries) are most resistant to picturing. There are few exceptions.

- A translation is defined by its restriction to any point. Therefore an arrow, which connects a point to its image, determines the translation. This gives rise to the well-known relation between translations and vectors.
- A symmetry with respect to a line is defined entirely by the line. The line perfectly represents the symmetry.
- The symmetry with respect to a point is completely determined by the point.

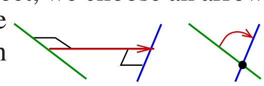
The relation between lines and points, on one hand, and symmetries with respect to them, on the other hand, is so intimate that lines and points have been entirely replaced by the symmetries in one of the approaches to foundations of geometry [**1**], [**3**].

Surprisingly, this tight relation extends to isometries of other types. In the plane (as well as in Euclidean, elliptic, hyperbolic, or Minkowski spaces of any dimension), any isometry is a composition of two involutions. Involutions are determined by their fixed point sets. Therefore, an isometry is encoded by an ordered pair of subspaces. Like an arrow representing a translation, the pair is not unique, but its nonuniqueness can be described and used. In particular, it can be used in pictorial rules for finding a composition of isometries. In dimensions < 4, the rules are especially simple, similar to the head to tail addition. This is the main content of the paper.

Splitting of a plane isometry into a composition of two involutions is well known. In more general situations it can be deduced from Wonenburger's theorem [**5**]: *any isometry of a nondegenerate inner product vector space over any field of characteristic* $\neq 2$ *can be presented as a composition of two linear involutory isometries.*

2. FRAMED ARROWS. Often an arrow in a mathematical picture portrays nothing but an ordered pair of points. Individual points are difficult to discern, and, in order to make the points more visible, they are connected with a segment; in order to show which point is first and which is second, the segment is turned into an arrow directed from the first point to the second one.

We need to portray ordered pairs of subspaces. If the subspaces happen to be points, we draw the arrow, as usual. In general, we draw the subspaces and connect them with an arrow for indicating the order. If the subspaces do not intersect, we choose an arrow perpendicular to both of them. If the subspaces intersect, we still need to show the order. To this end, we connect them with an arc-arrow.



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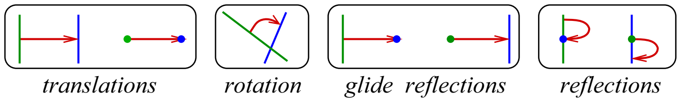
In Euclidean space, any affine subspace is the fixed point set for a unique involution isometry—the symmetry about the subspace. We will call this isometry a *flip* and the subspace, a *flipper.* The author apologizes for these neologisms: a more conventional word *symmetry* has a much broader meaning, while *reflection* is narrower, because it presumes that the fixed point set has a codimension one.

An ordered pair of flippers A and B is called a *biflipper* and denoted by $\overrightarrow{AB}$. It encodes $F_B \circ F_A$, where F_A and F_B are flips in A and B, respectively.

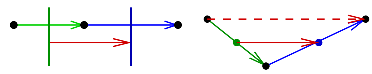
Biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be *equivalent* if $F_B \circ F_A = F_D \circ F_C$.

3. IN THE PLANE.

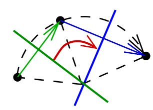


These biflippers encode all the plane isometries (except the identity).

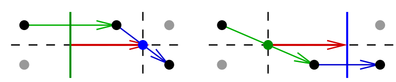
- The first two biflippers encode translation in the direction of the arrow by the distance *twice the length* of the arrow. The reader can easily verify this.



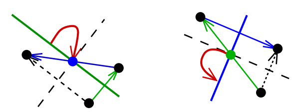
- The third biflipper encodes the rotation about the intersection point of the lines in the direction from the first to the second line, by the angle twice greater than the angle between the biflipper's lines. (*Warning:* often a rotation is shown by a similar picture, but with twice greater angle between the lines. Another difference: in a usual picture the direction of the arc-arrow shows if the rotation is clockwise or counter-clockwise, while in a biflipper it shows the ordering of lines.)



- The fourth and fifth biflippers encode the glide reflection, which is the reflection in the line containing the arrow followed by the translation in the direction of the arrow by the distance twice the length of the arrow.



- The sixth and seventh biflippers encode the reflection in the line perpendicular to the drawn one erected from the point. These are degenerate versions of the fourth and fifth biflippers, in the same sense as reflections in lines are degenerate glide reflections. Another encoding for reflections by biflippers comes from using the identity as a flip in the whole plane.



- The identity also is an isometry and it is encoded by any framed arrow consisting of two identical spaces. The identity is a special case (degeneration) for translations as well as for rotations.

Recall that biflippers that encode the same isometry are called equivalent. For different biflippers, equivalence may look quite different. Biflippers encoding translations are equivalent if and only if they can be obtained from each other by a translation.

A biflipper defining a glide reflection looks like an arrow decorated at one of its end points (no matter which of two) with a perpendicular line. Two such biflippers are equivalent if and only if the arrows lie on the same line and have the same length and direction. In other words, the arrows can be obtained from each other by a translation along this line. The equivalence classes of arrows are known as *sliding vectors.*

Biflippers defining rotations are equivalent if and only if they can be obtained from each other by a rotation about the intersection point of the biflipper's lines (the rotation center).

Biflippers defining reflections are equivalent if and only if they can be obtained from each other by a translation in the direction perpendicular to the line (i.e., along the reflection's axis).

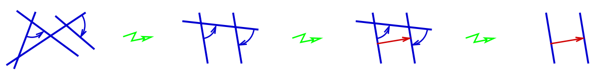
4. HEAD TO TAIL COMPOSITION LAWS. If $S = I \circ J$, $T = J \circ K$, where J is an involution, then, obviously, $S \circ T = I \circ J^2 \circ K = I \circ K$.

Therefore, if the head B of a biflipper $\overrightarrow{AB}$ which encodes an isometry T coincides with the tail of a biflipper $\overrightarrow{BC}$ which encodes an isometry S, then $S \circ T$ is encoded by the biflipper $\overrightarrow{AC}$. This scheme gives rise to graphical head to tail composition rules. They depend on the types of the composed isometries. It's remarkable that such a rule exists for any pair of isometries of the Euclidean plane and many other similar setups.

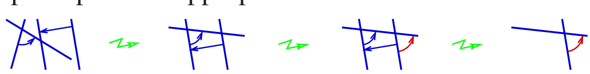
Translations. Choose biflippers formed of points and move one of the biflippers by an appropriate translation so that the head of the first of them would coincide with the tail of the second one. After that this looks as the usual head to tail addition of vectors.

Rotations. Take some biflippers encoding the rotations. Turn the biflippers about their centers in order to make the second line in the first of them coinciding with the first line in the second. If the other two framing lines are not parallel, then draw an arc-arrow connecting the first line of the first framed arrow to the second line of the second biflipper. Erase the coinciding lines and the old arc-arrows. The composition is a rotation. Notice that this head to tail construction gives the center of the composition.

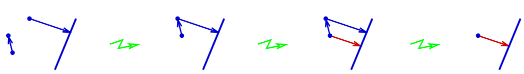
If the tail line of the first biflipper is parallel to the head line of the second biflipper, then after erasing everything except these tail and head lines and connecting them with an arrow, we obtain a biflipper of a translation.



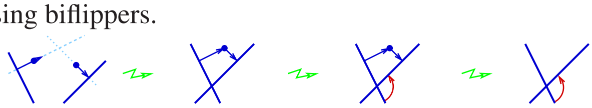
Translation followed by rotation. Choose a biflipper representing the translation formed of lines. Turn a biflipper representing the rotation so that its first line would be perpendicular to the direction of the translation (i.e., parallel to the lines forming the biflipper representing the translation). By a translation of the biflipper representing the translation, superimpose the appropriate lines.



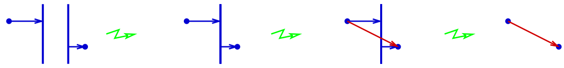
Translation followed by glide reflection. Let us compose a translation T and glide reflection G. The $G \circ T$ is a glide reflection. Its axis can be obtained by applying T^{-1} to the axis of G. Here is a head to tail proof of this.



Two glide reflections with nonparallel axes. The composition is a rotation. Here is how to find it using biflippers.



Two glide reflections with parallel axes. The composition is a translation.

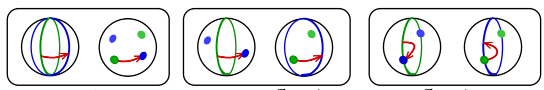


Exercise 1. Prove the following (see [**4**], Theorem 4.): *A head to tail composition rule exists for any pair of plane isometries. In other words, any isometries* S *and* T *of the Euclidean plane are presented by biflippers* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$*, respectively.*

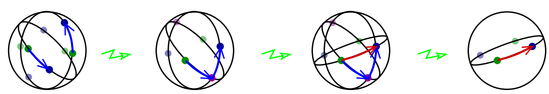
Exercise 2. Find head to tail composition rules for the pairs of plane isometries that have not been discussed above.

5. ON OTHER SURFACES. Flips and flippers. Slightly extending terminology introduced above, we will call a subset A of a metric space X a *flipper* in X, if there exists a unique involution isometry $F : X \to X$ such that A is the fixed point set of F. Then F is called a *flip* in A and is denoted by F_A. Flips and flippers are in 1-1 correspondence. An ordered pair of flippers is called a *biflipper* and defines an isometry, as above.

On the 2-sphere there are two kinds of flippers: great circles and pairs of antipodal points. Any isometry of the sphere is defined by a biflipper. Biflippers of rotations are equivalent if they can be obtained from each other by a rotation about the same axis. A head to tail rule exists for any pair of isometries. We show it for two rotations. Head to tail rules for other pairs are similar.



rotations *rotary reflections* *reflections*

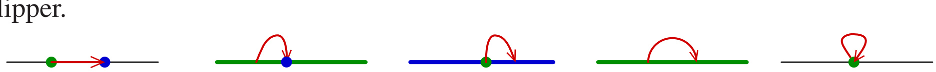


In the twofold covering Spin(3) of $SO(3)$, biflippers are covered by Hamilton's presentations of unit quaternions as vector-arcs, see [**4**].

On the elliptic plane (i.e., the projective plane with the metric induced from the covering 2-sphere.) Any flipper on the elliptic plane consists of a projective line and a point polar to each other. Since it is determined by the point, one may forget about the projective line. Any isometry is defined by a biflipper, which is determined by any straight arrows connecting the corresponding points. Two arrows determine the same isometry if they can be obtained from each other by sliding along their lines. Thus isometries of the projective plane are described by sliding vectors with obvious head to tail addition.

For biflippers *on the hyperbolic plane and 3-space*, see [**4**].

6. IN OTHER DIMENSIONS. On line. Any isometry of a line is either a reflection in point or a translation. A translation is encoded by a biflipper made of two points. A reflection can be presented by a biflipper only if the whole line is considered a flipper.



translation *reflections in points* *the identity*

Multiplication of biflippers. For any isometries $S : \mathbb{R}^k \to \mathbb{R}^k$ and $T : \mathbb{R}^l \to \mathbb{R}^l$, the direct product $S \times T : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \times \mathbb{R}^l : (x, y) \mapsto (S(x), T(y))$ is an isometry of $\mathbb{R}^{k+l} = \mathbb{R}^k \times \mathbb{R}^l$. If $A \subset \mathbb{R}^k$ and $A' \subset \mathbb{R}^l$ are affine subspaces, then $A \times A'$ is an affine subspace of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ and $F_A \times F_{A'} = F_{A \times A'}$.

Theorem 1. *For any biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$ encoding isometries $S : \mathbb{R}^k \to \mathbb{R}^k$ and $T : \mathbb{R}^l \to \mathbb{R}^l$, the isometry $S \times T : \mathbb{R}^{k+l} \to \mathbb{R}^{K+l}$ is encoded by $\overrightarrow{(A \times C)(B \times D)}$.*

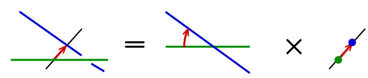
Proof. Notice that if $F, F' : \mathbb{R}^k \to \mathbb{R}^k$ commute and $G, G' : \mathbb{R}^l \to \mathbb{R}^l$ commute, then $F \times G$ commutes with $F' \times G'$. Therefore

$$\begin{aligned} F_{B\times D} \circ F_{A\times C} &= (F_B \times \mathrm{id}) \circ (\mathrm{id} \times F_D) \circ (F_A \times \mathrm{id}) \circ (\mathrm{id} \times F_C) \\ &= (F_B \times \mathrm{id}) \circ (F_A \times \mathrm{id}) \circ (\mathrm{id} \times F_D) \circ (\mathrm{id} \times F_C) \\ &= ((F_B \circ F_A) \times \mathrm{id}) \circ (\mathrm{id} \times (F_D \circ F_C)) \\ &= (F_B \circ F_A) \times (F_D \circ F_C) = S \times T. \end{aligned}$$

■

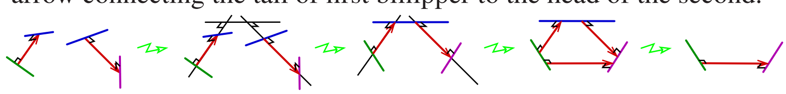
In the n-space. As follows from a well-known classification of isometries of $\mathbb{R}^n$ (see, e.g., M. Berger [**2**]), any isometry of $\mathbb{R}^n$ is isometric to a direct product of isometries with factors of dimension at most two. Therefore, any isometry of $\mathbb{R}^n$ can be encoded by a product of biflippers in $\mathbb{R}^1$ and $\mathbb{R}^2$.

In the 3-space. Biflippers can be easily classified as products of biflippers in the line and plane. See [**4**], section 10. Here we restrict our attention to screw motions and rotary reflections. Screw motions form an open dense subset in the space of all orientation preserving isometries of the 3-space. Rotary reflections form an open dense subset in the space of all orientation reversing isometries of the 3-space.

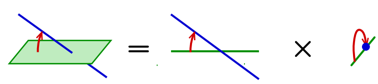


Screw motion is a composition of a rotation and a translation along the axis of the rotation. It is a direct product of a plane rotation by a translation of line. Its biflippers are pairs of skew lines. The axis of a screw motion is the common perpendicular of the lines that form its biflipper. Two such biflippers are equivalent if and only if the lines forming them have a common perpendicular and the biflippers can be obtained from each other by a screw motion whose axis is this common perpendicular.

Here is a head to tail rule for composing screw motions. Take any biflippers representing the screw motions. By moving along and rotating about their axes, superimpose the head line of the first biflipper and the tail line of the second one with the common perpendicular of the axes. Erase the common perpendicular and old arrows and draw the new arrow connecting the tail of first biflipper to the head of the second.



Rotary reflection is a composition of rotation about a line with reflection in a plane perpendicular to the axis of rotation. It is a direct product of a plane rotation and a reflection of line in a point. Its biflippers are formed of line and plane transversal to each other. The angle of the rotation is twice the angle between the line and plane, the axis is contained in the plane and perpendicular to the line; the plane of reflection contains the line of biflipper.



If the axes of two rotary reflections are skew and the common perpendicular to the axes does not pass through the fixed points of both rotary reflections, then they have no biflippers with a common flipper. Thus a head to tail rule for composing of these rotary reflections does not exists. Nonetheless, a biflipper of the composition can be easily obtained graphically from biflippers of the rotary reflections.

Problem. Head to tail rules exist for any isometries only in low dimensions. Find a general algorithm for constructing a biflipper which represents a composition of isometries in terms of their biflippers.

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A Partial Solution of Representing 1 as a Sum of n Distinct Unit Fractions

The equation $\sum_{k=1}^{n} \frac{1}{x_k} = 1$ appears as the central study of several papers. With respect to n, a lot of partial solutions to this equation exist in the literature [**1**]. In this paper, we present a nonalgorithmic solution based on a simple formula which is proved by induction.

Theorem 1. *For every $n \geq 2$, the identity $\sum_{k=1}^{n-1} \frac{k}{(k+1)!} + \frac{1}{n!} = 1$ holds.*

Proof. Since $\frac{1}{2!} + \frac{1}{2!} = \frac{1}{2} + \frac{1}{2} = 1$ the identity is true for $n = 2$. Suppose the theorem holds true for n which means $1 - \frac{1}{n!} = \sum_{k=1}^{n-1} \frac{k}{(k+1)!}$.

We have $\sum_{k=1}^{n} \frac{k}{(k+1)!} + \frac{1}{(n+1)!} = \sum_{k=1}^{n-1} \frac{k}{(k+1)!} + \frac{n}{(n+1)!} + \frac{1}{(n+1)!}$ which using the induction step is equal to $1 - \frac{1}{n!} + \frac{n}{(n+1)!} + \frac{1}{(n+1)!} = 1$. This proves the theorem for $n + 1$ so the theorem holds true for every $n \geq 2$. ■

We can see that if $2 \leq k < m < n$, then $\frac{(k+1)!}{k} < \frac{(m+1)!}{m} < n!$ holds. Hence we have the following.

Corollary. *Let $n \geq 3$. A solution of the equation $\sum_{k=1}^{n} \frac{1}{x_k} = 1$ with $x_1 < x_2 < \cdots < x_n$ is given by $x_k = \frac{(k+1)!}{k}$ for every k with $1 \leq k \leq n - 1$ and $x_n = n!$.*

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