FROM THE SIXTEENTH HILBERT PROBLEM TO TROPICAL GEOMETRY

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Part 1. A story of mystery, mistakes and solution

How important to read classics. In 2000 I was invited to give a talk on the 16th Hilbert problem. In order to prepare the talk, I read the text of the problem and was astonished. I realized how difficult was to formalize the problem and how much the modern understanding differs from Hilbert's one. The most striking discovery: **the sixteenth Hilbert's problem was solved long ago**.

1. Let us read the Sixteenth Hilbert Problem

The title of the corresponding part of Hilbert's talk [9] is
16. Problem of the topology of algebraic curves and surfaces.
Hilbert started with reminding of a background result:

> *The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack (Mathematische Annalen, vol. 10).*

Here Hilbert referred to the following *Harnack inequality.*[1]

$$\left(\begin{array}{c}\text{The number of connected components}\\ \text{of a plane projective real}\\ \text{algebraic curve of degree } n\end{array}\right) \le \frac{(n-1)(n-2)}{2}+1.$$

Digression on the nature of the Harnack Inequality. The Harnack Inequality can be proved by very different arguments. This unveils the dual nature of the subject.

Harnack's proof. Assume the contrary: let a curve A of degree n has $\#(ovals) > M = \frac{(n-1)(n-2)}{2}+1$. By an *oval* of a non-singular real algebraic plane projective curve A one means a connected component

[1]The words *Harnack inequality* are confusing: there are other, more famous Harnack inequalities concerning values of a positive harmonic function.

C of the set of real points of A zero homologous modulo 2 (that is C realizes $0 \in H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$).

Draw a curve B of degree $n-2$ through M points chosen on M ovals of A and $n-3$ points on one more connected component.

A curve of degree $n-2$ is defined by an equation with $\frac{(n-1)n}{2}$ coefficients. The condition that it passes through a given point is a linear equation on the coefficients. Hence a curve can be drawn indeed through

$$\frac{(n-1)n}{2} - 1 = \frac{(n-1)(n-2)}{2} + n - 1 - 1 = M + n - 3$$

points. On the other hand, let us estimate the number of intersection points. Since each oval is met an even number of points, this number is at least

$$2M + n - 3 = (n-1)(n-2) + 2 + n - 3 = n^2 - 2n + 1 > n(n-2).$$

However, by the Bezout Theorem, the number of intersection points cannot be greater than $n(n-2)$. $\square$

Klein's proof. The Harnack Inequality is an immediate corollary of the following theorem applied to the complexification of the curve and the complex conjugation involution on it:

Theorem. *Let S be an orientable closed connected surface, $\sigma : S \to S$ an orientation reversing involution, and F the fixed point set of σ. Then $\#$connected components$(F) \leq$ genus$(S) + 1$.*

Lemma. *Under the assumptions of Theorem above,*

$$\#\text{connected components}(S \smallsetminus F) \leq 2.$$

Proof of Lemma. Let A be a connected component of $S \smallsetminus F$. Then $\mathrm{Cl}(A) \cup \sigma(A)$ is a closed surface. Hence $\mathrm{Cl}(A) \cup \sigma(A) = S$. If $A \neq \sigma(A)$, then $\#$connected components $(S \smallsetminus F) = 2$. If $A = \sigma(A)$, then $\#$connected components $(S \smallsetminus F) = 1$. $\square$

Proof of Theorem. A curve with

$$\#\text{connected components } (S) > genus(S) + x$$

divides S to $> x + 1$ components. This follows immediately from the classical definition of genus. $\square$

Which of the proofs of the Harnack Inequality do you prefer? Harnack's proof is confined in the real domain and relies on the Bezout Theorem. Klein's proof relies on simple topological considerations, which run in the complexification. Harnack was a graduate student of

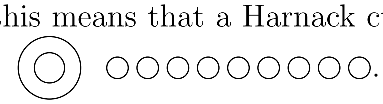
Klein, and the Harnack Inequality belongs to his thesis. It was published in Mathematische Annalen [7]. Later, in the same volume of Mathematische Annalen, Klein published [21] his own proof.

Relative position of branches. Let us return ***back to Hilbert's text.*** He continued:

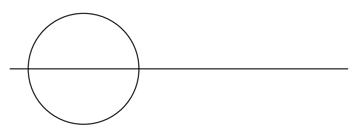
> *There arises the further question as to the relative position of the branches in the plane.*

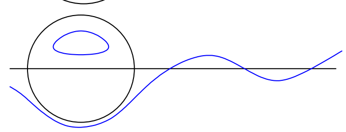
This question was raised by Hilbert in his paper [8]. Harnack, in the paper [7] mentioned above constructed curves with the *maximal* number of components for each degree. However, his curves are very special:

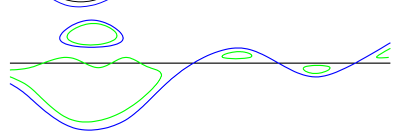
- The depth of each of their nests ≤ 2.
- A Harnack curve of degree $\underline{n}$ has $\frac{3n^2-6n}{8}+1$ outer and $\frac{n^2-6n}{8}+1$ inner ovals.

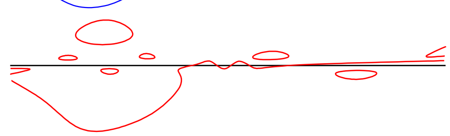
In degree 6 this means that a Harnack curve has 10 outer ovals and 1 inner oval: .

Harnack's construction [7].

Take a line and circle: 

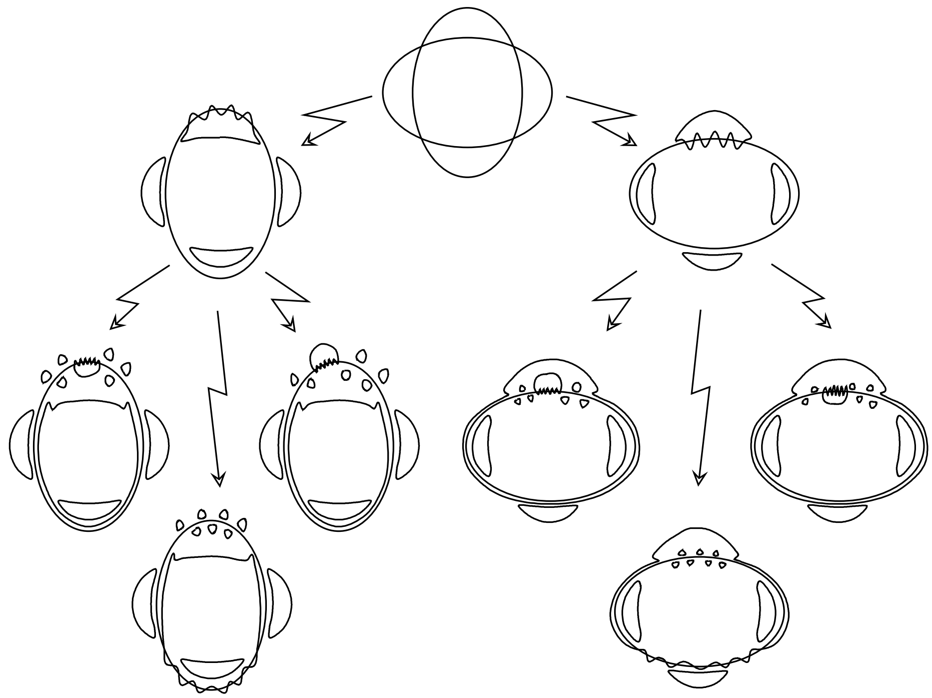
Perturb their union: 

Perturb the union of the result and the line: 

Perturb the union of the result and the line: 

And so on...

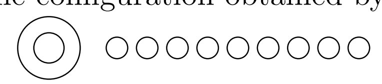
Hilbert's construction. Hilbert in his paper [8] suggested another construction:

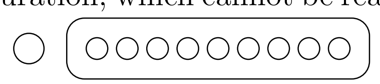


An ellipse does here what the line did in Harnack's construction.

Hilbert sextics. Each Hilbert's curve of degree 6 has one of the following two configurations of ovals:

(1) the configuration obtained by Harnack:



(2) a new configuration, which cannot be realized by Harnack's construction: 

Hilbert worked hard, but could not construct curves of degree 6 with 11 connected components positioned with respect to each other in any other way. ***He concluded that this is impossible.*** and turned to proof of impossibility:

> *As to curves of the 6-th order, I have satisfied myself–by a complicated process, it is true–that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.*

In other words, *only* mutual positions of ovals realized by Harnack's and Hilbert's constructions are *possible*.

Hilbert's "*complicated process*" allows one to answer to virtually all questions on topology of curves of degree 6. Now it is called *Hilbert-Rohn-Gudkov method.*

Hilbert-Rohn-Gudkov method involves a detailed analysis of singular curves which could be obtained by continuous deformation from a given nonsingular one.

The Hilbert-Rohn-Gudkov method required complicated fragments of *singularity theory*, which had not been elaborated at the time of Hilbert.

Hilbert's arguments were ***full of gaps***. His approach was realized completely ***only 69 years later*** by D.A.Gudkov. In *1954* Gudkov, in his Candidate dissertation (Ph.D.), ***proved*** Hilbert's statement about topology of sextic curves with 11 components. *15 years later*, in his Doctor dissertation, Gudkov ***disproved*** it and found the final answer.

Call for an attack. A "***complicated process***" ***could not*** really ***satisfy*** Hilbert. Desperately wishing to understand the real reasons of this very mysterious phenomenon, Hilbert called for attack:

> *A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest,*

Why did Hilbert distinguish curves with maximal number of branches? *Extremal cases of inequalities* had been known to be of *extreme interest*. Hilbert deeply appreciated this paradigm of the calculus of variations.

Now people (especially, specialists) tend to widen the content of Hilbert's 16th problem as just a call for *study of the topology of* **all** *real algebraic varieties*. To support this view, they cite also the next piece of Hilbert's text:

> *and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.*

The word *corresponding* is crucial here. Without it, this would really be a mere call to study the topology of real algebraic surfaces. So, what is *"the corresponding"*? Hilbert continues:

> *Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4-th order in three dimensional space can really have (Cf. Rohn, "Flächen vierter Ordnung" 1886).*

Solutions. Now we know that *the maximum number of connected components of a quartic surface in the 3-dimensional projective space is 10.*

This was proven in 1972 by *V.M.Kharlamov* in his Master thesis [13], in the *breakthrough* of 1969-72, which solved the sixteenth Hilbert problem.

*All the questions contained, explicitly or implicitly, in the sixteenth problem have been answered by **D.A.Gudkov, V.I.Arnold, V.A.Rokhlin** and **V.M.Kharlamov** in this breakthrough.*

In 1969, D.A.Gudkov [3] found the final answer to the question about *position of real branches of maximal curves of degree 6*.

V.I.Arnold [1] and V.A.Rokhlin [32] found in 1971-72 a ***conceptual cause*** of the phenomenon which struck Hilbert.

V.M.Kharlamov [17] completed by 1976 the *"corresponding investigation"* of nonsingular quartic surfaces.

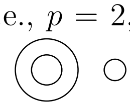
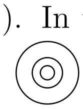
All in all this gives good reasons to consider the sixteenth Hilbert problem ***solved***. However, I am not aware about any publication, where it is claimed.

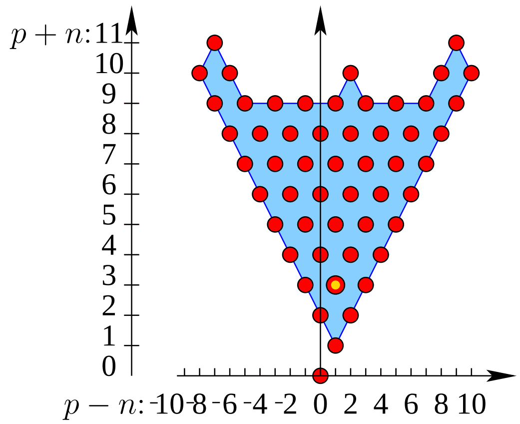
Unusual? The solution was initiated by completion of long difficult technical work, which looked like a final point. It followed by opening a new world with a relations to the complex domain, 4-dimensional topology, complex algebraic geometry.

The sixteenth Hilbert problem served the ***symbol*** of the breakthrough. Nobody wanted to ***dispose*** the symbol. Nobody cared to report that the puzzle had been solved.

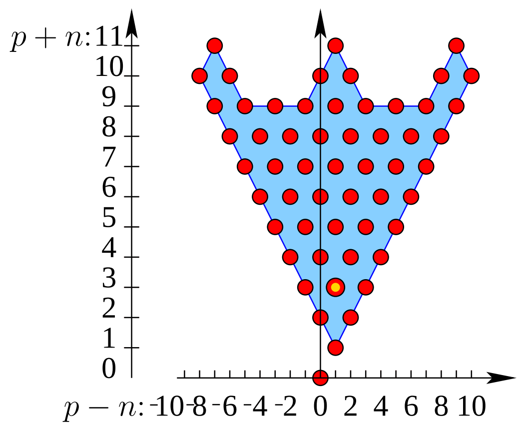
2. Breakthrough

Isotopy classification of nonsingular sextics. In 1969, ***Gudkov*** completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6. The project started in 1948. The adviser of Gudkov, A.A.Andronov, proposed him to *develop theory of degrees of coarseness for real algebraic curves* similar to the one that he developed in the theory of dynamical systems. I.G.Petrovsky suggested to unite this with study of sextics.

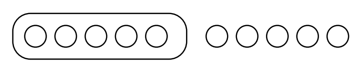
In 1954 Gudkov defended PhD. About 12-14 years later he prepared a publication. The summary of the results can be presented in the following table, where the ordinate is the total number of ovals, the abscissa is the difference between the numbers p and n of even and odd ovals of the curve (an oval is even or odd if it lies inside of, respectively, even or odd number of other ovals). The coordinates characterize uniquely an isotopy type of a nonsingular curve of degree 6, except the situation when $p + n = 3$ and $p - n = 1$ (i.e., $p = 2$, $n = 1$). In the latter situation there are two isotopy types:  and .



The ***referee did not like*** it. He suggested to make it ***more symmetric***.



Gudkov found a ***mistake*** and the ***final answer***.

Gudkov's M-curve [4], [5]. The missing curve  disproved Hilbert's statement.

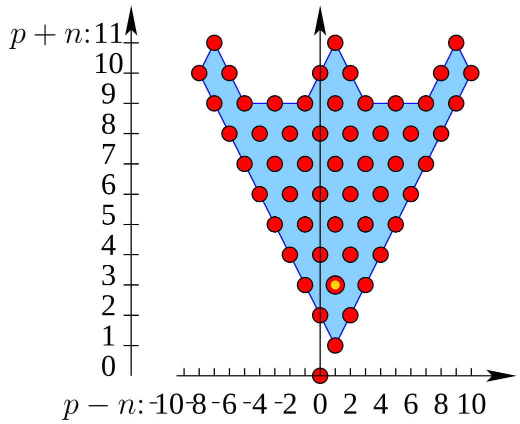
By the way, in the first version of his paper with formulations of the problems Hilbert was more cautious and correct:

> *As to curves of the 6-th order, I have satisfied myself–by a complicated process, it is true–that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another.*

He stopped here. Later he added words, which made the statement incorrect:

> *but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.*

Gudkov's conjecture. Symmetric top of the table



forced Gudkov [4] to formulate:

Gudkov's Conjecture. *For any curve of even degree $d = 2k$ with maximal number of ovals,* $p - n \equiv k^2 \mod 8$.

It was this conjecture that inspired the breakthrough.

Arnold's congruence. In 1971 Arnold [1] proved a *half* of Gudkov's conjecture: the same congruence, but modulo 4: $p - n \equiv k^2 \mod 4$.

Arnold's proof works for a larger class of curves: for any nonsingular curve of *type I* – a curve whose real ovals divide the Riemann surface of its complex points. The proof relies on the topology of the configuration formed in the complex projective plane $\mathbb{C}P^2$ by the complexification $\mathbb{C}A$ of the curve and the real projective plane $\mathbb{R}P^2$.

Complexification. Curve A of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on the projective plane, where F is a real homogeneous polynomial of degree d. If F is generic, then the equation $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of circles smoothly embedded in $\mathbb{R}P^2$, and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since d is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. *They are well-defined, as* $F(\lambda x) = \lambda^{2k} F(x)$. *Choose F to have $\mathbb{R}P^2_+$ orientable.* $p - n = \chi(\mathbb{R}P^2_+)$.

Let p be the number of *even* ovals, that is the number of connected components of $\mathbb{R}P^2_+$, n be the number of *odd* ovals, thet is the number of holes in $\mathbb{R}P^2_+$.

How to complexify $\mathbb{R}P^2_+$? In other words, how to complexify the notion of manifold with boundary? How to complexify inequality $F(x) \geq 0$?

Arnold: *Complexification of inequality is two-fold branched covering!*

Indeed, $F(x) \geq 0 \Leftrightarrow \exists y \in \mathbb{R} : F(x) = y^2$. $F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{C}Y$ in 3-variety

$$E = (\mathbb{C}^3 \smallsetminus 0) \times \mathbb{C}/(x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^k y).$$

Projection $\mathbb{C}Y \to \mathbb{C}P^2 : [x_0, x_1, x_2, y] \mapsto [x_0{:}x_1{:}x_2]$ is a two-fold covering branched over $\mathbb{C}A$. It maps $\mathbb{R}Y$ onto $\mathbb{R}P^2_+$. Let $\tau : \mathbb{C}Y \to \mathbb{C}Y$ be the automorphism of this branched covering, it is an involution with $\operatorname{fix}(\tau) = \mathbb{C}A$.

In homology. One can show that $\pi_1(\mathbb{C}Y) = 0$. This simplifies algebra, makes it commutative. Further, $H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}$, $H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0$.

$H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}$. This is the *scene of our algebraic action*. Here are *its decorations*:

- Intersection form $H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta$, it is a symmetric bilinear unimodular form.
- Involution $\tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y)$.
- Form of involution τ $H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta)$. This is also a symmetric bilinear unimodular form.
- Homology classes $[\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y)$. We orient $\mathbb{R}Y$. $[\infty]$ is the preimage of a generic projective line under $\mathbb{C}Y \to \mathbb{C}P^2$. $[\mathbb{C}A] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2$ for any ξ. Because $X \cap \tau(X) = (X \cap \mathbb{C}A) \cup$ (even number of points). $[\mathbb{C}A] = k[\infty]$; $k[\infty] \equiv [\mathbb{R}Y] \mod 2$, if $\mathbb{R}A$ divides $\mathbb{C}A$. Hence $[\mathbb{R}Y] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2$ for any ξ, if $\mathbb{R}A$ divides $\mathbb{C}A$.

Proof of Arnold's congruence. Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form. $w \in \mathbb{Z}^r$ is a characteristic class of Φ, if $\Phi(x, x) \equiv \Phi(x, w) \mod 2$ for any $x \in \mathbb{Z}^r$. Any unimodular symmetric bilinear form has a characteristic class. Any two characteristic classes are congruent modulo 2.

Lemma. For any two characteristic classes w, w' of a form Φ

$$\Phi(w', w') \equiv \Phi(w, w) \mod 8$$

Proof. $w' = w + 2x$ for some $x \in \mathbb{Z}^r$. Hence $\Phi(w', w') = \Phi(w, w) + 4\Phi(x, w) + 4\Phi(x, x)$, but $\Phi(x, x) \equiv \Phi(x, w) \mod 2$. Therefore $\Phi(w', w') \equiv \Phi(w, w) + 8\Phi(x, x) \mod 8$. □

Back to $\mathbb{C}Y$: As we have seen $[\mathbb{C}A]$ and $[\mathbb{R}Y]$ are characteristic for $\circ_\tau$, if $\mathbb{R}A$ divides $\mathbb{C}A$. Therefore $[\mathbb{C}A] \circ_\tau [\mathbb{C}A] \equiv [\mathbb{R}Y] \circ_\tau [\mathbb{R}Y] \mod 8$. $[\mathbb{C}A] \circ_\tau [\mathbb{C}A] = [\mathbb{C}A] \circ [\mathbb{C}A] = k[\infty] \circ k[\infty] = k^2[\infty] \circ [\infty] = 2k^2$. $[\mathbb{R}Y] \circ_\tau [\mathbb{R}Y] = -[\mathbb{R}Y] \circ [\mathbb{R}Y] = -(-\chi(\mathbb{R}Y)) = \chi(\mathbb{R}Y) = 2\chi(\mathbb{R}P^2_+) = 2(p - n)$. Because multiplication by $\sqrt{-1}$ is antiisomorphism between tangent and normal fibrations of $\mathbb{R}A$ + Poincaré-Hopf. Finally, we get $2k^2 \equiv 2(p - n) \mod 8$, that is $p - n \equiv k^2 \mod 4$. Provided $\mathbb{R}A$ bounds in $\mathbb{C}A$. In particular, if $p + n = g + 1$. □

Gudkov-Rokhlin congruence. Soon after Arnold's paper [1], Rokhlin published a paper [31], *Proof of Gudkov's conjecture*. He extended his famous topological theorem on divisibility by 16 of signature of a smooth closed 4-manifold with Spin structure, and deduced from this extension the Gudkov congruence. The deduction was wrong. The mistake was discovered and fixed much later by Alexis Marin [22].

Why did it take that long (8 years) to find the mistake? Because another proof of the Gudkov conjecture became available shortly after publication of the wrong one.

Four months after publication of [31], Rokhlin published [32] a generalization of Gudkov conjecture to maximal varieties of any dimension with a simple and correct general proof.

Rokhlin's Theorem. *Let A be a non-singular real algebraic variety of even dimension with* $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}A;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H_*(\mathbb{C}A;\mathbb{Z}_2)$. [2] *Then* $\chi(\mathbb{R}A) \equiv \sigma(\mathbb{C}A) \mod 16$.

Between the two papers by Rokhlin, [31] and [32], there was a paper [13] by Kharlamov with the upper bound (=10) for the number of connected components of a quartic surface.

The role of complexification. *Hilbert's puzzle had been solved!* The answer is in the complexification.

Gudkov's conjecture and its high-dimensional generalization proven by Rokhlin explain all the phenomena which had struck Hilbert and motivated his sixteenth problem. They are *real manifestations of fundamental topological phenomena located in the complexification*.

Hilbert never showed a slightest sign that he had expected a progress via getting out of the real world into the realm of complex.

Felix Klein did. He consciously looked for interaction of real and complex pictures as early as in 1876.

Mystery of the 16th Hilbert problem that emerged when the problem was solved **is in its number!** The number **sixteen** plays a very special role in the topology of real algebraic varieties.

Rokhlin's paper with his proof of Gudkov's conjecture and its generalizations is entitled: *"Congruences modulo* **sixteen** *in the* **sixteen***th Hilbert problem"*.

Many of subsequent results in this field have also the form of congruences modulo **16**. It is difficult to believe that Hilbert was aware of phenomena that would not be discovered until some seventy years later. Nonetheless, **16** was the number chosen by Hilbert.

[2] This is an extremal case of the Smith-Thom inequality according to which $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}A;\mathbb{Z}_2) \le \dim_{\mathbb{Z}_2} H_*(\mathbb{C}A;\mathbb{Z}_2)$ for any A.

The second part. Hilbert's sixteenth problem does not stop where I stopped citation, it has the second half:

> *In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form* $\frac{dy}{dx} = \frac{Y}{X}$,
> *where X and Y are rational integral functions of the nth degree in x and y.*
> *Written homogeneously, this is*
> $$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0,$$
> *where X, Y, and Z are rational integral homogeneous functions of the nth degree in x, y, z, and the latter are to be determined as functions of the parameter t.*

There is still quite a little progress in the second half of the sixteenth problem. Hilbert's hope for a similarity between the two halves has not realized.

Finiteness for the number of limit cycles for each individual equation has been proven. But even for $n = 2$, the **maximal number** of limit cycles is **still unknown**. See a nice survey [10] by Ilyashenko.

Success of the first part. In comparison to the second half, the first half of the 16th Hilbert problem was extremely successful:

It contained difficult concrete problems (maximal sextic curves, number of components of a quartic surface) which have been solved.

It attracted attention to a difficult field in the core of Mathematics.

Topological problems are the roughest and allow one to treat complicated objects unavailable for investigation from more refined viewpoints.

Although the concrete questions contained in the first half have been completely solved, the subject has little chances to be completed. As a "thorough investigation", the problem can hardly be solved.

3. Post Solution

What has happened since then? Use of complexification made possible to find numerous *restrictions on the topology of real algebraic varieties.*

Besides the congruence modulo 4, Arnold proved in the same paper [1] several *inequalities* on numerical characteristics of mutual position of ovals. He found a few useful ways to translate geometric phenomena in the real domain to the complex domain and back.

Kharlamov [14], [16], Gudkov and Krakhnov [6], Nikulin [29], Fiedler [2] and Mikhalkin [24] proved *congruences modulo various powers of 2* similar to the Gudkov-Rokhlin congruence.

Rokhlin [33] observed that a curve of type I brings a distinguished pair of *orientations* which come from the complexification and discovered a topological restriction on them. He suggested to change the main object of study: *Add to topology of the real variety the topology of its position in the complexification.*

Rigid isotopy is a deformation of a nonsingular real algebraic variety in the corresponding class of nonsingular real algebraic varieties.

Complex picture enhances our vision of rigid isotopies. Rokhlin [33] observed that curves of degree 5 with 4 ovals may be of type I or type II. This is why these curves are not rigid isotopic, although their real parts have the same topology.

Plane projective nonsingular curves of degree ≤ 4 were classified up to rigid isotopy by XX century.

Curves of degree 4, by Zeuthen. The rigid isotopy classification coincides with the isotopy classification.

Curves of degree 5 – Kharlamov [19]. The rigid isotopy classes are defined by the topology of real part and the type.

Curves of degree 6 – Nikulin [28]. As in degree 5, the rigid isotopy classes are defined by the topology of real part and the type.

Surfaces of degree 4 – Nikulin [28] and Kharlamov [18].

Constructions of curves similar Harnack's and Hilbert's constructions mentioned above can**not** give nonsingular plane curves of all isotopy types for degree ≥ 7

I had to develop [36], [37] a technique for perturbations of curves with complicated singularities and gluing of algebraic varieties.

This technique was used to obtain the isotopy classification of nonsingular projective curves of degree 7 (Viro [34]) and counter-examples to the Ragsdale conjecture (Viro [34], Itenberg [12]).

This technique is the subject of the second part of this talk. See also surveys [40] and [12].

To what degree? Often people ask: *To what degree the Hilbert sixteenth problem has been solved?*

The problem was **not** to give the topological classification of real algebraic curves of some specific degree. However, one may ask: *For what degrees the classification problems on topology of real algebraic varieties are solved?*

Isotopy classification problem of nonsingular plane projective curves of degree n has been solved for $n \le 7$.

For $n \le 5$ it was easy, solved in XIX century.

For $n = 6$ in 1969 by **Gudkov** [3].

For $n = 7$ in 1979 by **Viro** [34].

For *maximal* curves the *isotopy* classification has almost been done in degree 8.

Only 6 isotopy types are questionable.

For *pseudoholomorphic M-curves* the *isotopy* classification has been done in degree 8 by **Orevkov** [30].

Rigid isotopy classification of nonsingular plane projective curves of degree n has been solved for $n \le 6$.

For $n \le 4$ in XIX century by **Zeuthen, Klein**.

For $n = 5$ in 1981 by **Kharlamov** [19].

For $n = 6$ in 1979 by **Nikulin** [28].

For nonsingular *surfaces* in the projective 3-space all the problems have been solved for degree ≤ 4.

For $n \le 2$ see textbooks on Analytic Geometry.

For $n = 3$ by *Klein*.

For $n = 4$ in the seventies by **Nikulin** [28] and **Kharlamov** [18], [20].

Other objects of real algebraic geometry also were studied: curves on surfaces, curves with symmetries, degenerations of curves and surfaces, surfaces of classical types (like rational, Abelian, Enriques and K3 surfaces), rational 3-varieties, singular points of real polynomial vector fields, critical points of real polynomials, real algebraic knots and links, amoebas of real and complex algebraic varieties, real pseudoholomorphic curves, tropical varieties, ...

Open problems.

(1) The *second half* of the sixteenth Hilbert problem!
(2) How many connected *components* can a *surface of degree 5* in the real projective 3-space have?

(3) *Rigid isotopy* classification of curves for degree 7.
(4) Are all nonsingular real projective curves of a given *odd degree* with *connected* set of real points *rigid isotopic* to each other?
(5) Find *algebraic expressions* for basic topological invariants of a real algebraic curve (and, further, hypersurface, ...) *in terms of its equation*.
(6) Sharp estimates in the theory of *fewnomials*.
(7) Develop *real algebraic knot theories*.
(8) Study metric characteristics of real algebraic curves.
(9) Formulate counter-parts of topological questions about real algebraic varieties for varieties over *other* non algebraically closed *fields*,
(10) and solve them!

Part 2. Patchworking algebraic varieties and Tropical Geometry

4. Patchwork

Restrictions and constructions. A work towards isotopy classification of real algebraic varieties of a specific type (say, non-singular plane projective curves of a fixed degree) splits into work in two directions: first, one should find restrictions on topology imposed by the algebraic nature; second, one should prove existence of algebraic varieties of all the classes satisfying the restrictions.

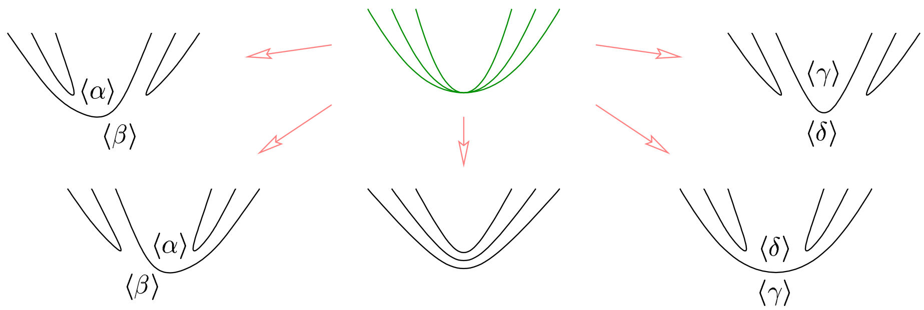
In the rest of the talk we consider the technique which is used in the constructions. I discovered this technique in 1979-1980. It proved to be useful for many other problems.

Construction of sextics. Here is how this technique works: 53 out of 56 topological types of non-singular plane projective curves of degree 6 can be realized by perturbation of the union of 3 ellipses tangent to each other at 2 points.

What can jump out of the points of tangency under perturbation? All possible isotopy types of the result are shown in Figure 1.

The two points of tangency can be perturbed simultaneously and independently. Figure 2 shows how curves of degree 6 with the maximal number of components of all three isotopy types are obtained.

Similarly non-singular curves of degree 7 of all topological types which were not realized by 1979 are obtained from four curves with two singular points of the same kind.



$\alpha =$	4	0	3	0	2	1	0	1	0	0
$\beta =$	0	4	0	3	0	1	2	0	1	0

$\gamma =$	3	2	1	0	0
$\delta =$	0	0	0	1	0

FIGURE 1. Perturbations of three branches tangent quadratically to each other at a point. Here $\langle n\rangle$ denotes n ovals which lie outside each other. Possible pairs (α,β), (γ,δ) are shown in the tables.

What lies behind these pictures? What are the equations of the curves?

Draw equations. Equations of curves are to be drawn on plane. A monomial $a_{kl}x^ky^l$ should be placed at $(k,l)\in\mathbb{R}^2$. A polynomial $a(x,y)=\sum_{kl}a_{kl}x^ky^l$ should sit on its *Newton polygon*

$$\Delta(a) = conv\{(k,l)\in\mathbb{R}^2 \mid a_{kl}\neq 0\}.$$

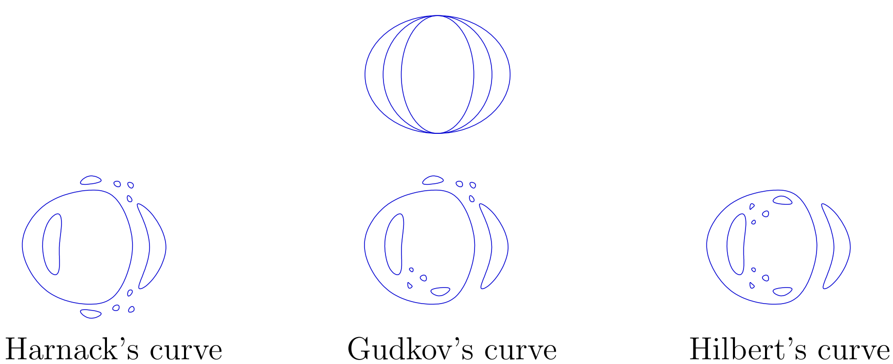
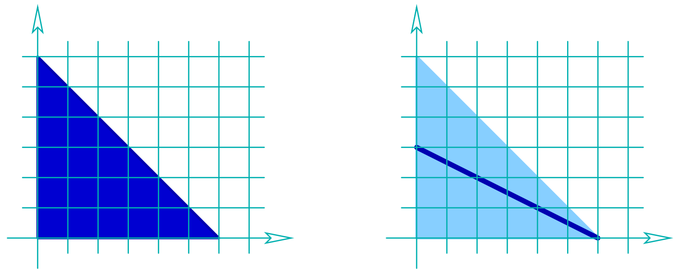


FIGURE 2. Obtaining of the M-sextics by small perturbation of the union of three ellipses.

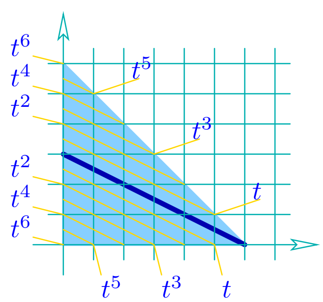
The Newton polygon for a generic polynomial of degree 6 is the triangle with vertices $(0,0),(6,0),(0,6)$.



However we started from the union of 3 ellipses. On $\mathbb{R}P^2$ it can be placed as the union of 3 parabolas $(y-ax^2)(y-bx^2)(y-cx^2)=0$. Then the Newton polygon is $[(6,0),(0,3)]$.

To perturb, we fill the two missing triangles with equations of curves we want to insert instead of neighborhoods of the singular points.

Introduce a small parameter $t>0$ to keep the new fragments of the polynomial in peace with each other. For sufficiently small t, the fragments defined by small terms are small, separated and do not spoil each other.



Log paper. A (double) logarithmic paper is a graph paper with logarithmic scales on both axes. The transition to the log paper corresponds to the change of coordinates:

$$\begin{cases} u = \ln x \\ v = \ln y. \end{cases}$$

How do graphs look on the *log paper*? The simplest special case: $y=ax^k$. We are forced to consider only *positive* x, y and hence assume that $a>0$. Then $v=\ln y=\ln(ax^k)=k\ln x+\ln a=ku+\ln a$, or $v=ku+b$, where $b=\ln a$. Thus $y=ax^k$ turns into $v=ku+b$.

Similarly, any binomial equation $y^l=ax^k$ defines line $lv=ku+b$.

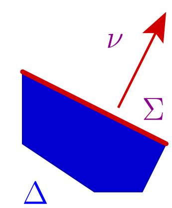
Logarithmic asymptotes. Let a be a real polynomial in x, y,

$$a(x,y) = \sum_{kl} a_{kl} x^k y^l$$

and V be the curve defined by $a(e^u, e^v) = 0$. Let Δ be the Newton polygon of a,

$$\Delta = conv\{(k,l) \in \mathbb{R}^2 \mid a_{kl} \neq 0\},$$

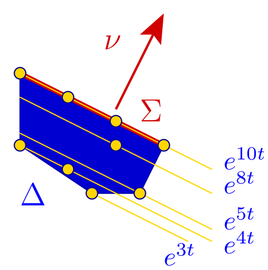
Σ be a side of Δ, and $\nu = (m,n)$ be an integer vector orthogonal to Σ.



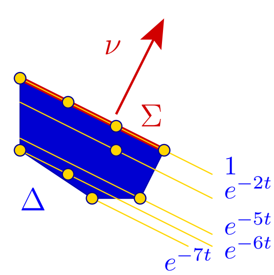
Go in the direction of ν looking at V. This movement corresponds to the following change of coordinates: $(u,v) \mapsto (mt+u, nt+v)$. Here is what happens to the equation:

$$a(e^u, e^v) = 0 \ \mapsto \ a(e^{mt+u}, e^{nt+v}) = 0,$$

that is $\sum a_{k,l} e^{ku+lv} = 0 \ \mapsto \sum (a_{k,l} e^{(km+ln)t}) e^{ku+lv} = 0$. All the coefficients tend to ∞. The distributions of the factors can be shown on the Newton polygon. The same factors appear on the line orthogonal to ν:

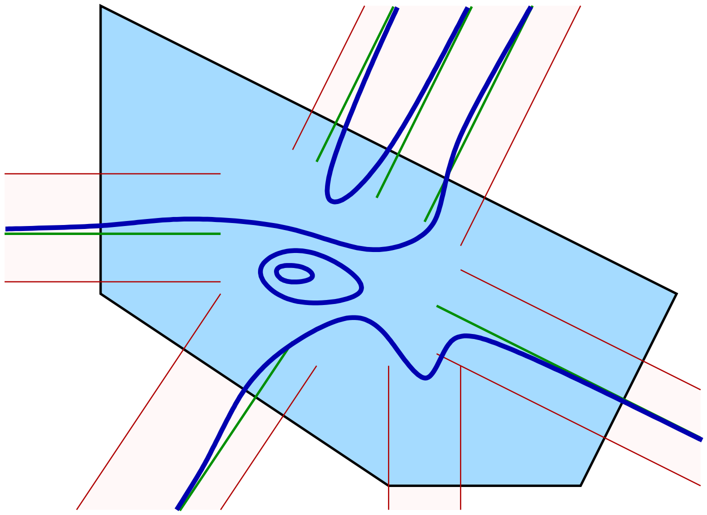


Calibrate!

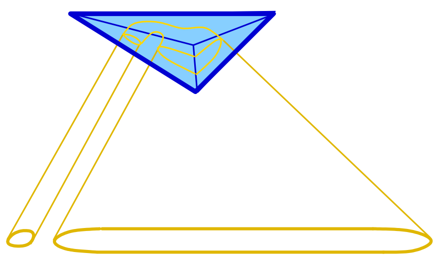


$a(e^{mt+u}, e^{nt+v}) = 0$ tends to $a^{\Sigma}(u,v) = \sum_{(k,l)\in\Sigma} a_{kl} e^{ku+lv} = 0$ as $t \to \infty$.

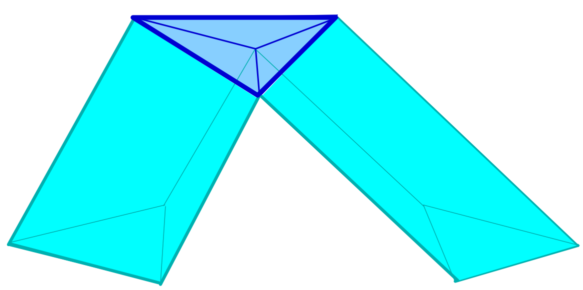
Picture of logarithmic asymptotes:



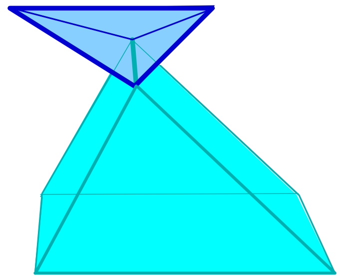
In high dimensions everything goes similarly. Consider a hypersurface defined by a generic polynomial. The principal part of the hypersurface fits inside of sufficiently expanded Newton polyhedron.



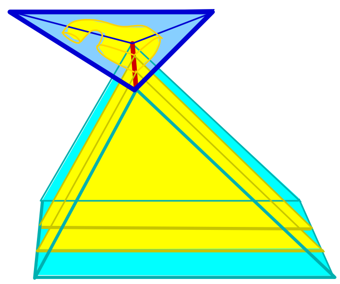
The space outside of Δ is divided into domains corresponding to the faces of Δ. A prism corresponds to a principal face.

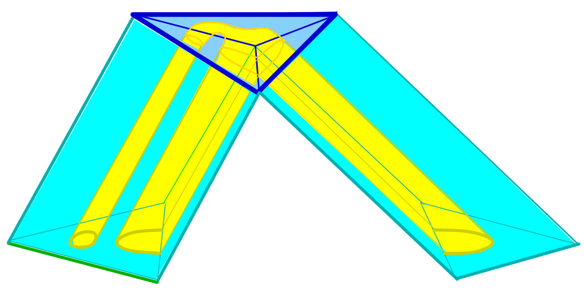


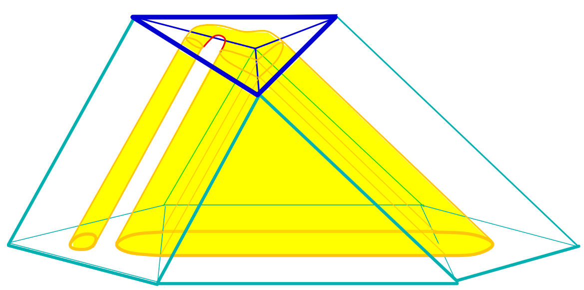
The domain corresponding to Σ has a shape of $\Sigma \times \Sigma^{\wedge}$, where $\Sigma^{\wedge}$ is the cone dual to Σ.



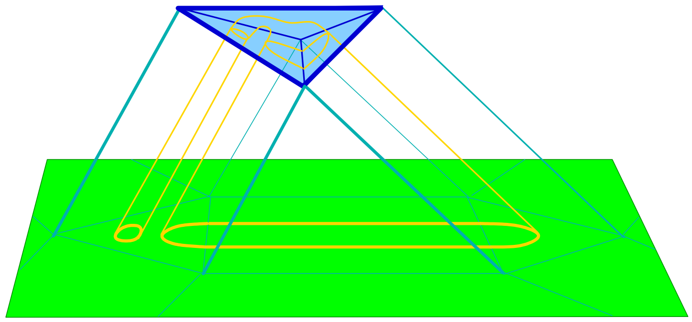
In the domain corresponding to face Σ the hypersurface is approximated by the hypersurface defined by the part of the polynomial sitting on Σ.







Consider a trace of the picture on a hyperplane which is far bellow the Newton Polyhedron.



The intersection of the hypersurface with the hyperplane is made of pieces corresponding to the faces of Δ looking down.

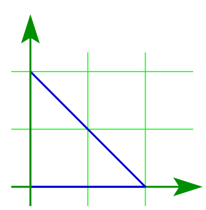
This can be used to *patchwork a hypersurface*. Just prepare pieces matching each other and put them on faces of a polyhedron.

Combinatorial patchwork. If only the smallest possible pieces are used, then the patchwork is nothing but combinatorics.

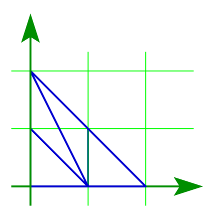
Initial data for combinatorial patchworking.

- m a positive integer (the degree of the curve),

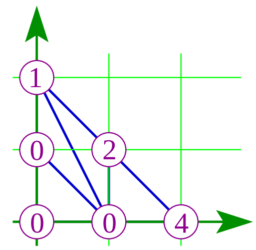
- Δ the triangle with vertices $(0,0)$, $(m,0)$, $(0,m)$, (In our example, $m = 2$.)



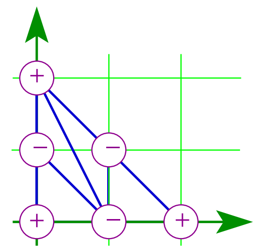
- τ a *convex* triangulation of Δ with integer vertices.



• $\nu : \Delta \longrightarrow \mathbb{R}_+$ a convex PL-function, such that triangles of τ are its domains of linearity.



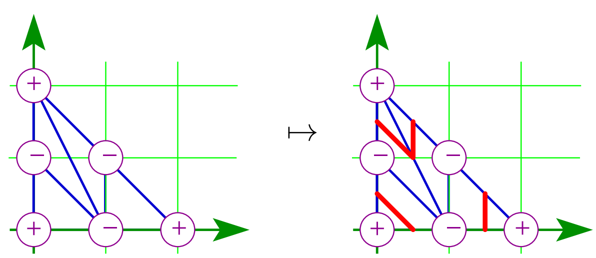
• $\sigma_{k,l}$ a sign ($+$ or $-$) at each vertex (k,l) of τ.



Patchworking of polynomials.

$$b_t(x,y) = \sum_{\substack{(k,l) \text{ runs over} \\ \text{vertices of } \tau}} \sigma_{k,l} t^{\nu(k,l)} x^k y^l.$$

Patchworking of PL-curve.

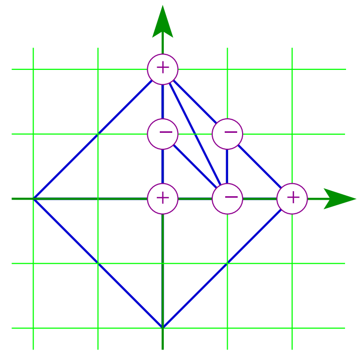


Combinatorial Patchwork Theorem. *Let m, Δ, τ, $\sigma_{k,l}$ and ν be initial data, b_t be the patchworked polynomial and $L \subset \Delta$ be the patchworked PL-curve.*

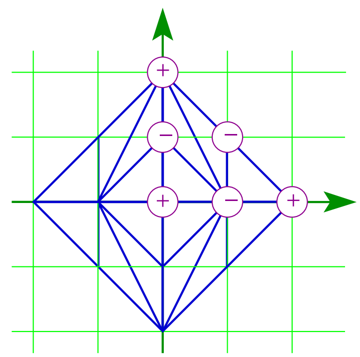
Then for a sufficiently small $t > 0$ the polynomial b_t defines in the first quadrant $\mathbb{R}^2_{++} = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$ a curve a_t such that the pair $(\mathbb{R}^2_{++},\ a_t)$ is homeomorphic to $(\operatorname{Int}\Delta,\ L \cap \operatorname{Int}\Delta)$.

Patchwork in all quadrants.

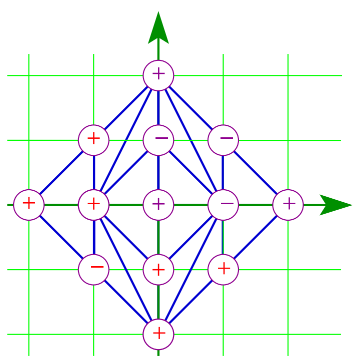
Adjoin to Δ its images $\Delta_x = s_x(\Delta)$, $\Delta_y = s_y(\Delta)$, $\Delta_{xy} = s_x \circ s_y(\Delta)$, where s_x, s_y are reflections against the coordinate axes. Put $A\Delta = \Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$.



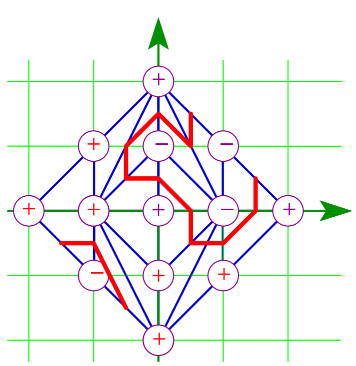
Extend τ to a symmetric triangulation $A\tau$ of $A\Delta$.



Extend $\sigma_{i,j}$ to a distribution of signs at the vertices of $A\tau$ by the rule: $\sigma_{i,j}\sigma_{\varepsilon i,\delta j}\varepsilon^i\delta^j = 1$, where $\varepsilon, \delta = \pm 1$. (In other words, passing from a vertex to its mirror image with respect to an axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign otherwise.)

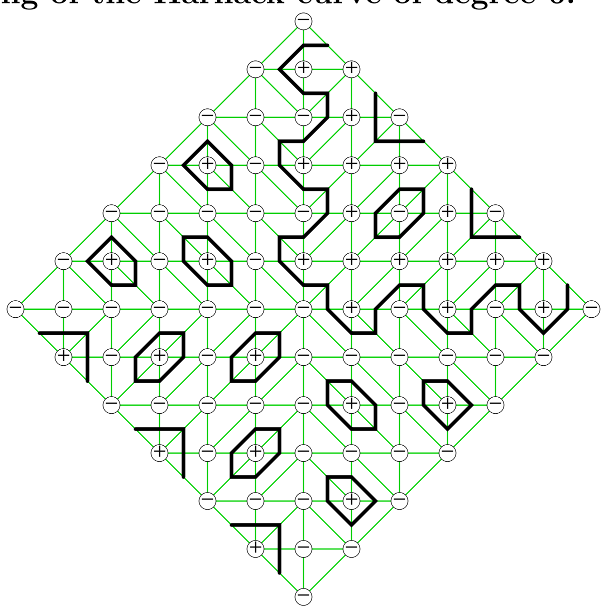


Draw the midlines.

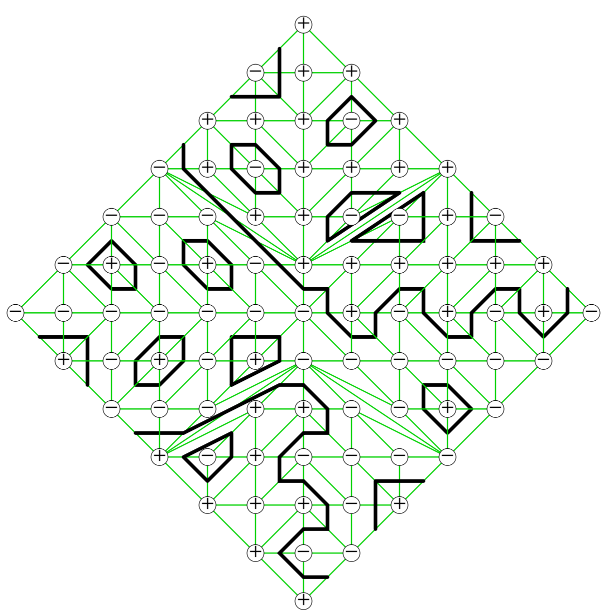


Addendum to the Patchwork Theorem. *Under the assumptions of Patchwork Theorem, for all sufficiently small $t > 0$ there exist a homeomorphism $A\Delta \to \mathbb{R}^2$ mapping AL onto the the affine curve defined by b_t and a homeomorphism $P\Delta \to \mathbb{R}P^2$ mapping PL onto the projective closure of this affine curve.*

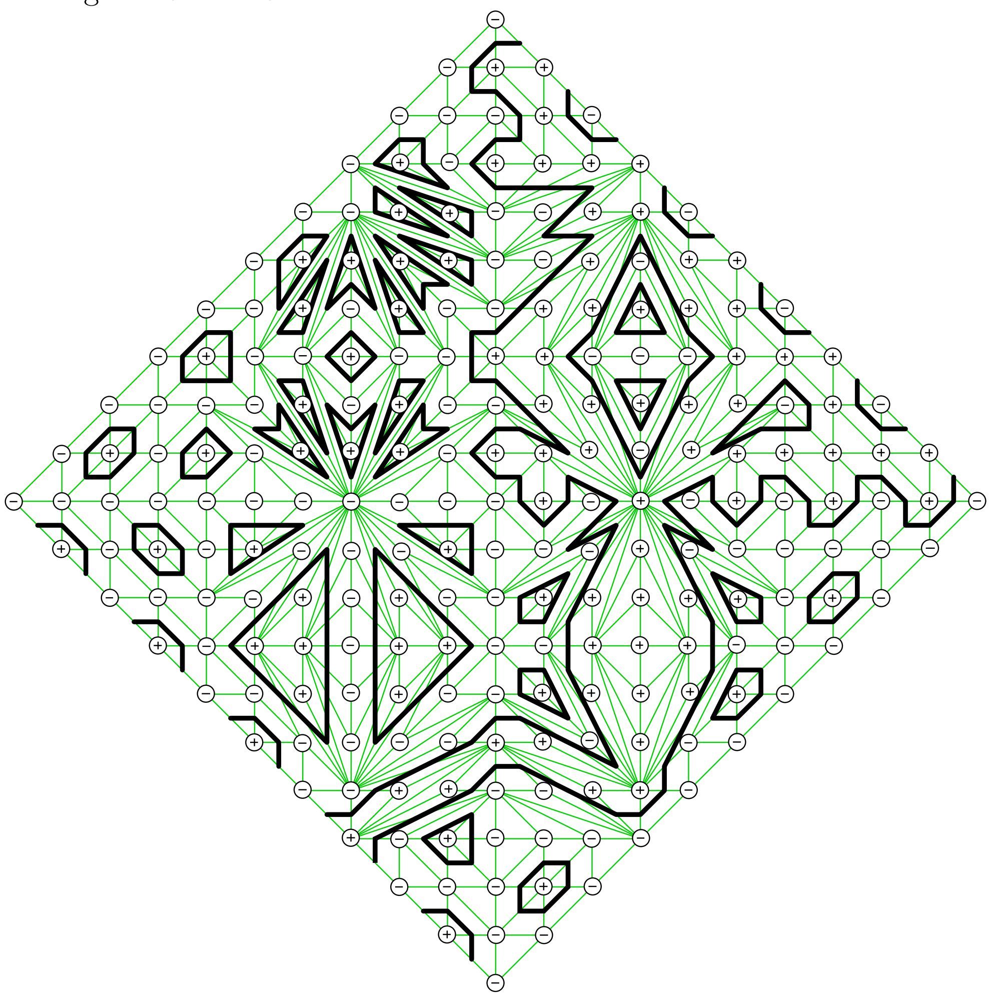
Patchworking of the Harnack curve of degree 6.



Gudkov's curve.



Curve of degree 10 with 32 odd ovals. Ilia Itenberg's patchwork [11] of a counter-example to the Ragsdale Conjecture. This is a curve of degree 10 with 32 odd ovals.



5. Tropical

Arnold's advice. In the late nineties Arnold proposed me to look into papers by Litvinov and Maslov on the *Idempotent Mathematics*. Arnold suggested that it may be related to *integrals against the Euler characteristic* aka in [39].

I could not find any relation to the integrals, but was not disappointed. This is really a fantastic matter.

Dequantization of positive real numbers is a family of semifields $\{S_h\}_{h\in[0,\infty)}$. As a set, $S_h = \mathbb{R}$ for each h.

The semiring operations $\oplus_h$ and $\odot_h$ in S_h are defined as follows:

$$a \oplus_h b = \begin{cases} h\ln(e^{a/h} + e^{b/h}), & \text{if } h > 0 \\ \max\{a, b\}, & \text{if } h = 0 \end{cases} \tag{1}$$

$$a \odot_h b = a + b \tag{2}$$

These operations depend *continuously* on h. For $h > 0$,

$$D_h : \mathbb{R}_{>0} \to S_h : x \mapsto h \ln x$$

is a *semiring isomorphism* of $\{\mathbb{R}_{>0}, +, \cdot\}$ onto $\{S_h, \oplus_h, \odot_h\}$.

Semiring S_h with $h > 0$ is a copy of $\mathbb{R}_{>0}$ with the usual operations.

$S_0 = \mathbb{R}_{\max+}$, a copy of $\mathbb{R}$ with addition $(a, b) \mapsto \max\{a, b\}$ and multiplication $(a, b) \mapsto a + b$. (This is an idempotent semiring, for the addition is idempotent: $a + a = a$ for any a.)

It is fashionable to consider any one parameter family of objects with all the objects but one isomorphic to each other and the one degenerated as a sort of quantization. Here it makes more sense than usually, because the deformation was discovered in relation with quantum mechanics.

Speaking mathematically: S_h is a continuous degeneration of the semiring $\mathbb{R}_{>0}$ to $\mathbb{R}_{\max,+}$

Speaking quantum: S_0 is a *classical* object (idempotent semifield $\mathbb{R}_{\max+}$, not that classical in mathematics), S_h with $h \neq 0$ are *quantum* objects (but very classical in mathematics), and the whole family S_h is a quantization of $\mathbb{R}_{\max,+}$

Litvinov-Maslov Correspondence Principle [23]. *There exists a (heuristic) correspondence, in the spirit of the correspondence principle in Quantum Mechanics, between important, useful and interesting constructions and results over the field of real (or complex) numbers (or the semiring of all nonnegative numbers) and similar constructions and results over idempotent semirings.*

Correspondences.

Integral $\int_X f(x)\,dx$	$\longleftrightarrow$	Supremum $\sup_X\{f(x)\}$
Fourier transform $\tilde{f}(\xi) = \int e^{ix\xi} f(x)\,dx$	$\longleftrightarrow$	Legendre transform $\tilde{f}(\xi) = \sup\{x \cdot \xi - f(x)\}$
Linear problems	$\longleftrightarrow$	Optimization problems
Polynomial over $\mathbb{R}_+$ $p(x) = \sum_k a_k x^k$	$\longleftrightarrow$	Convex PL-function $M_p(u) = \max_k \{ku + \ln a_k\}$

The dequantization continuously deforms the graph of polynomial on log paper to the tropical graph of the same polynomial. The deformation consists of the graphs of the same polynomial $\sum_k \ln(a_k)x^k$, but on S^2_h with varying $h \in [0,1]$.

Tropical algebra is how one calls the algebra $\mathbb{R}_{\max,+}$ that is the set $\mathbb{R}$ with operations $(a,b) \mapsto \max\{a,b\}$ and $(a,b) \mapsto a+b$ (or rather an isomorphic algebra $\mathbb{R}_{\min,+}$).

The adjective *tropical* was coined by French mathematicians in honor of *Imre Simon*, who resides in São Paolo, Brazil and was one of the pioneers in *algebra* of $\mathbb{R}_{\max,+}$. This is a *semi-ring*. Everything is as in a ring, but it has neither subtraction, nor 0.

Adjoin $-\infty$ to $\mathbb{R}_{\max,+}$ as 0 and denote the result by $\mathbb{T}$. This is a semi-field.

Tropical polynomials. A polynomial over $\mathbb{T}$ is a convex PL-function with integral slopes. Indeed, a monomial $ax_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$ is $a + k_1x_1 + k_2x_2 + \dots + k_nx_n$, a linear function $a + \langle k, x\rangle$.

A tropical polynomial is a finite tropical sum of tropical monomials, that is the maximum of a finite collection of linear functions.

Tropical geometry is an algebraic geometry over $\mathbb{T}$.

Algebraic geometry is based on polynomials. The tropical geometry is based on convex PL-functions with integral slopes.

It would be exotic and needless if there were no relations to the classical algebraic geometry, which is provided by the Litvinov-Maslov dequantization.

Real algebraic geometry as quantized PL-geometry

$$
\begin{tikzcd}
\begin{array}{c}\text{Real}\\ \text{polynomials}\\ \text{with positive coefficients}\end{array} \arrow[d, "\text{generate}"] & \begin{array}{c}\text{Convex}\\ \text{PL-functions}\\ \text{with integral slopes}\end{array} \arrow[l, "\text{quantization}"'] \arrow[d, "\text{generate}"'] \\
\begin{array}{c}\text{Real}\\ \text{Algebraic Geometry}\\ \text{in positive quadrant}\end{array} & \text{Tropical Geometry} \arrow[l, "\text{quantization}"]
\end{tikzcd}
$$

Additional simple tricks, like the transition above from patchworking in the first quadrant to patchworking in all quadrant, allow one to remove in the left hand side of this diagram all mentions of positivity.

Furthermore, one can incorporate into the picture complex numbers with arbitrary arguments. See [27].

Combinatorial patchworking can be presented as a construction of real tropical plane curve and its dequantization. See details in [41]

Applications of Tropical Geometry (besides combinatorial patchworking) lie in enumerative geometry, both real and complex. See [25], [26], [27]. So far all applications are based on the fact that it is easier to construct tropical varieties rather than algebraic varieties, and deformation quantization allows one to keep many interesting properties of the varieties under consideration.

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