CONGRUENCE MODULO 8 FOR REAL ALGEBRAIC CURVES OF DEGREE 9

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1. Introduction and statement of the result. Let A be a non-singular real algebraic curve of degree m in $\mathbf{RP}^2$. Its connected components are embedded circles. Those of them whose complement in $\mathbf{RP}^2$ is not connected are called *ovals*. One says that an oval u lies *inside* an oval v if u is contained is the orientable component of the complement of v. A union of d ovals $v_1, \dots, v_d$ such that v_i is inside v_{i+1}, $1 \le i < d$, is called a *nest of the depth d*. An oval is called *exterior* if it does not lye inside any other oval; an oval is called *empty* if there is no other ovals inside it. An oval is called *even* if it is contained inside an even number of other ovals, and *odd* otherwise. Denote by p and n the number of even and odd ovals respectively. One says that A is an *M-curve* if it has the maximal possible number of connected components which equals $M(m) = (m-1)(m-2)/2 + 1$. If A has $M(m) - i$ connected components then it is called an *$(M-i)$-curve* Let $\mathbf{C}A$ be the complexification of A. If $\mathbf{C}A \setminus A$ is not connected, A is a curve *of type I*; if $\mathbf{C}A \setminus A$ is connected then A is a curve *of type II.*

For curves of an even degree $m = 2k$, in some cases, the difference $p - n$ satisfies congruences. For example,
Gudkov-Rohlin congruence $p - n \equiv k^2 \mod 8$ for M-curves,
Gudkov-Krahnov-Kharlamov congruence $p - n \equiv k^2 \pm 1 \mod 8$ for $(M-1)$-curves,
Kharlamov-Marin congruence $p - n \not\equiv k^2 + 4 \mod 8$ for M-curves of type II, and
Arnold congruence $p - n \equiv k^2 \mod 4$ for curves of type I.
These statements do not extend to curves of *odd* degrees. So, for an M-curve of any odd degree $2k+1$ with $k \ge 3$, the residue $p - n \mod 8$ may take any values congruent to $k \mod 2$. As far as we know, the following theorem is the first result of this kind.

Theorem 1. *Let A be a curve of degree $m = 2k+1 = 4d+1$ which has 4 paiwise distinct nests of the depth d. Then*

$$\begin{array}{lll} \textit{if } A \textit{ is an } M\textit{-curve then} & p - n \equiv k(k+1) & \mod 8; \quad (1) \\ \textit{if } A \textit{ is an } (M-1)\textit{-curve then} & p - n \equiv k(k+1) \pm 1 & \mod 8; \\ \textit{if } A \textit{ is an } (M-2)\textit{-curve of type II then} & p - n \not\equiv k(k+1) + 4 & \mod 8; \\ \textit{if } A \textit{ is a curve of type I then} & p - n \equiv k(k+1) & \mod 4; \end{array}$$

It is clear that (1) for $d = 2$ is equivalent to the fact that the number of exterior empty ovals of an M-curve of degree 9 with 4 nests is divisible by 4. This was conjectured by Korchagin [2]. Theorem 1 is obtained below (see Sect. 4) as a consequence of Kharlamov-Viro congruence [1] which generalizes the classical congruences to the case of singular curves of even degrees.

2. Brown - van der Blij invariant. By a *quadratic space* we mean a triple $(V, \circ, q)$ composed of a vector space V over the field $\mathbf{Z}_2$, a bilinear form $V \times V \to \mathbf{Z}_2$, $(x, y) \mapsto x \circ y$, and a function $q : V \mapsto \mathbf{Z}_4$ which is quadratic with respect to $\circ$ in the sense that $q(x + y) = q(x) + q(y) + 2x \circ y$. A quadratic space is determined by its *Gram matrix* with respect to a base $e_1, \dots, e_n$ of V, i.e. the matrix $Q = (q_{ij})$ where $q_{ii} = q(e_i)$ and $q_{ij} = e_i \circ e_j$ for $i \neq j$ (the diagonal entries are defined mod 4, the others mod 2; note that $q(x) \equiv x \circ x \mod 2$). It is easy to see that by elementary changes of the base, one can put the Gram matrix to the block-diagonal form $\operatorname{diag}(d_1, \dots, d_t) \oplus Q_1 \oplus \dots \oplus Q_s$ where each block Q_i is either $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. If all $d_i \neq 2$, we say that the form q is *informative* and in this case we define its *Brown - van der Blij invariant* $B(q) = \sum B(d_i) + \sum B(Q_i) \mod 8$ where $B(0) = 0$, $B(1) = 1$, $B(-1) = -1$, $B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$, and $B\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4$.

3. Kharlamov-Viro congruence for nodal curves. Let A be a curve in $\mathbf{RP}^2$ of degree $2k$ defined by $f = 0$ and let each of its singular points is the point of transverse intersection of two smooth real local branches. A is called an *M-curve* (a curve *of type I*) if the normalization of any its irreducible component is an M-curve (a curve of type I). A curve which is not of type I, is of type II. Let $x_1, \dots, x_s$ be the singular points and Γ_A be the union of the connected components of A passing through them. Let $b = 0$ if $\mathbf{RP}^2_+ = \{f \geq 0\}$ is contractable in $\mathbf{RP}^2$ and $b = (-1)^k$ otherwise. Let $C_1, \dots, C_r$ be the components of $\mathbf{RP}^2 \setminus \Gamma_A$ where $f > 0$ near Γ_A.

Let us define a quadratic space $(V, \circ, q)$ as follows. Let $(V_0, \circ, q_0)$ be the quadratic space with the orthogonal base $e_1, \dots, e_s$ such that $q_0(e_1) = \dots = q_0(e_s) = -1$. Set $c_i = \sum_{j \in \alpha_i} e_j$ where $\{x_j\}_{j \in \alpha_i}$ are the singular points through which ∂C_i passes only once. In the cases when either A is contractible in $\mathbf{RP}^2$ or, as in Sect. 4, there is a *branch of* A (i.e. a smoothly immersed circle) which is non-contractible in $\mathbf{RP}^2$, we define $V \subset V_0$ as the subset generated by $c_1, \dots, c_r$ and we set $q = q_0|_V$.

In the case when A is not contractible in $\mathbf{RP}^2$ but all its branches are, let us choose a simple closed curve in A which is not contractible in $\mathbf{RP}^2$. Let $(V_0', \circ, q_0')$ be the quadratic space with the base $(e_0, \dots, e_s)$ which contains V_0 as a quadratic subspace ($q_0'|_{V_0} = q_0$) and let $q_0'(e_0) = (-1)^k$, $e_0 \circ e_j = 0$ iff $L \sim 0$ in $H_1(RP^2_+, RP^2_+ \setminus x_j)$. Let $V \subset V_0'$ be the subspace generated by $c_1, \dots, c_r$, and $e_0 + \sum_{j \in \alpha_0} e_j$ where $\alpha_0 = \{j \mid L \sim 0 \text{ in } H_1(RP^2_-, RP^2_- \setminus x_j)\}$, and let $q = q_0'|_V$.

Theorem 2. *Suppose that each branch of A which is conractable in $\mathbf{RP}^2$ cuts other branches at $n \equiv 0 \mod 4$ singular points and each branch which is not contractible in $\mathbf{RP}^2$, at $n \equiv (-1)^{k+1} \mod 4$ singular points. If A is an M-curve then $\chi(\mathbf{RP}^2_+) \equiv k^2 + B(q) + b \mod 8$ and also the corresponding analogues of Gudkov-Krahnov-Kharlamov, Kharlamov-Marin, and Arnold congruences take place.*

Theorem 2 is a corollary of Theorem (3.B) on curves with arbitrary singularities from the paper by Kharlamov and Viro [1]. Theorem 2 is formulated here because there are mistakes in [1] in the discussion of the corresponding particular case (4.I), (4.J) of Theorem (3.B).

4 Proof of Theorem 1. Let us choose any three pairwise distinct nests of the depth d and a point inside the innermost oval of each of them. Theorem 1 follows from Theorem 2 applied to the union of A and the three straight lines passing through the three chosen points. Indeed, the union of the three chosen lines and the non-contractible branch of A divides $\mathbf{RP}^2$ into 4 triangles and 3 quadrangles (curvilinear). All ovals not belonging to the three chosen nests lye in the quadrangles (otherwise would exist a conic having too many intersections with A). Therefore, after the suitable choice of the sign, one has $\chi(\mathbf{RP}^2_+) = \chi(\bigcup \overline{C}_j) + p' - n'$ where p' and n' are the numbers of even and odd ovals, not belonging to the three chosen nests. $B(q)$ can be computed according to Sect. 2.

References

1. V.M. Kharlamov, O.Ya. Viro, *Extension of the Gudkov-Rohlin congruence*, Lect. Notes. Math. **1346** (1988), 357–406.
2. A.B. Korchagin, *Construction of new M-curves of 9th degree*, Lect. Notes. Math. **1524** (1991), 407–426.

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