1. Introduction and statement of the result. Let $A$ be a non-singular real algebraic curve of degree $m$ in $\mathbb{RP}^2$. Its connected components are embedded circles. Those of them whose complement in $\mathbb{RP}^2$ is not connected are called ovals. One says that an oval $u$ lies inside an oval $v$ if $u$ is contained in the orientable component of the complement of $v$. A union of $d$ ovals $v_1, \ldots, v_d$ such that $v_i$ is inside $v_{i+1}$, $1 \leq i < d$, is called a nest of the depth $d$. An oval is called exterior if it does not lie inside any other oval; an oval is called empty if there is no other ovals inside it. An oval is called even if it is contained inside an even number of other ovals, and odd otherwise. Denote by $p$ and $n$ the number of even and odd ovals respectively. One says that $A$ is an $M$-curve if it has the maximal possible number of connected components which equals $M(m) = (m - 1)(m - 2)/2 + 1$. If $A$ has $M(m) - i$ connected components then it is called an $(M - i)$-curve. Let $CA$ be the complexification of $A$. If $CA \setminus A$ is not connected, $A$ is a curve of type I; if $CA \setminus A$ is connected then $A$ is a curve of type II.

For curves of an even degree $m = 2k$, in some cases, the difference $p - n$ satisfies congruences. For example, Gudkov-Rohlin congruence $p - n \equiv k^2 \mod 8$ for $M$-curves, Gudkov-Krahnov-Kharlamov congruence $p - n \equiv k^2 \pm 1 \mod 8$ for $(M - 1)$-curves, Kharlamov-Marin congruence $p - n \not\equiv k^2 + 4 \mod 8$ for $M$-curves of type II, and Arnold congruence $p - n \equiv k^2 \mod 4$ for curves of type I. These statements do not extend to curves of odd degrees. So, for an $M$-curve of any odd degree $2k + 1$ with $k \geq 3$, the residue $p - n \mod 8$ may take any values congruent to $k \mod 2$. As far as we know, the following theorem is the first result of this kind.

Theorem 1. Let $A$ be a curve of degree $m = 2k + 1 = 4d + 1$ which has 4 pairwise distinct nests of the depth $d$. Then

if $A$ is an $M$-curve then $p - n \equiv k(k + 1) \mod 8$; (1)
if $A$ is an $(M - 1)$-curve then $p - n \equiv k(k + 1) \pm 1 \mod 8$;
if $A$ is an $(M - 2)$-curve of type II then $p - n \not\equiv k(k + 1) + 4 \mod 8$;
if $A$ is a curve of type I then $p - n \equiv k(k + 1) \mod 4$;

It is clear that (1) for $d = 2$ is equivalent to the fact that the number of exterior empty ovals of an $M$-curve of degree 9 with 4 nests is divisible by 4. This was conjectured by Korchagin [2]. Theorem 1 is obtained below (see Sect. 4) as a consequence of Kharlamov-Viro congruence [1] which generalizes the classical congruences to the case of singular curves of even degrees.
2. Brown - van der Blij invariant. By a quadratic space we mean a triple
\((V, \circ, q)\) composed of a vector space \(V\) over the field \(\mathbb{Z}_2\), a bilinear form \(V \times V \to \mathbb{Z}_2\),
\((x, y) \mapsto x \circ y\), and a function \(q : V \to \mathbb{Z}_4\) which is quadratic with respect to \(\circ\) in
the sense that \(q(x + y) = q(x) + q(y) + 2x \circ y\). A quadratic space is determined by
its Gram matrix with respect to a base \(e_1, \ldots, e_n\) of \(V\), i.e. the matrix
\(Q = (q_{ij})\) where \(q_{ii} = q(e_i)\) and \(q_{ij} = e_i \circ e_j\) for \(i \neq j\) (the diagonal entries are defined
mod 4, the others mod 2; note that \(q(x) \equiv x \circ x \mod 2\)). It is easy to see that by
elementary changes of the base, one can put the Gram matrix to the block-diagonal
form \(\text{diag}(d_1, \ldots, d_s) \oplus Q_1 \oplus \cdots \oplus Q_s\) where each block \(Q_i\) is either \((\begin{smallmatrix}0 & 1 \\ 1 & 0 \end{smallmatrix})\),
or \((\begin{smallmatrix}2 & 1 \\ 1 & 2 \end{smallmatrix})\).

If all \(d_i \neq 2\), we say that the form \(q\) is informative and in this case we define its
Brown - van der Blij invariant
\(B(q) = \sum B(d_i) + \sum B(Q_i) \mod 8\) where \(B(0) = 0\),
\(B(1) = 1\), \(B(0) = -1\), \(B\left(\begin{smallmatrix}0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = 0\), and \(B\left(\begin{smallmatrix}2 & 1 \\ 1 & 2 \end{smallmatrix}\right) = 4\).

3. Kharlamov-Viro congruence for nodal curves. Let \(A\) be a curve in \(\mathbb{RP}^2\)
of degree \(2k\) defined by \(f = 0\) and let each of its singular points is the point of
transverse intersection of two smooth real local branches. \(A\) is called an \(M\)-curve
(a curve of type I) if the normalization of any its irreducible component is an \(M\)-
curve (a curve of type II). Let \(x_1, \ldots, x_s\) be the singular points and \(\Gamma_A\) be the union of the connected components of \(A\)
passing through them. Let \(b = 0\) if \(\mathbb{RP}^2 = \{f \geq 0\}\) is contractable in \(\mathbb{RP}^2\)
and \(b = (-1)^k\) otherwise. Let \(C_1, \ldots, C_r\) be the components of \(\mathbb{RP}^2 \setminus \Gamma_A\),
where \(f > 0\) near \(\Gamma_A\).

Let us define a quadratic space \((V, \circ, q)\) as follows. Let \((V_0, \circ, q_0)\) be the quadratic
space with the orthogonal base \(e_1, \ldots, e_s\), such that \(q_0(e_1) = \cdots = q_0(e_s) = -1\).
Set \(c_i = \sum_{j \in \alpha_i} e_j\) where \(\{x_j\} \subset \alpha_i\) are the singular points through which \(\partial C_i\) passes
only once. In the cases when either \(A\) is contractible in \(\mathbb{RP}^2\) or, as in Sect. 4, there
is a branch of \(A\) (i.e. a smoothly immersed circle) which is non-contractible in \(\mathbb{RP}^2\),
we define \(V \subset V_0\) as the subset generated by \(c_1, \ldots, c_r\) and we set \(q = q_0|_V\).

In the case when \(A\) is not contractible in \(\mathbb{RP}^2\) but all its branches are, let us
choose a simple closed curve \(C\) in \(A\) which is not contractible in \(\mathbb{RP}^2\). Let \((V_0', \circ, q_0')\)
be the quadratic space with the base \((e_0, \ldots, e_s)\) which contains \(V_0\) as a quadratic
subspace \((q_0'|_{V_0} = q_0)\)
and let \(q_0'(e_0) = (-1)^k\), \(e_0 \circ e_j = 0\) iff \(L \sim 0\) in \(H_1(\mathbb{RP}_+^2, \mathbb{RP}_+^2 \setminus x_j)\).
Let \(V \subset V_0'\) be the subspace generated by \(c_1, \ldots, c_r\), and \(e_0 + \sum_{j \in \alpha_0} e_j\) where
\(\alpha_0 = \{j | L \sim 0\} \subset H_1(\mathbb{RP}_+^2, \mathbb{RP}_+^2 \setminus x_j)\},\) and let \(q = q_0'|_V\).

**Theorem 2.** Suppose that each branch of \(A\) which is contractible in \(\mathbb{RP}^2\) cuts
other branches at \(n \equiv 0 \mod 4\) singular points and each branch which is not contractible in \(\mathbb{RP}^2\),
at \(n \equiv (-1)^{k+1} \mod 4\) singular points. If \(A\) is an \(M\)-curve then
\(\chi(\mathbb{RP}_+^2) \equiv k^2 + B(q) + b \mod 8\) and also the corresponding analogues of Gudkov-
Kharlamov-Kharlamov-Marin, and Arnold congruences take place.

Theorem 2 is a corollary of Theorem (3.B) on curves with arbitrary singularities
from the paper by Kharlamov and Viro [1]. Theorem 2 is formulated here because
there are mistakes in [1] in the discussion of the corresponding particular case (4.I),

4 Proof of Theorem 1. Let us choose any three pairwise distinct nests of the
depth \(d\) and a point inside the innermost oval of each of them. Theorem 1 follows from
Theorem 2 applied to the union of \(A\) and the three straight lines passing
through the three chosen points. Indeed, the union of the three chosen lines and the non-contractible branch of $A$ divides $\mathbb{RP}^2$ into 4 triangles and 3 quadrangles (curvilinear). All ovals not belonging to the three chosen nests lie in the quadrangles (otherwise would exist a conic having too many intersections with $A$). Therefore, after the suitable choice of the sign, one has $\chi(\mathbb{RP}^2) = \chi(\bigcup \mathcal{C}_j) + p' - n'$ where $p'$ and $n'$ are the numbers of even and odd ovals, not belonging to the three chosen nests. $B(q)$ can be computed according to Sect. 2.

References