Topology Vol. 31, No. 4, pp. 865–902, 1992
Printed in Great Britain.

0040–9383/92 \$5.00+.00

STATE SUM INVARIANTS OF 3-MANIFOLDS AND QUANTUM 6*j*-SYMBOLS

V.G. TURAEV and O.Y. VIRO

(*Received* 11 *October* 1990*; in revised form* 24 *September* 1991)

IN THE 1980s the topology of low dimensional manifolds has experienced the most remarkable intervention of ideas developed in rather distant areas of mathematics. In the 4-dimensional topology this process was initiated by S. Donaldson. He applied the theory of the Yang–Mills equation and instantons to study 4-manifolds. In dimension 3 a similar breakthrough was made by V. Jones. He discovered his famous polynomial of links in 3-sphere S^3 via an astonishing use of von Neumann algebras. It has been soon understood that deep notions of statistical mechanics and quantum field theory stay behind the Jones polynomial (see [8,16,18]). The relevant basic algebraic structures turn out to be the Yang–Baxter equation, the R-matrices, and the quantum groups (see [5–7]). This viewpoint, in particular, enables one to generalize the Jones polynomial to links in arbitrary compact oriented 3-manifolds (see [13]).

In this paper we present a new approach to constructing "quantum" invariants of 3-manifolds. Our approach is intrinsic and purely combinatorial. The invariant of a manifold is defined as a certain state sum computed on an arbitrary triangulation of the manifold. The state sum in question is based on the so-called quantum $6j$-symbols associated with the quantized universal enveloping algebra $U_q(sl_2(\mathbb{C}))$ where q is a complex root of 1 of a certain degree $r > 2$ (see [9]). The state sum on a triangulation X of a compact 3-manifold M is defined, roughly speaking, as follows. Assume for simplicity that M is closed, i.e. $\partial M = \varnothing$. We consider "colorings" of X which associate with edges of X elements of the set of colors $\{0, 1/2, 1, \dots, (r-2)/2\}$. Having a coloring of X we associate with each 3-simplex of X the q-$6j$-symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in \mathbb{C}$$

where (i, l), (j, m), (k, n) are the pairs of colors of opposite edges of this simplex. We multiply these symbols over all 3-simplexes of X and sum up the resulting products (with certain weights) over colorings of X.

The main point of the construction outlined above is independence of the state sum of the choice of triangulation. This is verified using a geometric technique developed by M.H.A. Newman [12] and J.W. Alexander [1] in the late 1920s. Alexander proved that some simple transformations of triangulations of polyhedra enable one to relate any two combinatorially equivalent triangulations. These Alexander transformations are infinite in number even in the case of 3-dimensional manifolds. However, in the case of triangulations of manifolds one may pass to the dual cell subdivisions. This passage transforms the Alexander moves into certain operations on cell complexes. These latter operations can be presented as compositions of certain local moves, which are finite in number in each dimension. In particular in dimension 3 there are three such moves. (Essentially these moves were considered by S. Matveev [11] and R. Piergallini [21] in their study of special spines of 3-manifolds). Thus, translating our state model into the "dual" language we have to check only 3 identities which happen to follow directly from the basic properties of q-$6j$-symbols.

The ideas outlined above lead not only to numerical invariants of 3-manifolds but rather to a 3-dimensional non-oriented topological quantum field theory (corresponding to the root of unity q; for a general discussion of topological quantum field theories see [2].) In particular, with each closed surface F we associate a finite-dimensional vector space $Q(F) = Q_q(F)$ over $\mathbb{C}$. The full modular group *Mod* F (the group of isotopy classes of degree ± 1 homeomorphisms $F \to F$) canonically acts in $Q(F)$. Note that to define $Q(F)$ we have to fix a triangulation of F and to show that *a posteriori* $Q(F)$ does not depend on the choice of triangulation up to a canonical isomorphism. In this respect our construction resembles very much the construction of simplicial homology.

It would be most important to relate our invariants of 3-manifolds with Witten's topological quantum field theory based on a Feynmann integral with non-abelian Chern–Simons action [18] and its mathematical counterpart introduced in [13]. In contrast to [18,13], our invariants are not sensible to orientations of manifolds. Moreover they are defined for non-oriented (and even non-orientable) manifolds. Note also that the action of *Mod* F in $Q(F)$ discussed above is an honest linear action in contrast to the projective action in [18,13]. These obsevations suggest that for orientable 3-manifolds our topological quantum field theory is related to $\mathcal{F} \otimes \bar{\mathcal{F}}$ where $\mathcal{F}$ is the theory constructed in [13] and overbar is the complex conjugation.

In a forthcoming paper of the first author our constructions will be used to produce invariants of links in compact 3-manifolds which are computed from triangulations of link exteriors and which generalize the Jones polynomial of links in the 3-sphere.

In the case $r = 3, q = \exp(\pm 2\pi\sqrt{-1/3})$, our invariants may be computed from standard cohomological invariants of manifolds. In particular, this computation shows non-triviality of our invariants.

Actual computation of our invariants from their definition is algorithmical but rather work-consuming. With this view we develop a dual approach to the invariants based on the theory of simple 2-skeletons of 3-manifolds. This theory generalizes the theory of special spines (see [4,11, 21]). Namely, we show that the invariants may be computed via a state sum model on any simple 2-skeletons. Usually it is easier to deal with simple 2-skeletons than with triangulations. Here the situation is similar to the one in homology theory where simplicial homology of polyhedra are computed in terms of cell decompositions. The difference however is that cell decompositions generalize triangulations whereas simple stratifications generalize the cell subdivisions of 3-manifolds which are dual to triangulations.

In particular this dual approach enables one to calculate our invariants from Heegaard diagrams.

Note that q-$6j$-symbols were used in [17] in a different manner to produce isotopy invariants of links in those 3-manifolds which are circle bundles over surfaces.

The paper consists of eight sections. In Section 1 we introduce our state sum models on triangulations of 3-manifolds. Section 1 begins with an axiomatic description of algebraic objects which are prerequisite for our approach to constructing invariants. We present the state sum model for closed 3-manifolds (this case is conceptually simpler) and then proceed to 3-manifolds with boundary.

In Section 2 we construct the relevant 3-dimensional topological quantum field theory and, in particular, define the corresponding representations of the modular groups.

Sections 3, 4 and 5 are devoted to proof of independence of the state sum on the choice of triangulation. In Section 3 we recall the Alexander theorem and translate it into the dual language. In Section 4 we introduce simple 2-polyhedra and study a version of our model on these polyhedra. In Section 5 we conclude the proof of the invariance of the state sum.

In Section 6 we develop an approach to the same invariants based on the theory of simple stratifications.

In Section 7 we show that the q-$6j$-symbols where q is a root of unity fit in the framework of our constructions.

In Section 8 we present calculation of $|M|$ for $M = S^3, \mathbb{R}P^3, L(3,1)$ and $S^1 \times S^2$.

Section 9 is concerned with the simplest case: when q is a cubic root of unity. In this case we give an interpretation of our state sum invariants in terms of cohomology. In Appendix 1 we prove a relative version of the Alexander theorem (used in Section 5).

In Appendix 2 we discuss simple spines of manifolds.

Topological part of this paper is written in PL-category. In particular, all manifolds as well as maps of polyhedra are piecewise linear.

1. STATE SUM INVARIANTS OF TRIANGULATED 3-MANIFOLDS

1.1 Initial data. In this subsection we describe our initial, purely algebraic data which will be used below to define an invariant of triangulated 3-manifolds.

Fix a commutative ring K with unity. Denote by K^* the group of invertible elements of K. Assume that we are given a finite set I, a function $i \mapsto w_i$: $w_i : I \to K^*$, and an element w of K^*. Assume that we have distinguished a set *adm* of unordered triples of elements of I. Here we put no condition on this set of triples; in particular, elements of a triple are permitted to coincide with each other. The triples belonging to this distinguished set will be said to be *admissible*.

An ordered 6-tuple $(i, j, k, l, m, n) \in I$ is said to be admissible, if the unordered triples

$$(i, j, k), (k, l, m), (m, n, i), (j, l, n)$$

are admissible. (A geometric motivation of this definition will be given in the next subsection.)

Assume that with each admissible 6-tuple $(i, j, k, l, m, n) \in I$ it is associated an element of K. We will denote this element by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

and call it the *symbol* of the 6-tuple. Assume finally the following symmetries of the symbol: for any admissible 6-tuple (i, j, k, l, m, n)

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} = \begin{vmatrix} l & m & k \\ i & j & n \end{vmatrix} = \begin{vmatrix} l & j & n \\ i & m & k \end{vmatrix}. \tag{1}$$

Note that if the 6-tuple (i, j, k, l, m, n) is admissible then the 6-tuples (j, i, k, m, l, n), (i, k, j, l, n, m), (i, m, n, l, j, k), (l, m, k, i, j, n) and (l, j, n, i, m, k) involved in (1) are also admissible.

Now we introduce some conditions on initial data.

Let us say that the initial data described above satisfy *the condition* $(*)$, if *for any* $j_1, j_2, j_3, j_4, j_5, j_6 \in I$ *such that the triples* $(j_1, j_3, j_4), (j_2, j_4, j_5), (j_1, j_3, j_6)$, *and* (j_2, j_5, j_6) *are admissible we have*

$$\sum_j w_j^2 w_{j_4}^2 \begin{vmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{vmatrix} \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{vmatrix} = \delta_{j_4, j_6}.$$

Here δ is the Kronecker delta. It is understood that we sum up over j such that the symbols involved in the sum are defined, i.e. the 6-tuples involved are admissible.

The initial data is said to satisfy *the condition* $(**)$ if *for any*

$$a, b, c, e, f, j_1, j_2, j_3, j_{23} \in I$$

such that the 6*-tuples*

$$(j_{23}, a, e, j_1, f, b) \text{ and } (j_3, j_2, j_{23}, b, f, c)$$

are admissible we hare

$$\sum_j w_j^2 \begin{vmatrix} j_2 & a & j \\ j_1 & c & b \end{vmatrix} \begin{vmatrix} j_3 & j & e \\ j_1 & f & c \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ a & e & j \end{vmatrix} = \begin{vmatrix} j_{23} & a & e \\ j_1 & f & b \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{vmatrix}.$$

Here, as above, we sum up over such j that all the symbols involved are defined.

Conditions $(*)$ and $(**)$ axiomatize the orthogonality and the Biedenharn–Elliot identities for q-$6j$-symbols.

The initial data is said to satisfy *the condition* $(***)$, if *for any* $j \in I$

$$w^2 = w_j^{-2} \sum_{k,l\,:\,(j,k,l)\in adm} w_k^2 w_l^2.$$

The initial data is said to be *irreducible*, if for any $j, k \in I$ there exists a sequence $l_1, l_2, \dots, l_n$ with $l_1 = j, l_n = k$ such that $(l_i, l_{i+1}, l_{i+2}) \in adm$ for any $i = 1, \dots, n-2$.

The following Lemma shows that in the case of irreducible initial data it suffices to verify the equality of the condition $(***)$ only for one value of j.

1.1.A LEMMA. *If the initial data is irreducible and satisfy the condition* $(*)$*, then* $w_j^{-2} \sum_{k,l\,:\,(j,k,l)\in adm} w_k^2 w_l^2$ *does not depend on* $j \in I$.

Proof. Irreducibility implies that it is sufficient to prove that

$$w_j^{-2} \sum_{k,l\,:\,(j,k,l)\in adm} w_k^2 w_l^2 = w_r^{-2} \sum_{k,l\,:\,(r,k,l)\in adm} w_k^2 w_l^2$$

for any $j, r \in I$ such that there exists $i \in I$ with $(i, j, r) \in adm$. Fix such (i, j, r). The condition $(*)$ implies that if $k, l \in I$ are such that the triple (j, k, l) is admissible then

$$w_j^{-2} = \sum_{\substack{m\,:\,(l,i,m)\in adm,\\ (m,r,k)\in adm}} w_m^2 \begin{vmatrix} l & i & m \\ r & k & j \end{vmatrix} \begin{vmatrix} r & i & j \\ l & k & m \end{vmatrix}. \tag{2}$$

Thus

$$\begin{aligned} w_j^{-2} \sum_{k,l\,:\,(j,k,l)\in adm} w_k^2 w_l^2 &= \sum_{\substack{k,l,m\,:\,(l,i,m)\in adm,\\ (m,r,k)\in adm,\\ (j,k,l)\in adm}} w_k^2 w_l^2 w_m^2 \begin{vmatrix} l & i & m \\ r & k & j \end{vmatrix} \begin{vmatrix} r & i & j \\ l & k & m \end{vmatrix} \\ &= \sum_{m,k\,:\,(m,r,k)\in adm} w_m^2 w_k^2 \Bigg(\sum_{\substack{l\,:\,(l,i,m)\in adm,\\ (j,k,l)\in adm}} w_l^2 \begin{vmatrix} l & i & m \\ r & k & j \end{vmatrix} \begin{vmatrix} r & i & j \\ l & k & m \end{vmatrix} \Bigg). \end{aligned}$$

Formula (2) with interchanged indices l, m and j, r permits to replace the expression in the brackets by w_r^{-2}. This gives the desired result. □

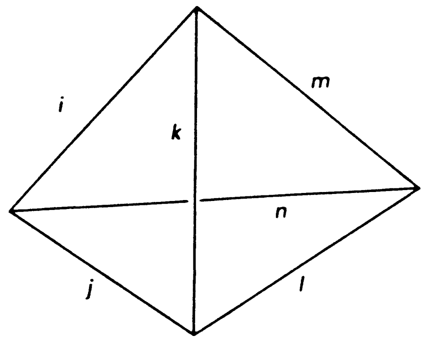

Fig. 1.

1.2 Colored tetrahedra. By a colored tetrahedron we will mean a 3-dimensional simplex with an element of the set I attached to each edge (see Fig. 1).

The element attached to an edge E is called the *color* of E. A colored tetrahedron is said to be *admissible* if for any its 2-face A the colors of the three edges of A form an admissible triple. Now we can explain geometrically the notion of admissible 6-tuple. A 6-tuple $(i, j, k, l, m, n) \in I^6$ is admissible iff the colored tetrahedron presented in Fig. 1 is admissible.

Each admissible colored tetrahedron T gives rise to a set of admissible 6-tuples. Namely, choose a 2-face A of T and write down the colors of the edges of A followed by the colors of the opposite edges of T. This gives an admissible 6-tuple, which depends, of course, on the choice of A and on the choice of order in the set of edges of A. Clearly the resulting 24 admissible 6-tuples may be obtained from each other by the obvious action of the symmetry group of T, which is the symmetric group S_4. Equalities (1) ensure that the symbols of these 6-tuples are equal to each other. Denote the common value of these symbols by $|T|$. Note that to define $|T| \in K$ we have not used an orientation of T.

1.3 State model for closed 3-manifolds. Let M be a closed triangulated 3-manifold. Let a be the number of vertices of M, let $E_1, \dots, E_b$ be the edges of M, and let $T_1, \dots, T_d$ be the 3-simplexes of M.

By a coloring of M we mean an arbitrary mapping

$$\varphi : \{E_1, E_2, \dots, E_b\} \to I.$$

A coloring is said to be *admissible* if for any 2-simplex A of M the colors of the three edges of A form an admissible triple. Denote the set of admissible colorings of M by $adm(M)$.

Each admissible coloring φ of M induces an admissible coloring of each 3-simplex T_i of M. Denote the resulting colored tetrahedron by T_i^φ.

For $\varphi \in adm(M)$ put

$$|M|_\varphi = w^{-2a} \prod_{r=1}^{b} w^2_{\varphi(E_r)} \prod_{t=1}^{d} |T_t^\varphi| \in K. \tag{3}$$

Put

$$|M| = \sum_{\varphi \in adm(M)} |M|_\varphi. \tag{4}$$

1.3.A THEOREM. *If initial data satisfy the conditions* $(*)$, $(**)$ *and* $(***)$, *then* $|M|$ *does not depend on the choice of triangulation of* M.

The proof of 1.3.A is given below in Section 5.

Theorem 1.3.A gives a scheme to define topological invariants of 3-manifolds. To realize it, one needs concrete initial data. Some initial data are given below in Section 7.

1.4 Relative case. Let M be a compact triangulated 3-manifold. Let a be the number of vertices of M. Suppose that e of them lie on the boundary ∂M. Let $E_1, \ldots, E_b$ be the edges of M, and let $T_1, \ldots, T_d$ be the 3-simplexes of M. Let exactly the first f of the edges lie on ∂M.

By a coloring and admissible coloring of M we shall mean just the same as in 1.3. By a coloring of ∂M we mean an arbitrary mapping

$$\alpha : \{E_1, E_2, \ldots, E_f\} \to I.$$

A coloring of ∂M is said to be *admissible* if for any 2-simplex A of ∂M the colors of the three edges of A form an admissible triple. Denote the set of admissible colorings of ∂M by $adm(\partial M)$.

For any admissible coloring $\varphi : \{E_1, E_2, \ldots, E_b\} \to I$ of M set

$$|M|_\varphi = w^{-2a+e} \prod_{r=1}^{f} w_{\varphi(E_r)} \prod_{s=f+1}^{b} w^2_{\varphi(E_s)} \prod_{t=1}^{d} \left|T_t^\varphi\right| \in K. \tag{5}$$

For $\alpha \in adm(\partial M)$ denote by $adm(\alpha, M)$ the set of all admissible colorings of M which extend α. Put

$$\Omega_M(\alpha) = \sum_{\varphi \in adm(\alpha, M)} |M|_\varphi.$$

If $adm(\alpha, M) = \varnothing$, i.e, α has no extension to M, then $\Omega_M(\alpha) = 0$ (as the sum of the empty set of summands).

1.4.A THEOREM. *If the initial data satisfy the conditions* $(*)$, $(**)$ *and* $(***)$, *then for any compact* 3-*manifold* M *with triangulated boundary and any admissible coloring* α *of* ∂M *all extensions of the triangulation of* ∂M *to* M *yield the same* $\Omega_M(\alpha)$.

This Theorem generalizes Theorem 1.3.A and is proven below in Section 5.

2. FUNCTORIAL NATURE OF THE INVARIANTS

2.1 Operator version of the invariant. For each triangulated closed surface F we define a K-module $C(F)$ to be the module freely generated over K by admissible colorings of F. One may equip $C(F)$ with the scalar product $C(F) \times C(F) \to K$ which makes the set of admissible colorings an orthonormal basis of $C(F)$.

If $F = \varnothing$, then we put $C(F) = K$ (in accordance with the generally accepted convention that there exists exactly one map $\varnothing \to I$).

Let $W = (M; i_+, i_-)$ be a cobordism between triangulated surfaces F_+ and F_-, i.e. M is a compact 3-manifold, $i_+ : F_+ \to \partial M$ and $i_- : F_- \to \partial M$ are embeddings with $\partial M = i_+(F_+) \cup i_-(F_-)$ and $i_+(F_+) \cap i_-(F_-) = \varnothing$. Define a homomorphism

$$\Phi_W : C(F_+) \to C(F_-)$$

by the formula

$$\Phi_W(\alpha) = \sum_{\beta \in adm(F_-)} \Omega_M(i_+(\alpha) \cup i_-(\beta))\beta$$

where α is an admissible coloring of F_+ and $i_+(\alpha) \cup i_-(\beta) \in adm(\partial M)$ is the coloring determined by α and β. In other words, Φ_W is the homomorphism which has, with respect to the natural bases of the spaces $C(F_+), C(F_-)$, the matrix with elements $\Omega_M(i_+(\alpha) \cup i_-(\beta))$.

The operator Φ_W can be considered as a generalization of the preceding invariants. Indeed, for a closed M, considered as a cobordism between empty surfaces, Φ_W acts in $K(= C(\varnothing))$ as multiplication by $|M|$. As for $\Omega_M(\alpha)$, with $\alpha \in adm(\partial M)$, they are the matrix elements for Φ_W where $W = (M; id\colon \partial M \to \partial M, \varnothing \to \partial M)$.

2.1.A COROLLARY OF 1.4.A. *For any cobordism $W = (M; i_+, i_-)$ between triangulated surfaces, the homomorphism $\Phi_W\colon C(F_+) \to C(F_-)$ does not depend on the extension of triangulations of F_+ and F_- to M involved in the definition of Φ_W.*

2.2 Multiplicativity of the invariants. It is well known that cobordisms can be considered as morphisms of a category. Objects of this category are closed manifolds. Each cobordism $W = (M; i_+, i_-)$ between surfaces F_+ and F_- is a morphism of this category from F_+ to F_-. The composition of cobordisms $W_1 = (M_1; i_1\colon F_1 \to \partial M, i_2\colon F_2 \to \partial M)$ and $W_2 = (M_2; j_2\colon F_2 \to \partial M, j_3\colon F_3 \to \partial M)$ is the cobordism $W_2 \circ W_1 = (M_1 \cup M_2; i_1, j_3)$ obtained from W_1 and W_2 by gluing along F_2.

The following theorem is a straightforward corollary of definitions.

2.2.A THEOREM. $\Phi_{W_2 \circ W_1} = \Phi_{W_2} \circ \Phi_{W_1}$.

2.3 Topological 3-dimensional quantum field theory. Theorem 2.2.A looks as the main condition for the correspondence $F \mapsto C(F)$, $W \mapsto \Phi_W$ to be a covariant functor from the category of cobordisms of triangulated surfaces to the category of K-modules. But it is not a functor since the other condition is not satisfied: for the unit cobordism [which is $(F \times [0,1]; F \times 0, F \times 1)$] the induced homomorphism sometimes is not identity.

However Theorem 2.2.A allows to improve this construction producing a functor. To do that, consider, for any triangulated closed surface F, the cobordism $id_F = (F \times [0,1]; i_0, i_1)$ where $i_t\colon F \to \partial(F \times [0,1])$ are defined by $i_t(x) = (x,t)$. Define a module $Q(F) = Coim(\Phi_{id_F}) = C(F)/Ker\,\Phi_{id_F}$. By 2.1.A it is well defined. Furthermore, any cobordism $W = (M; i_+\colon F_+ \to \partial M, i_-\colon F_- \to \partial M)$ is homeomorphic to the composition $W \circ id_{F_+}$. Therefore $\Phi_W = \Phi_W \circ \Phi_{id_{F_+}}$ and $Ker\,\Phi_W \supset Ker\,\Phi_{id_{F_+}}$. Consequently $\Phi_W\colon C(F_+) \to C(F_-)$ induces a K-linear homomorphism $Q(F_+) \to Q(F_-)$. We will denote it by Ψ.

The identity $\Phi_{W_2 \circ W_1} = \Phi_{W_2} \circ \Phi_{W_1}$ implies that $\Psi_{W_1 \circ W_2} = \Psi_{W_2} \circ \Psi_{W_1}$. Furthermore, $\Psi_{id_F} = id_{Q(F)}$, since Ψ_{id_F} is monomorphism (by the definition of $Q(F)$) and $\Psi_{id_F} \circ \Psi_{id_F} = \Psi_{id_F}$. Thus

$$F \to Q(F),\; W \mapsto \Psi_W$$

is a functor from the category of cobordisms of triangulated surfaces to the category of K-modules.

(*Remark.* This argument is fairly general. Let us call a mapping of a category $\wp$ to a category $\mathcal{D}$ a *semifunctor*, if it satisfies the first condition of the definition of a functor: namely, it sends a composition of morphisms $\wp$ to the composition of their images in $\mathcal{D}$. Suppose that $\mathcal{D}$ is abelian. Assign to each object of $\wp$ the coimage of the identity morphism of the image of this object in $\mathcal{D}$. This operation is extended naturally to an honest functor from $\wp$ to $\mathcal{D}$.)

Although $Q(F)$ is defined in terms of a triangulation of F, it does not depend on the triangulation in the following sense. For any two triangulations of F there exists a triangulation of $F \times [0, 1]$ coinciding on $F \times 0$ and $F \times 1$ with these given triangulations. It determines an isomorphism between the $Q(F)$'s which are defined via these triangulations of F. By 2.2.A, this isomorphism does not depend on the choice of the triangulation of $F \times [0, 1]$. We will identify the spaces $Q(F)$ defined via different triangulations of F by these isomorphisms.

Thus we have, for any initial data, the functor $F \mapsto Q(F)$, $W \mapsto \Psi_W$ from the category of cobordisms of (topological, i.e. non-triangulated) surfaces to category of K-modules. Following to a modern terminology (see [2]), it can be called a *topological* $(2+1)$-*dimensional quantum field theory*. Note however that originally this term is applied to a functor from the category of *oriented* cobordisms (of *oriented* surfaces).

2.4 Actions of modular groups. The functor of the preceding Subsection determines naturally representations of modular groups ($=$ mapping class groups of closed surfaces $=$ groups of isotopy classes of homeomorphisms of surfaces).

Let F be a closed surface, $h : F \to F$ homeomorphism. Fix some triangulation of F. Define a homomorphism $h_\# : C(F) \to C(F)$ by

$$h_\#(\alpha) = \sum_{\beta \in adm(F)} \Omega_{F\times[0,1]}(i_0(\beta) \cup i_1 h(\alpha))\beta$$

where $i_t : F \to \partial(F \times [0, 1])$ are defined by $i_t(x) = (x, t)$ and α is an admissible coloring of the triangulation of F. In other words, $h_\#$ is the homomorphism $\Phi_{(F\times[0,1];i_0,i_1\circ h)}$ induced by the cobordism $(F \times [0, 1]; i_0, i_1 \circ h)$.

As follows from 2.2.A, $(h \circ g)_\# = h_\# \circ g_\#$.

By the same reason as for Φ_W above, $h_\#$ induces a homomorphism $Q(F) \to Q(F)$. We denote this induced homomorphism by $h_\#$. The identity $(h \circ g)_\# = h_\# \circ g_\#$ implies $(h \circ g)_* = h_* \circ g_*$. Furthermore, $id_* = \Psi_{id_F} = id$. Therefore $(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = id_* = id$, and thus h_* is an isomorphism for any homeomorphism h.

If homeomorphisms h and g are isotopic, then $h_\# = g_\#$ and therefore $h_* = g_*$. Indeed,

$$\Omega_{F\times[0,1]}(i_0(\beta) \cup i_1 h(\alpha)) = \Omega_{F\times[0,1]}(i_0(\beta) \cup i_1 g(\alpha))$$

since using an isotopy between h and g it is easy to define a self-homeomorphism of $F \times [0, 1]$ which is identity on $F \times 0$ and maps $i_1 h(\alpha)$ to $i_1 g(\alpha)$.

Thus for any closed surface F we have a representation of the mapping class group of F in $Q(F)$.

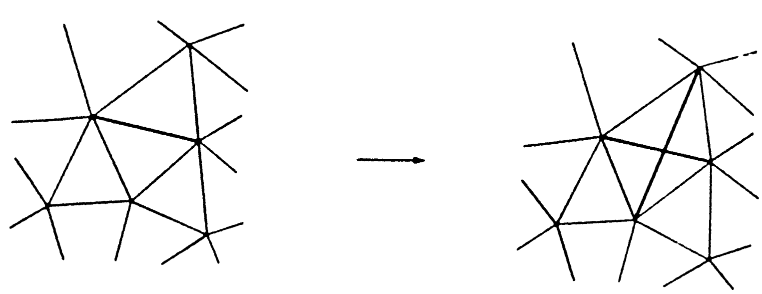

Fig. 2.

2.5 A refinement of the theory. Assume that there exists a function $c: I \to \mathbb{Z}_2$ such that for any admissible triple (i, j, k)

$$c(i) + c(j) + c(k) = 0.$$

Then each coloring of a 3-manifold M composed with c is a 1-cocycle of M. For any $h \in H^1(M; \mathbb{Z}_2)$ one can define a state sum invariant of h summing up our state sum terms (5) over all colorings which induce cocycles representing h. This refines the theory introduced above.

3. TRANSFORMATIONS OF A TRIANGULATION AND ITS DUAL

3.1 Alexander Theorem. To prove independence of the results of contructions of Sections 1 and 2 on triangulations (Theorems 1.3.A, 1.4.A and 2.1.A) we use the technique of Alexander [1] relating different triangulations of a manifold.

Let X be a polyhedron with a triangulation T, let E be its (open) simplex and $b \in E$. Remind that the (*closed*) *star* of a simplex E is the union of all closed simplexes containing E, it is denoted by $St_T E$. The transformation of T which replaces the star $St_T E$ by the cone over the boundary of $St_T E$ centered in b is called a *star subdivision* of T along E. (Simplexes of the initial triangulation which do not belong to the star of E also belong to the new triangulation of X.) Star subdivisions were introduced by Alexander [1]. We will call them *the Alexander moves*. Figure 2 shows a star subdivision along an edge in the 2-dimensional situation.

J.W. Alexander [1] used previous results of M.H.A. Newman [12] to prove the following theorem.

3.1.A THEOREM. *For any polyhedron P, which is dimensionally homogeneous* (*i.e. is a union of some collection of closed simplexes of the same dimension*)*, any two triangulations of P can be transformed one to another by a finite sequence of Alexander moves and transformations inverse to Alexander moves.*

3.2 Relative version. To prove 1.4.A we need the following relative version of the Alexander theorem.

3.2.A THEOREM. *Let P be a dimensionally homogeneous polyhedron and Q its subpolyhedron. Any two triangulations of P coinciding on Q can be transformed one to another by a sequence of Alexander moves and transformations inverse to Alexander moves, which do not change the triangulation of Q.*

The proof of 3.2.A is given in Appendix 1.

3.3 Dual picture of the Alexander move. The local picture of the Alexander move along a simplex E of a triangulated space P is determined by the combinatorics of the star of E in P. In particular, if P is a 3-manifold and $dim\, E = 1$ then this local picture is determined by the position of E with respect to the boundary: E may be contained in ∂P or not, and by the number of 3-simplexes containing E. Thus the number of the moves is actually infinite, which makes it rather difficult to verify directly the invariance of our state sums under the Alexander moves.

In the frameworks of triangulations we can not factorize the Alexander moves into more elementary ones, which would be finite in number. (How to do this for a kind of singular triangulations, is discussed in Appendix 2.)

To circumphere this problem, we pass to the dual picture for the moves. Recall that each combinatorial triangulation of a manifold M induces a relative cell subdivision of the pair $(M, \partial M)$. This subdivision is said to be dual to the original triangulation. It is constructed as follows. With each strictly increasing sequence $A_0 \subset A_1 \subset \cdots \subset A_m$ of simplexes of M one associates an m-dimensional linear simplex $[A_0, A_1, \ldots, A_m]$ in M whose vertices are the barycenters of $A_0, A_1, \ldots, A_m$. For a simplex A of M denote by A^* the union of all simplexes $[A_0, A_1, \ldots, A_m]$ with $A_0 = A$. It is well known (and easy to visualize if $\mathit{dim}\, M = 3$) that A^* is a combinatorial cell of dimension $\mathit{dim}\, M - \mathit{dim}\, A$. This cell is called the *barycentric star* of A. It intersects A transversally in the barycenter of A. The cells $\{A^*\}_A$, where A runs over all simplexes of M, form a relative cell subdivision of the pair $(M, \partial M)$.

A reader whose topological background does not contain these notions, can just look at Fig. 3, where the pieces of barycentric stars contained in one tetrahedron are drawn boldface. As the whole 3-manifold is a union of tetrahedra, its barycentric star subdivision is the union of subdivisions of Fig. 3.

Now let us visualize the transformation of the barycentric star subdivision corresponding to the Alexander move along an edge E not contained in the boundary of the manifold. Consider, first, a simpler 2-dimensional picture shown in Fig. 4. The barycentric star E^* of the edge E is replaced by a quadrangle. In the 3-dimensional case shown in Fig. 5.1 the barycentric star E^* of the edge E is a plaque. In result of the Alexander move the halves E_1 and E_2 of E come up. The corresponding plaques E_1^* and E_2^* have the same number of sides as E^* and are positioned on both sides of it. Except E_1 and E_2, the only new edges emerging under the Alexander move are the edges connecting the new vertex $E_1 \cap E_2$ with the vertices of the link of E. Their barycentric stars are quadrangles joining the corresponding sides of E_1^* and E_2^*. Together with E_1^* and E_2^* they constitute a prism. Thus the Alexander move replaces a plaque E^* by a prism. This is the only change.

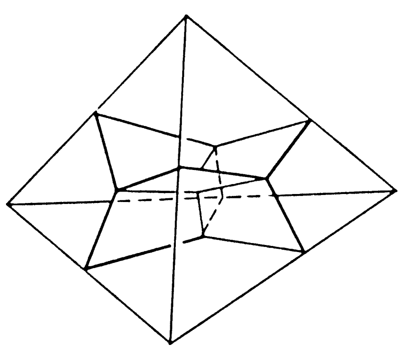

Fig. 3.

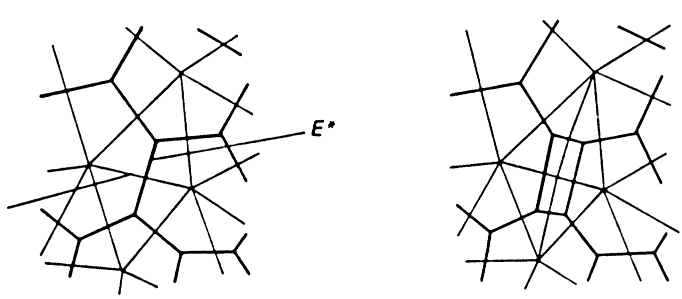

Fig. 4.

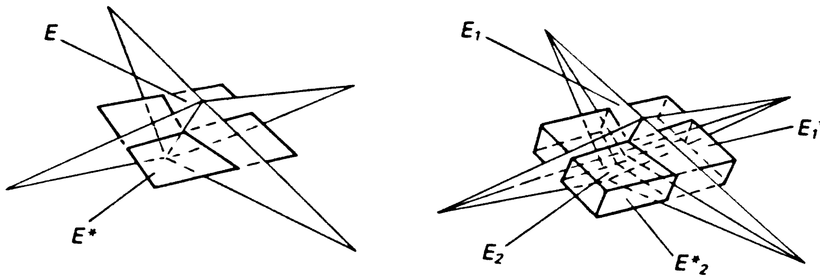

Fig. 5.1.

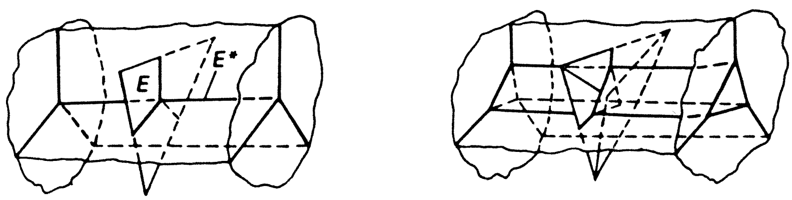

Fig. 5.2.

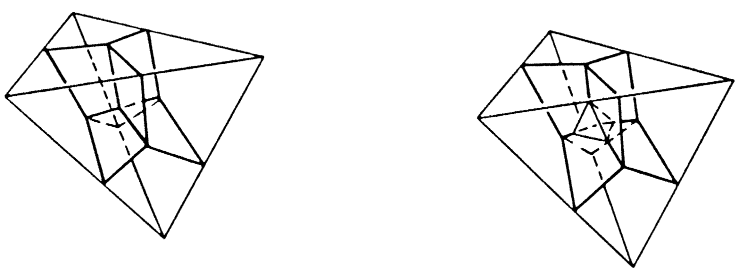

Fig. 5.3.

As we mentioned above the Alexander moves in dimension 3 along edges form an infinite series of moves. In our dual picture the members of this series differ from each other in the number of sides of E^*.

In the case of 3-manifolds when E does not lie in the boundary and $dim\, E = 2$ or $dim\, E = 3$ the dual pictures of the star subdivisions are shown in Figs 5.2 and 5.3.

3.4 Factorizing the Alexander move. Let E be an edge of a triangulation of a 3-manifold M and let E do not lie on ∂M. Consider the dual picture of the Alexander move along E. It is easy to imagine a process which creates gradually from the old barycentric star subdivision the new one. In Fig. 6.1 such a process is shown. It starts with creating a small bubble in the center of E^*. Then we puff up this bubble. In some moment it reaches the boundary of E^*. This happens in an internal point of some side of E^*. Then the base circle of the bubble crosses vertices, one by one. Stop when the bubble engulfs the whole E^*. At this moment our prism is ready: it consists of the old E^*, the bubble surface and the parts of the old 2-strata adjacent to E^* contained inside the bubble.

The similar processes corresponding to the cases $dim\, E = 2$, $dim\, E = 3$ are shown in Figs 6.2 and 6.3.

It is clear that the processes shown in Fig. 6 may be decomposed into sequences of elementary (local) events shown in Figs 7, 8, and 9. However these local modifications do not proceed inside the class of barycentric star subdivisions of triangulations. Thus to appeal to these processes we have to enlarge the class of objects on which our state sums are defined. We shall do that in the next Section.

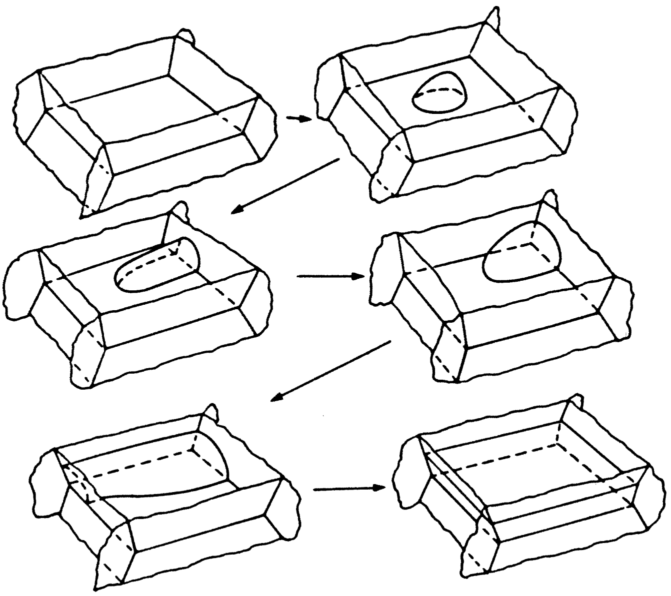

Fig. 6.1.

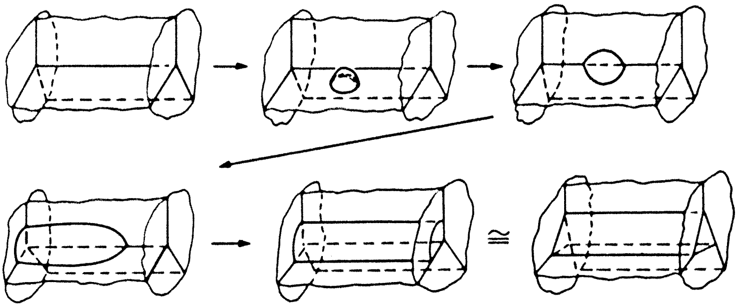

Fig. 6.2.

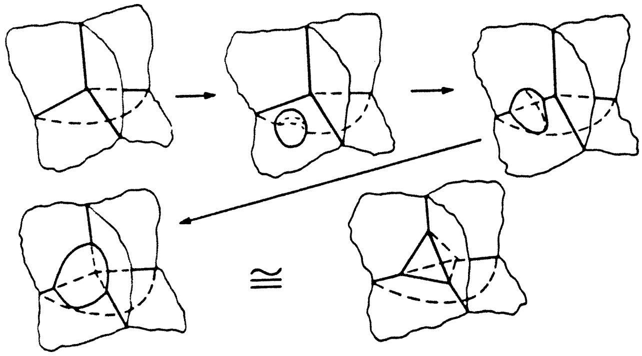

Fig. 6.3.

Fig. 7.

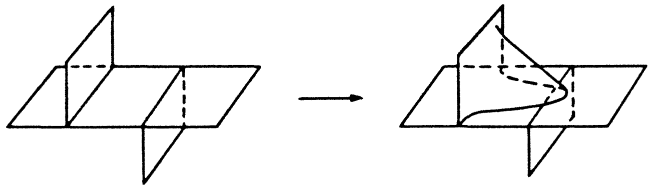

Fig. 8.

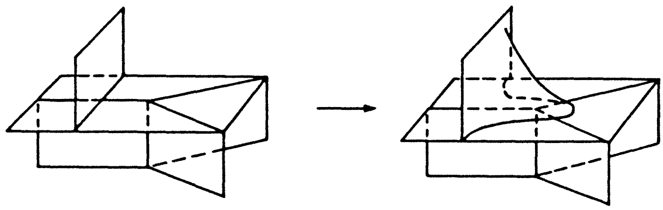

Fig. 9.

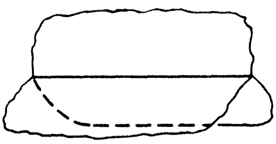

Fig. 10.

4. SIMPLE 2-POLYHEDRA AND A STATE SUM MODEL

4.1 Simple graphs and simple 2-polyhedra. By a *simple graph* we mean a finite graph (= finite 1-dimensional CW-complex) such that each point of it has a neighborhood homeomorphic either to $\mathbb{R}$ or to the union of 3 half-lines meeting in their common end point. Each simple graph is naturally stratified with strata of dimension 1 being the connected components of the set of points which have neighborhoods homeomorphic to $\mathbb{R}$. The 0-strata of a simple graph Γ are the (3-valent) vertices of Γ. The 1-strata homeomorphic to $\mathbb{R}$ are called *edges* and 1-strata homeomorphic to S^1 *loops* of Γ. (Thus some components of our simple graphs eventually contain no vertex.)

A 2-dimensional polyhedron X is called *simple* 2-*polyhedron* (*with boundary*), if each point of X has a neighborhood homeomorphic either to

(1) the plane $\mathbb{R}^2$, or
(2) the union of 3 halfplanes meeting each other in their common boundary line (see Fig. 10), or
(3) the cone over the 1-skeleton of a tetrahedron (see Fig. 11), or
(4) the halfplane $\mathbb{R}^2_+$, or
(5) the union of 3 copies of the quadrant $\{(x, y) \in \mathbb{R}^2\colon\ x \geqslant 0, y \geqslant 0\}$ meeting each other in the copies of the halfline $x = 0$ (see Fig. 12).

The set of points of a simple polyhedron X which have no neighborhoods of types (l), (2), (3) is called the *boundary* of X and denoted by ∂X. It is a simple graph.

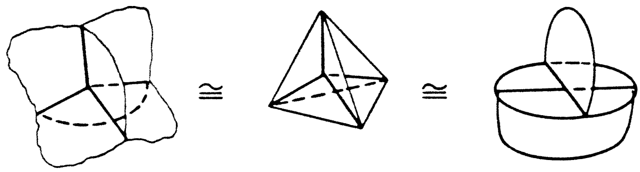

Fig. 11.

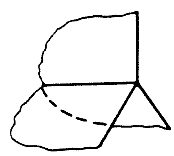

Fig. 12.

Each simple 2-dimensional polyhedron is naturally stratified. In this stratification each stratum of dimension 2 (a 2-face) is a connected component of the set of points having neighborhoods homeomorphic to $\mathbb{R}^2$. Strata of dimension 1 are of the following two types: internal 1-strata which are connected components of the set of points without neighborhood homeomorphic to $\mathbb{R}^2$, but with neighborhoods as in Fig. 10, and 1-strata of the boundary. Strata of dimension 0 are of two types also: internal 0-strata which are the points without neighborhoods of the types (1), (2), (4) and (5), but with neighborhoods as in Fig. 11, and the vertices of the boundary.

Simple 2-polyhedra appear naturally as 2-skeletons of those cell subdivisions of compact 3-manifolds which are dual to triangulations.

Remark. Simple 2-polyhedra are also called fake surfaces. This class of 2-polyhedra is interesting from many viewpoints. For example, they are generic in the following senses:

(1) They are obtained by gluing surfaces with boundary to other surfaces or simple 2-polyhedra by generic mappings of boundary components.
(2) They make a dense subset in the space of all metric 2-polyhedra (with respect to the Hausdorff metric).
(3) By local operations, preserving simple homotopy type, one can transform any compact 2-polyhedron into a simple one (which, in the metric case, can be made arbitrarily close to the original 2-polyhedron).

4.2 State model for simple 2-polyhedra. Let X be a simple 2-dimensional polyhedron (may be with non empty boundary). Let $x_1, \ldots, x_d$ be the vertices of $X - \partial X$, let $E_1, \ldots, E_f$ be the edges of ∂X and let $\Gamma_1, \ldots, \Gamma_b$ be the 2-strata of X. By a coloring of X we mean an arbitrary mapping

$$\varphi : \{\Gamma_1, \Gamma_2, \ldots, \Gamma_b\} \to I.$$

The coloring is said to be *admissible*, if for any edge E of $X - \partial X$ the colors of the three 2-strata incident to E form an admissible triple. Denote the set of admissible colorings of X by $adm(X)$.

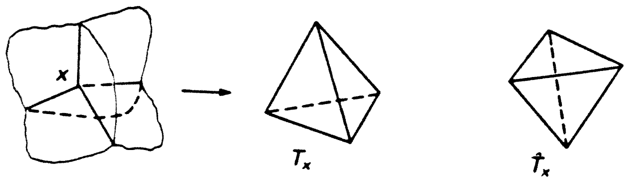

Fig. 13.

By a coloring of a simple graph Γ we shall mean any mapping of the set of its 1-dimensional strata to I. A coloring of a simple graph is said to be *admissible*, if for each vertex the colors of the edges adjacent to it make an admissible triple. The set of admissible colorings of a simple graph Γ will be denoted by $adm(\Gamma)$. Any coloring of a simple 2-polyhedron X induces in a natural way a coloring of its boundary ∂X: a 1-stratum of ∂X takes the color of the 2-stratum of X in whose boundary this 1-stratum is contained. Evidently, if the coloring of X is admissible, then the induced coloring of ∂X is admissible too. The map $adm(X) \to adm(\partial X)$ defined by this construction will be denoted by ∂.

With each vertex x of $X - \partial X$ we associate a tetrahedron T_x whose vertices and edges correspond respectively to germs of 1-strata and 2-strata of X incident to x (see Fig. 13). The 1-skeleton of T_x is nothing but the polyhedral link of x in X. Let $\hat{T}_x$, be the dual tetrahedron, i.e. the tetrahedron whose vertices, edges and faces correspond respectively to faces, edges and vertices of T_x. Thus edges and faces of $\hat{T}_x$, correspond to germs of 2-strata and 1-strata of X incident to x. Each admissible coloring φ of X induces an admissible coloring of $\hat{T}_x$: the color of the edge of $\hat{T}_x$ corresponding to a 2-stratum Γ of X is defined to be $\varphi(\Gamma) \in I$. Denote the resulting admissibly colored tetrahedron by $\hat{T}_x^{\varphi}$. For $\varphi \in adm(X)$ put

$$|X|_{\varphi} = w^{-2\chi(X)+\chi(\partial X)} \prod_{r=1}^{b} w_{\varphi(\Gamma_r)}^{2\chi(\Gamma_r)} \prod_{s=1}^{f} w_{\partial\varphi(E_s)}^{\chi(E_s)} \prod_{t=1}^{d} \left|\hat{T}_{x_t}^{\varphi}\right| \in K \tag{6}$$

where χ is the Euler characteristic, and $w_{\varphi(\Gamma_r)}^{2\chi(\Gamma_r)}$, $w_{\partial\varphi(E_s)}^{\chi(E_s)}$ mean $w_{\varphi(\Gamma_r)} \in K$ to degree $2\chi(\Gamma_r)$ and $w_{\partial\varphi(E_s)}$ to degree $\chi(E_s)$ respectively. (Strata are thought to be open, so if E_s is homeomorphic to $\mathbb{R}^1$ then $\chi(E_s) = -1$ and if E_s is homeomorphic to S^1 then $\chi(E_s) = 0$.) Put

$$|X| = \sum_{\varphi \in adm(X)} |X|_{\varphi}. \tag{7}$$

For any admissible coloring α of ∂X put

$$\Omega_X(\alpha) = \sum_{\varphi:\, \partial(\varphi)=\alpha} |X|_{\varphi}. \tag{8}$$

If $\{\varphi\colon\ \partial(\varphi) = \alpha\} = \varnothing$, then $\Omega_X(\alpha) = 0$.

4.2.A LEMMA. *Let a simple 2-polyhedron X be the union of simple 2-polyhedra Y and Z and let each component of $T = Y \cap Z$ be a component of both ∂Y and ∂Z. Then for any admissible coloring β of ∂X*

$$\Omega_X(\beta) = \sum_{\alpha \in adm(T)} \Omega_Y(\alpha \cup (\beta\,|_{Y\cap\partial X}))\Omega_Z(\alpha \cup (\beta\,|_{Z\cap\partial X})) \tag{9}$$

where $\alpha \cup (\beta\,|_{Y\cap\partial X})$ and $\alpha \cup (\beta\,|_{Z\cap\partial X})$ are the colorings of ∂Y and ∂Z induced by α, β (note that $\partial Y = T \cup (Y \cap \partial X)$ and $\partial Z = T \cup (Z \cap \partial X)$).

Proof. Lemma 4.2.A is a direct consequence of the equality

$$|X|_{\varphi} = |Y|_{\varphi|Y}|Z|_{\varphi|Z} \tag{10}$$

which holds for any $\varphi \in adm(X)$. Formula (10) follows straightforwardly from the definition of $|X|_{\varphi}$ and additivity of Euler characteristic. Indeed, a face Γ of X is the union of some faces

$\Gamma'_1, \dots, \Gamma'_m$ of Y, some faces $\Gamma''_1, \dots, \Gamma''_v$ of Z, and some 1-strata $E_{i_1}, \dots, E_{i_w}$ of T. Therefore $\chi(\Gamma)$ is the sum of Euler characteristics of these pieces, and

$$2\chi(\Gamma) = 2\chi(\Gamma'_1) + \cdots + 2\chi(\Gamma'_u) + \chi(E_{i_1}) + \cdots + \chi(E_{i_w}) + 2\chi(\Gamma''_1) + \cdots + 2\chi(\Gamma''_v)$$
$$+\chi(E_{i_1}) + \cdots + \chi(E_{i_w}).$$

For similar reasons,

$$-2\chi(X) + \chi(\partial X) = (-2\chi(Y) + \chi(\partial Y)) + (-2\chi(Z) + \chi(\partial Z)). \qquad \square$$

4.3 Local moves on simple 2-polyhedra. In the second half of eighties Matveev [10] and Piergallini [21] introduced several transformations of simple 2-dimensional polyhedra. Each of these transformations replaces a standard fragment of a simple polyhedron by some other standard fragment. In Figs 8 and 9 above we show two Matveev–Piergallini transformations. The transformation shown in Fig. 8 will be called the *lune move* and denoted by $\mathcal{L}$. The transformation shown in Fig. 9 will be called *the Matveev move* and denoted by $\mathcal{M}$. Note that these transformations do not change homotopy (and even simple homotopy) type of 2-polyhedron.

We also need the transformation shown in Fig. 7. This move adds two new disk 2-strata and one circle 1-stratum and punctures one old 2-stratum. We shall call this move a *bubble move* and denote it by $\mathcal{B}$.

Note that the moves $\mathcal{M}, \mathcal{L}, \mathcal{B}$ preserve the boundary.

4.4 Invariance of the state sum with respect to local moves

4.4.A LEMMA. *Let X be a simple 2-polyhedron and α be an admissible coloring of ∂X. If the initial data satisfies the condition* $(*)$ *then $|X|$ and $\Omega_X(\alpha)$ are invariant under $\mathcal{L}$.*

Proof. Let X' be a polyhedron obtained from X by $\mathcal{L}$. Then $X = Y \cup Z$ and $X' = Y' \cup Z$ where $Y \cap Z = \partial Y = \partial Z$, $Y' \cap Z = \partial Y' = \partial Z$ and Y, Y' are simple 2-polyhedra with boundary shown in Fig. 14.

By Lemma 4.2.A it is sufficient to prove that

$$\Omega_Y(\beta) = \Omega_{Y'}(\beta) \tag{11}$$

for any $\beta \in adm(\partial Y) = adm(\partial Y')$.

Let $\Gamma, \Gamma', \Gamma''$ be the faces of Y' and E', E'' the edges of $\partial Y'$ pointed out in Fig. 14. Denote by x and y the vertices of Y' which appear in Fig. 14. Fix some $\beta \in adm(\partial Y)$ and put $j_4 = \beta(E')$, $j_6 = \beta(E")$.

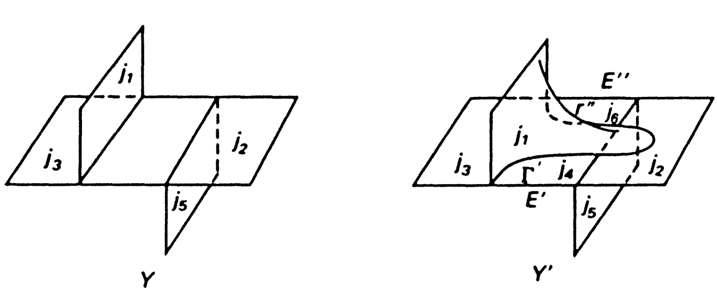

Fig. 14.

Consider the case $j_4 \neq j_6$. Let $\varphi \in adm(Y')$ with $\partial\varphi = \beta$. Denote by j_1, j_2, j_3, j_5 the φ-colors of the faces of Y' which appear in Fig. 14. The element $|Y'|_\varphi$ is the product of a certain factor which does not depend on the choice of φ and the factor

$$w^2_{\varphi(\Gamma)}\left|\hat{T}^\varphi_x\right|\left|\hat{T}^\varphi_y\right| = w^2_{\varphi(\Gamma)}\begin{vmatrix} j_2 & j_1 & \varphi(\Gamma) \\ j_3 & j_5 & j_4 \end{vmatrix}\ \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & \varphi(\Gamma) \end{vmatrix}.$$

Summing up these expressions over all $\varphi \in \partial^{-1}(\beta)$ we get zero because of condition $(*)$ and the assumption $j_4 \neq j_6$. Thus

$$\Omega_{Y'}(\beta) = \sum_{\varphi\,:\,\partial(\varphi)=\beta} |Y'|_\varphi = 0.$$

On the other hand, $\Omega_Y(\beta) = 0$, since $j_4 \neq j_6$ implies that no coloring of Y induces β.

Assume now that $j_4 = j_6$. In this case there is a unique coloring ψ of Y which induces β. See Fig. 14. By the definition

$$\Omega_Y(\beta) = |Y|_\psi = w^{-4} w_{j_1} w_{j_2} w_{j_3} w_{j_5}.$$

On the other hand,

$$\begin{aligned}\Omega_{Y'}(\beta) &= \sum_{\varphi\,:\,\partial(\varphi)=\beta} |Y'|_\varphi \\ &= w^{-4} w_{j_1} w_{j_2} w_{j_3} w_{j_5} w^2_{j_4} \sum_{\varphi\,:\,\partial(\varphi)=\beta} w^2_{\varphi(\Gamma)}\left|\hat{T}^\varphi_x\right|\left|\hat{T}^\varphi_y\right| \\ &= w^{-4} w_{j_1} w_{j_2} w_{j_3} w_{j_5} \sum_j w^2_{j_4} w^2_j \begin{vmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{vmatrix}\ \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{vmatrix}.\end{aligned}$$

By the condition $(*)$ the sum in the latter expression equals $\delta_{j_4 j_6} = 1$. Thus $\Omega_{Y'}(\beta) = \Omega_Y(\beta)$. This finishes the proof. □

4.4.B LEMMA. *Let X be a simple 2-polyhedron and α be an admissible coloring of ∂X. If the initial data satisfies the condition* $(**)$ *then $|X|$ and $\Omega_X(\alpha)$ are invariant under the move $\mathcal{M}$.*

The proof of 4.4.B is quite similar to the proof of 4.4.A. Here five tetrahedra are involved into play: two tetrahedra correspond to the two vertices of X and three tetrahedra correspond to the three vertices of X'. In Fig. 15 we present a convenient notation for colors of faces which converts $(**)$ into an equality similar to (11). □

4.4.C LEMMA. *Let X be a simple 2-polyhedron and α be an admissible coloring of ∂X. If the initial data are irreducible and satisfies the condition* $(**)$ *then $|X|$ and $\Omega_X(\alpha)$ are invariant under the move $\mathcal{B}$.*

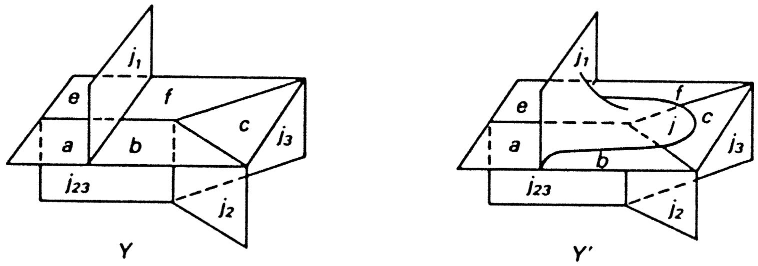

Fig. 15.

Proof. Let X' be a polyhedron obtained from X by the bubble more. Then $X = Y \cup Z$ and $X' = Y' \cup Z$ where $Y \cap Z = \partial Y = \partial Z$, $Y' \cap Z = \partial Y' = \partial Z$ and Y, Y' are simple 2-polyhedra with boundary shown in Fig. 16. By 4.2.A it is sufficient to prove that $\Omega_Y(\beta) = \Omega_{Y'}(\beta)$ for any $\beta \in adm(\partial Y) = adm(\partial Y')$. If the color of the boundary circle ∂Y with respect to β is j, then

$$\Omega_Y(\beta) = w^{-2} w_j^2 \text{ and } \Omega_{Y'}(\beta) = \sum_{k,I\,:\,(j,k,l)\in adm} w^{-4} w_k^2 w_l^2.$$

The result follows from the condition $(***)$. $\square$

4.5 Digression: a two-dimensional polyhedral quantum field theory. A cobordism between simple graphs Γ and Δ is a simple 2-polyhedron X with embeddings $i : \Gamma \to \partial X$, $j : \Delta \to \partial X$ such $i(\Gamma) \cup j(\Delta) = \partial X$, $i(\Gamma) \cap j(\Delta) = \varnothing$, and $i(\Gamma)$, $j(\Delta)$ are unions of components of ∂X. It is easy to see that any two simple graphs are cobordant in this sense; so the corresponding cobordism group is trivial.

There is an obvious composition operation for cobordisms of simple graphs: if (X, i, j) is a cobordism between Γ and Δ and (Y, k, l) a cobordism between Δ and Σ, then $(X \cup_{kj^{-1}\,:\,j(\Delta)\to k(\Delta)} Y; i, l)$ is a cobordism between Γ and Σ. Simple graphs are objects and their cobordisms (considered up to homeomorphisms identical on the boundary) are morphisms of a category called the *category of simple* 2*-polyhedra* and denoted by $\underline{S}$.

For each simple graph Γ we define the K-module $C(\Gamma)$ to be the module freely generated over K by the admissible colorings of Γ. One may equip $C(\Gamma)$ with the scalar product $C(\Gamma) \times C(\Gamma) \to K$ which makes the set of admissible colorings an orthonormal basis of $C(\Gamma)$. If $\Gamma = \varnothing$, then $C(\Gamma) = K$.

For any simple 2-polyhedron X the mapping $\alpha \mapsto \Omega_X(\alpha)$ uniquely extends to a K-linear homomorphism $C(\partial X) \to K$, which will be denoted also by Ω_X.

To each cobordism $U = (X; i, j)$ between simple graphs Γ and Δ we associate a homomorphism $\Phi_U : C(\Gamma) \to C(\Delta)$ defined on the generators by the formula

$$\Phi_U(\alpha) = \sum_{\beta\in adm(\Delta)} \Omega_X(i(\alpha) \cup j(\beta))\beta. \tag{12}$$

The identity morphisms in the category $\underline{S}$ of simple 2-polyhedra are trivial cobordisms $id_\Gamma = (\Gamma \times [0,1]; i_0, i_1)$ where $i_t : \Gamma \to \Gamma \times [0,1]$: $x \mapsto (x,t)$. As follows directly from definition, $\Phi_{id_\Gamma = id}$. This observation together with the following Theorem 4.5.A mean that we have a functor $\Gamma \mapsto C(\Gamma)$, $U \mapsto \Phi_U$ from the category $\underline{S}$ to the category K-$\underline{Mod}$ of K-modules. In analogy to topological quantum field theories it can be called *a polyhedral* 2*-dimensional quantum field theory*.

4.5.A THEOREM. *If U is a cobordism between simple graphs Γ and Δ and V a cobordism between simple graphs Δ and Σ, then*

$$\Phi_V \circ \Phi_U = \Phi_{V\circ U} : C(\Gamma) \to C(\Sigma)$$

where $V \circ U$ is the composition of cobordisms U and V.

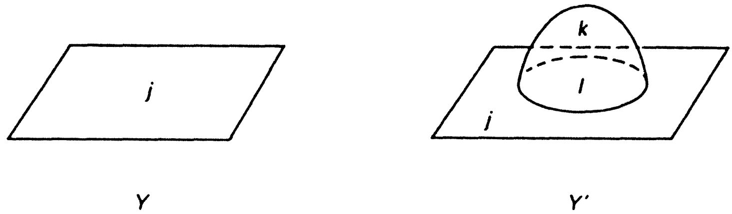

Fig. 16.

Theorem 4.5.A follows straightforwardly from Lemma 4.2.A.

None of the moves $\mathcal{M}, \mathcal{L}, \mathcal{B}$ (see Section 4.3 above) changes the boundary of a simple 2-polyhedron. Therefore application of these moves to a cobordism between two simple graphs gives cobordisms between the same graphs. Denote by $\underline{Q}$ the quotient category of $\underline{S}$ constructed by identifying morphisms of $\underline{S}$ which can be obtained one from another by some sequence of moves $\mathcal{L}^{\pm 1}, \mathcal{M}^{\pm 1}$, and $\mathcal{B}^{\pm 1}$. The objects of $\underline{Q}$ are the objects of $\underline{S}$ (simple graphs) though a morphism of $\underline{Q}$ is a class of cobordisms convertible to each other by moves $\mathcal{L}^{\pm 1}$, $\mathcal{M}^{\pm 1}$, and $\mathcal{B}^{\pm 1}$.

From lemmas of the preceding Subsection it follows

4.5.B. *The functor $\Gamma \mapsto C(\Gamma), U \mapsto \Phi_U$ from the category $\underline{S}$ of simple 2-polyhedra to the category K-Mod of K-modules induces a functor $\underline{Q} \to K$-$\underline{Mod}$.*

5. PROOF OF INVARIANCE THEOREMS

5.1 Dual colorings. Let M be a compact triangulated 3-manifold, X be the union of the (closed) barycentric stars of all edges of M. It is obvious that X is a simple 2-polyhedron with boundary $\partial X = X \cap \partial M$. Each coloring φ of M induces a dual coloring φ^* of X by the formula

$$\varphi^*(E^*) = \varphi(E)$$

where E is an edge of M and E^* is the dual 2-cell of X. This establishes a bijective correspondence between colorings of M and those of X.

It is straightforward to observe that φ is admissible if and only if φ^* is admissible. Therefore, the formula $\varphi \mapsto \varphi^*$ induces a bijection $adm(M) \to adm(X)$.

5.1.A LEMMA. *For any $\varphi \in adm(M)$*

$$|M|_\varphi = |X|_{\varphi^*}.$$

Proof. The Lemma follows directly from the definitions. One should take into account that all 2-strata of X are open 2-cells and all 1-strata of X are open edges (not loops). Thus $\chi(\Gamma) = 1$ for any 2-strata Γ of X, and $\chi(E) = -1$ for any 1-strata E of X. Furthermore, if a (respectively, e) is the number of vertices of M (respectively, of ∂M) then

$$\begin{aligned} \chi(\partial X) &= \chi(\partial M) - e \\ &= 2\chi(M) - e, \\ \chi(X) &= \chi(M) + a - e. \end{aligned}$$

Thus

$$\chi(\partial X) - 2\chi(X) = -2a + e. \qquad \square$$

5.2 Proof of Theorems 1.3.A and 1.4.A. We have just to combine the results obtained above. By the Alexander Theorem 3.1.A and its relative version 3.2.B, it is sufficient to prove that state sums of 1.3.A and 1.4.A are not changed by the Alexander moves along simplexes not lying on the boundary of the manifold. By 5.1.A these sums coincide with the ones defined in Section 4 for the 2-skeletons of the barycentric star subdivisions. By 3.4 it is sufficient to prove that these sums are not changed by moves $\mathcal{B}, \mathcal{L}$ and $\mathcal{M}$ applied to these 2-skeletons. And this has been proved in 4.4.

6. DUAL APPROACH TO THE INVARIANTS OF 3-MANIFOLDS

6.1 Spines and simple stratifications of 3-manifolds. A polyhedron X is called a *spine* of a compact manifold M with non-empty boundary if there exists an embedding $i: X \to M$ such that M collapses to $i(X)$.[†]

In the case of closed M a polyhedron X is called a spine of M if it is a spine of M with an open ball removed.

A spine of a compact 3-manifold which is a simple polyhedron with empty boundary is called a simple spine of this 3-manifold.

A 2-dimensional polyhedron is said to be *cellular* if each stratum of its natural stratification is homeomorphic to Euclidean space of dimension 2, 1 or 0.

Remark. Casler [4] who first considered simple cellular 2-polyhedra called them *standard* polyhedra. Matveev [10,11] used the term *special polyhedron*. Note that his definition slightly differs from that of Casler: he omitted the condition that the 1-strata are cells. However he meant the same notion, as follows from the fact that the main theorem of [11] is not valid for the lens space $L(3,1)$ if one admits simple spines with disk 2-strata and closed 1-strata.

6.1.A THEOREM (Casler [4]). *Any compact 3-manifold has a simple cellular spine.*

Note that a regular neighborhood U of any polyhedron X embedded in a 3-manifold M with $\partial M \cap X = \varnothing$ is homeomorphic to the cylinder of some map $\pi: \partial U \to X$. It is easy to see that in the case when X is a simple polyhedron without boundary, the map π can be taken to be a (*topological*) *immersion* in the sense that each point of ∂U has a neighborhood in ∂U mapping homeomorphically onto its image in X. As a summary we formulate the following assertion.

6.1.B. *Any compact 3-manifold M with non empty boundary is homeomorphic to the cylinder of a topological immersion of ∂M onto an arbitrary simple spine of M.*

(Here the condition $\partial M \cap X = \varnothing$ does not appear since any spine can be pushed off a collar of ∂M.)

Note that 6.1.B gives a way of description of 3-manifolds, which is related to the simple spine presentations. It is outlined in Appendix 2.

A simple spine of a 3-manifold M and a simple spine of M with several open balls removed will be called a *simple 2-skeleton* of M. For example, for any compact 3-manifold M the union of the barycentric stars of all r-simplexes of $M - \partial M$ with $r > 0$ is a simple 2-skeleton of M.

Another important special class of simple 2-skeletons is related with Heegaard diagrams. Namely, for any Heegaard surface F in a closed 3-manifold M and any complete systems $\{m_1, \dots, m_g\}$, $\{m'_1, \dots, m'_g\}$ of meridian disks of the handlebodies bounded by F in M such that the boundaries of these disks are transversal to each other (i.e. constitute a Heegaard diagram of M), the union

$$F \cup m_1 \cup \dots \cup m_g \cup m'_1 \cup \dots \cup m'_g$$

is a simple 2-skeleton of M.

[†] Remind the notion of collapse of a polyhedron to a subpolyhedron. Suppose K is a polyhedron and σ is a (closed) simplex of K with face τ. If τ is the proper face of no simplex in K except σ (and in particular σ is a face of no simplex in K and $\dim \tau = \dim \sigma - 1$), then one says that there is an elementary collapse from K to $K - (\mathit{Int}\,\sigma \cup \mathit{Int}\,\tau)$ [where $\mathit{Int}\,\alpha$ means $\alpha -$ (faces of α)]. If L is a subpolyhedron of a polyhedron K and there are polyhedra $K = K_0 \supset K_1 \supset \dots \supset K_n = L$ such that there is an elementary collapse from K_{i-1} to K_i, $i = 1, 2, \dots, n$, then one says that K *collapses* to L.

6.2 Matveev–Piergallini theorem and its corollaries. Local moves $\mathcal{M}, L$ on simple 2-polyhedra were introduced by Matveev and Piergallini with a view towards investigation of simple cellular spines of 3-manifolds.[‡]

6.2.A. *Any two simple 2-skeletons of a compact 3-manifold can be transformed one to another by a sequence of the moves $\mathcal{M}^{\pm 1}$, $\mathcal{L}^{\pm 1}$ and $\mathcal{B}^{\pm 1}$.*

To prove 6.2.A we use the following theorem of Matveev [11] and Piergallini [21].

6.2.B THEOREM. *Any two simple cellular spines of a 3-manifold can be transformed one to another by a sequence of moves $\mathcal{M}^{\pm 1}$ and $\mathcal{L}^{\pm 1}$.*

Reduction of 6.2.A to 6.2.B. Take any two simple 2-skeletons of a 3-manifold. By several bubble moves make them to be spines of the same manifold (the initial manifold with some collection of open balls removed). Furthermore make, if necessary, bubble moves to produce 1-strata. Then applying $\mathcal{L}$ several times, transform the *simple* spines obtained into *simple cellular* spines. Now we are in the situation of 6.2.B.

6.3 Digression: gluing simple polyhedra. Let X be a simple polyhedron without boundary, Γ be a simple graph. A topological immersion $\varphi : \Gamma \to X$ is said to be *generic*, if the following conditions are fulfilled:

(1) all vertices of Γ are mapped to 2-strata of X,
(2) no vertex of X is contained in $\varphi(\Gamma)$,
(3) the restrictions of φ to 1-strata of Γ are transversal to 1-strata of X. i.e. the inverse image of any 1-stratum of X is finite and at each point of it φ goes from one germ of 2-stratum of X to another,
(4) φ has no triple points,
(5) each double point s of φ is a transversal intersection of 1-strata of Γ and lies in a 2-stratum of X.

All these conditions obviously are conditions of general position. In particular any map $\Gamma \to X$ can be approximated by generic topological immersion.

6.3.A. *Let X be a simple polyhedron without boundary, K a simple polyhedron, L a component of ∂K and $\varphi : L \to X$ a generic topological immersion. Then the space $X \cup_\varphi K$ is a simple polyhedron with the boundary $\partial K - L$.*

It is clear that any simple polyhedron without boundary can be obtained from a closed surface by successive gluing of surfaces with boundary along generic immersions of their boundary curves.

‡ The bubble move applied to a simple cellular polyhedron gives a simple polyhedron, which is however not cellular. That is why Matveev and Piergallini did not consider this move.

6.4 Spines in relative situation. Let M be a compact 3-manifold with some simple graph Γ embedded in ∂M. A simple polyhedron X with boundary is called a *simple spine of the pair* (M, Γ), if there exists an embedding $i : X \to M$ such that M collapses to $i(X)$ and $i(\partial X) = \Gamma = i(X) \cap \partial M$. A simple spine of (M, Γ) or $(M -$ (several open balls), $\Gamma)$ is called a *simple* 2*-skeleton of* (M, Γ).

6.4.A THEOREM. *For any compact 3-manifold M and any simple graph* $\Gamma \subset \partial M$ *there exists a simple spine of* (M, Γ).

Proof. Let X be a simple spine of M. By 6.1.B there exists an immersion $\pi : \partial M \to X$ such that M is homeomorphic to the cylinder of π. As it follows from 6.3.A and the fact that π is a topological immersion, after some small isotopy of Γ in ∂M, the space $X \cup_{\pi|\Gamma : \Gamma \times 1 \to X} \Gamma \times [0, 1]$ is a simple polyhedron. It is obviously a simple spine of (M, Γ). □

The simple spines constructed in the proof of 6.4.A have an additional property. If one removes from a simple spine of this kind all the strata whose closure intersects with ∂M, then the result will be a simple spine of M. Let us call a spine of (M, Γ) of this type a *collar* spine of (M, Γ). For each such spine the union of strata whose closure intersects ∂M is a cylinder over Γ. It is clear that each collar spine can be obtained by the construction of the proof of 6.4.A. Collar spines of (M, Γ) and $(M -$ (several open balls), $\Gamma)$ are called *collar* 2*-skeletons* of (M, Γ).

6.4.B THEOREM. *Any two collar spines of a pair* (M, Γ)*, where* M *is a compact 3-manifold and* $\Gamma \subset \partial M$ *is a simple graph, can be transformed one to another by a sequence of moves* $\mathcal{M}^{\pm 1}$ *and* $\mathcal{L}^{\pm 1}$ *with the intermediate results also being collar spines.*

Proof. Let S_1 and S_2 be collar spines of (M, Γ), and X_1, X_2 be the corresponding simple spines of M. So

$$S_1 = x_i \cup_{\pi_t|\Gamma \times 1} \Gamma \times [0, 1]$$

where $\pi_i : \partial M \to X_i$ are the corresponding topological immersions (with $\partial M \times [0.1] \cup_{\pi_i} X_i$ homeomorphic to M). By 6.2.B there exists a sequence of moves $\mathcal{M}^{\pm 1}, \mathcal{L}^{\pm 1}$ transforming X_1 to X_2. This sequence can be easily realized inside M in the following obvious sense: there exists a family of spines X_t with $t \in [1, 2]$ of M embedded in $\mathit{Int}\, M$ such that for all but finite set $t_1, t_2, \dots, t_q$ of values of t the polyhedron X_t is simple, the family X_t with $t \in (t_i, t_{i+1})$ is an isotopy and the passing by t through each of t_i gives a Matveev–Piergallini move of X_t.

Topological immersions π_1, π_2 can be obviously included into a continuous family $\pi_t : \partial M \to X_t$ with $t \in [1, 2]$ of topological immersions such that for any $t \in [1, 2]$ the space

$$\partial M \times [0, 1] \cup_{\pi_t : \partial M \times 1 \to X_t} X_t$$

is homeomorphic to M. By a small isotopy of $\Gamma \subset \partial M$, which does not change the topological types of spaces

$$X_i \cup_{\pi_t|\Gamma \times 1} \Gamma \times [0, 1]$$

with $i = 1, 2$, the family of 2-polyhedra

$$S_t = X_t \cup_{\pi_t|\Gamma \times 1} \Gamma \times [0, 1]$$

can be made such that for all but finite set $t_1', t_2', \dots, t_r'$ of values of t the polyhedron S_t is simple, the family S_t with $t \in (t_i, t_{i+1})$ is an isotopy and passing by t through each of t_i, gives a Matveev–Piergallini move of S_t. □

6.4.C COROLLARY. *Any two collar 2-skeletons of a pair* (M, Γ)*, where* M *is a compact 3-manifold and* $\Gamma \subset \partial M$ *is a simple graph, can be transformed one into another by a sequence of moves* $\mathcal{M}, \mathcal{L}$ *and* $\mathcal{B}$ *(and their inverses), with the intermediate results also being collar 2-skeletons.*

6.5 Semifunctor "Skeleton". A closed (topological) surface with an embedded simple graph will be called a *marked surface*. A marked surface is said to be *completely marked* if each component of the complement of the graph is homeomorphic to $\mathcal{R}^2$.

Define the category $\underline{MC}$ whose objects are completely marked surfaces and morphisms are (compact 3-dimensional) cobordisms between the underlying (non-marked) surfaces. Denote by $\underline{C}$ the category of nonmarked surfaces and cobordisms between them.

Assign to a marked surface its simple graph and to a cobordism between two marked surfaces a collar simple 2-skeleton of this cobordism. It determines a semifunctor $Ske : \underline{MC} \to \underline{Q}$ where $\underline{Q}$ is the quotient category of the category $\underline{S}$ of cobordisms of simple graphs introduced in 4.5 above. Composition of this skeleton semifunctor with the functor $Q \to K\text{-}\underline{Mod}$ introduced in 4.5 can be factorized through the forgetful functor $\underline{MC} \to \underline{C}$. The semifunctor $C \to K\text{-}\underline{Mod}$ obtained is a functor, which coincides with the functor (topological quantum field theory) defined in 2.3.

We obtain thus a new description of this functor: it assigns to a closed surface F a K-module $Q(F)$ which can be obtained as a quotient module of $C(\Gamma)$ where Γ is any simple graph embedded into F in such a way that each component of its complement in F is homeomorphic to $\mathbb{R}^2$. The factorization should be done by the kernel of the homomorphism induced by the trivial cobordism. To each cobordism it assigns the homomorphism induced by a collar simple 2-skeleton of this cobordism. In particular, to any closed 3-manifold M it assigns a homomorphism $K \to K$ (since $Q(\varnothing) = K$) acting as multiplication by number $|M|$ which can be calculated by formulae (7), (6) applied to any simple skeleton X of M.

6.6 Non-functorial generalization. The condition that surfaces are *completely* marked has appeared in the definition of $\underline{MC}$ to define a *functor*. But in some situations noncomplete marking naturally arise. For example, if M is the complement of a regular neighborhood of a link. Then with each framing of the link one associates a graph Γ on ∂M consisting of the longitudes of the link components. Then the state sum invariant of a simple 2-skeleton of (M, Γ) is an invariant of the initial framed link.

7. QUANTUM 6j-SYMBOLS

The $6j$-symbols play an important role in the representation theory of semi-simple Lie algebras. The q-analogs (or q-deformations) of $6j$-symbols for the Lie algebra $sl_2(\mathbb{C})$ were introduced in [3] and related to the representation theory of the algebra $U_q(sl_2(\mathbb{C}))$ in [9]. Here we present certain results of [9] specialized to the case when q is a complex root of unity.

7.1 Introduction of the "initial data". Fix an integer $r \geqslant 3$ and denote by I the set of integers and half-integers $\{0, 1/2, 1, 3/2, \dots, (r-3)/2, (r-2)/2\}$. Fix a root of unity q_0 of degree $2r$ such that $q_0^2 = q$ is a primitive root of unity of degree r. For an integer $n \geqslant 1$ set

$$[n] = \frac{q_0^n - q_0^{-n}}{q_0 - q_0^{-1}} \in \mathbb{R}.$$

Set

$$[n]! = [n][n-1]\dots[2][1].$$

In particular, $[1]! = [1] = 1 \in \mathbb{R}$. Put also $[0]! = [0] = 1$. Note that $[r] = 0$ and $[n] \neq 0$ for $n = 0, 1, \dots, r-1$.

A triple $(i, j, k) \in I$ will be called *admissible* if $i + j + k$ is an integer and

$$i \leqslant j + k, \quad j \leqslant i + k, \quad k \leqslant i + j, \quad i + j + k \leqslant r - 2.$$

For an admissible triple (i, j, k) put

$$\Delta(i, j, k) = \left(\frac{[i + j - k]!\,[i + k - j]!\,[j + k - i]!}{[i + j + k + 1]!}\right)^{1/2}.$$

Note that the expression in the round brackets presents a real number. By the square root $x^{1/2}$ of a real number x we will mean the positive root of $|x|$ multiplied by $\sqrt{-1}$ if $x < 0$.

For any admissible 6-tuple $(i, j, k, l, m, n) \in I^6$ one defines $(q-6j)$-symbol and Rakah–Wigner $(q-6j)$-symbol denoted respectively by

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}.$$

These symbols are related by the following formula

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} = [2k+1]^{1/2}[2n+1]^{1/2}\sqrt{-1}^{2(l+m+2k-i-j)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}.$$

The Rakah–Wigner symbol is computed by the following formula

$$\begin{aligned} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} &= \Delta(i, j, k)\Delta(i, m, n)\Delta(j, l, n)\Delta(k, l, m) \\ &\quad \times \sum_z (-1)^z [z+1]! \big\{[z - i - j - k]!\,[z - i - m - n]!\,[z - j - l - n]! \\ &\quad \times [z - k - l - m]! \\ &\quad \times [i + j + l + m - z]!\,[i + k + l + n - z]!\,[j + k + m + n - z]!\big\}^{-1}. \end{aligned}$$

Here z runs over non-negative integers such that all expressions in the square brackets are non-negative i.e.

$$\begin{gathered} \min(i + j + l + m, i + k + l + n, j + k + m + n) \geqslant z, \\ z \geqslant \max(i + j + k, i + m + n, j + l + n, k + l + m). \end{gathered}$$

We define

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}. \tag{15}$$

The equalities (1) follow directly from definitions.

For $i \in I$ put $w_i = (\sqrt{-1})^{2i}[2i+1]^{1/2}$. It is easy to show that

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} = w_k w_n \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \tag{16}$$

Our initial data consists of the set I, the function $i \mapsto w_i\colon I \to \mathbb{C} - 0$, the admissible triples and the symbol $|\,|$ described above, and w equal to either $\sqrt{2r}/|q_0 - q_0^{-1}|$ or $-\sqrt{2r}/|q_0 - q_0^{-1}|$.

7.2 THEOREM. *The initial data just described is irreducible and satisfy the conditions* $(*)$, $(**)$ *and* $(***)$ *of Section* 1.1.

Proof. It is easy to check that for any $j \in \{0, 1/2, 1, \dots, (r-3)/2, (r-2)/2\}$ the triple $j, |j-k|, k)$ is admissible. Therefore the initial data is irreducible. The substitution (16) transforms the formulas 6.16 and 6.18 of [9] (formulated in terms of the $(q-6j)$-symbols $\{\,\}$) respectively into (*) and (**). Now let us check (***) with $j = 0$ i.e. prove that

$$w^2 = w_0^2 \sum_{k,l:\ (0,k,l)\in adm} w_k^2 w_l^2. \tag{17}$$

Clearly, $w_0 = 1$. Further, by the definition of *adm* above,

$$\big\{(k,l):\ (0,k,l) \in adm\big\} = \big\{(k,k):\ k = 0, 1/2, 1, \dots, (r-2)/2\big\}.$$

Thus we have to prove that

$$w^2 = \sum_{l=1}^{r-1} w^4_{(l-1)/2}. \tag{18}$$

By the definition of w_k above,

$$w^4_{(l-1)/2} = \frac{\big(q_0^l - q_0^{-l}\big)^2}{\big(q_0 - q_0^{-1}\big)^2}.$$

Since $q_0^{2r} = 1$,

$$\sum_{l=1}^{r-1} q_0^{2l} = \sum_{l=1}^{r-1} q_0^{-2l} = -1.$$

Therefore the right hand side of (18) equals

$$-2r/\big(q_0 - q_0^{-1}\big)^2 = w^2. \qquad \square$$

In certain special cases one may simplify the right hand side of (15). Consider for instance a 6-tuple $(i, j, k, l, m, n) \in I^6$ with $n = 0$. Such a 6-tuple is admissible if and only if $i = m$, $j = l$ and the triple (i, l, k) is admissible. One easily computes

$$\left\{\begin{matrix} i & j & k \\ j & i & 0 \end{matrix}\right\}^{RW} = \frac{(-1)^{i+j+k}}{[2i+1]^{1/2}[2j+1]^{1/2}}$$

and

$$\left|\begin{matrix} i & j & k \\ j & i & 0 \end{matrix}\right| = \frac{\sqrt{-1}^{\,-2(i+j)}}{[2i+1]^{1/2}[2j+1]^{1/2}}. \tag{19}$$

Remark. The initial data introduced in 7.1 may be equipped with a function $c: I \to \mathbb{Z}_2$ satisfying the condition of Section 2.5. Namely $c(i) = 2i\,(mod\, 2)$. Thus the corresponding topological quantum field theory can be refined along the lines of Section 2.5.

8. CALCULATIONS FOR SIMPLEST CLOSED 3-MANIFOLDS

8.1 Summary of results. In this section we calculate $|M|$ for several closed manifolds M, which allow simple skeleton without 0-dimensional strata. Since the calculation in those cases does not involve $6j$-symbols, we are able to formulate results in terms of w and w_i, and, for the initial data of Section 7, to find $|M|$ for all values of q_0.

8.1.A. For any initial data

$$\left|S^3\right| = w^{-4} \sum_{i \in I} w_i^4, \tag{20}$$

$$\left|\mathbb{R}P^3\right| = w^{-2} \sum_{i \in I} w_i^2, \tag{21}$$

$$|L(3,1)| = w^{-2} \sum_{\substack{i \in I:\\ (i,i,i) \in adm}} w_i^2, \tag{22}$$

$$\left|S^1 \times S^2\right| = w^{-2} \sum_{i \in I:} N(i) w_i^2, \tag{23}$$

where $N(i)$ is the number of $j \in I$ such that $(i, j, j) \in I$.

8.1.B. For the initial data of Section 7

$$\left|S^3\right| = w^{-2} = -\left(q_0 - q_0^{-1}\right)^2 / 2r, \tag{24}$$

$$\left|\mathbb{R}P^3\right| = \begin{cases} \dfrac{(q_0 - 1)(q_0^{-1} - 1)}{r}, & \text{if } (-q_0)^r = -1, \\ 0, & \text{if } (-q_0)^r = 1, \end{cases} \tag{25}$$

$$|L(3,1)| = \frac{\left(q_0^{[(r-2)/3]+1} - q_0^{-[(r-2)/3]-1}\right)^2}{-2r}, \tag{26}$$

$$\left|S^1 \times S^2\right| = 1. \tag{27}$$

Theorem 8.1.B shows that $|S^3|$, $|\mathbb{R}P^3|$ and $|L(3,1)|$ considered as functions of q_0 on the set of complex roots of unity are not continuous. Indeed for any $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $M = S^3, \mathbb{R}P^3$ or $L(3,1)$ $\lim_{q_0 \to \zeta} |M|_{q_0} = 0$. Therefore these functions are not restrictions of rational functions. Simple renormalization by constant can not improve the situation, as the case of $S^1 \times S^2$ shows.

The rest of Section 8 is devoted to proof of 8.1.A and 8.1.B.

8.2 Sphere. One can take sphere S^2 as a simple skeleton of S^3. The colorings of S^2 correspond to colors $i \in I$: the only 2-stratum can be colored with any color. For coloring φ_i corresponding to i formula (6) gives $|S^2|_{\varphi_i} = w^{-4} w_i^4$. That proves formula (20). Formulae (18) and (20) imply (24).

8.3 Real projective space. A projective plane $\mathbb{R}P^2$ can be taken as a simple skeleton of $\mathbb{R}P^3$. The colorings of $\mathbb{R}P^2$ correspond to colors $i \in I$: the only 2-stratum can be colored with any color. For coloring φ_i corresponding to i, formula (6) gives $|\mathbb{R}P^2|_{\varphi_i} = w^{-2} w_i^2$. The only difference with the case of 8.2 is that $\chi(\mathbb{R}P^2) = 1$ while $\chi(S^2) = 2$. That proves (21).

For the initial data of Section 7 from (21) it follows that

$$|\mathbb{R}P^3| = w^{-2} \sum_{l=1}^{r-1} w_{(l-1)/2}^2 = \frac{(q_0 - q_0^{-1})^2}{-2r} \sum_{l=1}^{r-1} (-1)^{l-1} \frac{q_0^l - q_0^{-1}}{q_0 - q_0^{-l}}$$
$$= \frac{q_0 - q_0^{-1}}{2r} \sum_{l=1}^{r-1} \left((-q_0)^l - (-q_0)^{-l}\right).$$

An easy calculation shows that the last sum is equal to $[(q_0 - 1)((-q_0)^r - 1)]/(q_0 + 1)$ that is zero in the case $(-q_0)^r = 1$ and equals $2(1 - q_0)/(1 + q_0)$ in the case $(-q_0)^r = -1$. Plugging these values proves (25).

8.4 Lens space $L(3,1)$. For $L(3,1)$ there is an obvious simple skeleton X homeomorphic to circle with disk adjoined by three-fold covering. The colorings of X correspond to colors $i \in I$ with $(i,i,i) \in adm$: the only 2-stratum can be colored with any color such that along the 1-stratum the admissibility condition is fulfilled. As in 8.3, for coloring φ_i, corresponding to i, formula (6) gives $|X|_{\varphi_i} = w^{-2} w_i^2$ and thus (22).

For the initial data of Section 7, from (22) it follows that

$$|L(3,1)| = w^{-2} \sum_{\substack{i \in I, \\ (i,i,i) \in adm}} w_i^2 = \frac{(q_0 - q_0^{-1})^2}{-2r} \sum_{k=0}^{(r-2)/3} \frac{q_0^{2k+1} - q_0^{-2k-1}}{q_0 - q_0^{-1}}$$
$$= \frac{q_0 - q_0^{-1}}{-2r} \sum_{k=0}^{(r-2)/3} \left(q_0^{2k+1} - q_0^{-2k-1}\right).$$

Calculation of the sum gives (26).

8.5 $S^1 \times S^2$. The manifold $S^1 \times S^2$ can be presented as a *prism manifold*, i.e. $K \cup_\pi S^1 \times D^2$ where K is Klein bottle, $S^1 \times D^2$ solid torus, and $\pi : \partial(S^1 \times D^2) \to K$ double covering. Therefore $X = K \cup 0 \times D^2$ is a simple skeleton of $S^1 \times S^2$. The meridian $0 \times \partial D^2$ of the solid torus is projected by π to a simple closed curve on K with complement $K - \pi(0 \times \partial D^2)$ homeomorphic to cylinder $I \times S^1$. Therefore X has two 2-strata: this cylinder and the meridian disk. A coloring of X is determined by the colors of the meridian disk and cylinder. Denote these colors by i and j respectively and the coloring by $\varphi_{i,j}$. Formula (6) gives $|X|_{\varphi_{i,j}} = w^{-2} w_i^2$. The right hand side does not depend on j. The number of colorings with a given i is equal to $N(i)$, since the condition $(i, j, j) \in I$ is admissibility conditions along the only 1-stratum of X. It proves (23).

A straightforward calculation shows that for the initial data of Section 7

$$N(i) = \begin{cases} 0, & \text{if } i \notin \mathbb{Z}, \\ r - 2i - 1, & \text{if } i \in \mathbb{Z}. \end{cases}$$

It follows that

$$\left|S^1 \times S^2\right| = \frac{(q_0 - q_0^{-1})^2}{-2r} \sum_{i=0}^{(r-2)/2} (r - 2i - 1) \frac{q_0^{2i+1} - q_0^{-2i-1}}{q_0 - q_0^{-1}}$$
$$= \frac{q_0 - q_0^{-1}}{-2r} \sum_{i=0}^{(r-2)/2} (r - 2i - 1)\left(q_0^{2i+1} - q_0^{-2i-1}\right).$$

Laborious, but straightforward evaluation shows that this expression equals 1 for all values of q_0.

9. THE CASE $r = 3$

In this section we explicitly describe the initial data introduced in Section 7 for the case $r = 3$ and compute the corresponding invariants of simple polyhedra and 3-manifolds.

9.1 The initial data. The set I consists of two elements 0 and $1/2$. Up to permutations there are only two admissible (unordered) triples: $(0,0,0)$ and $(0,1/2,1/2)$.

Let q_0 be a root of 1 of degree 6 with $q_0^2 \neq 1$. Put $\varepsilon = q_0 + q_0^{-1}$. It is easy to check that either $Re\, q_0 > 0$ and $\varepsilon = 1$ or $Re\, q_0 < 0$ and $\varepsilon = -1$.

We have

$$\begin{aligned} w_0 &= (\sqrt{-1})^0[1]^{1/2} = 1, \\ w_{1/2} &= (\sqrt{-1})[2]^{1/2} = \sqrt{-1}(q_0 + q_0^{-1})^{1/2} = \varepsilon^{1/2}\sqrt{-1}, \\ w &= \pm\sqrt{2}. \end{aligned}$$

It is easy to verify that each admissible 6-tuple may be transformed by the action of the symmetric group S_4 mentioned in Section 1.1 into one of the following three 6-tuples: $(0,0,0,0,0,0)$, $(1/2,1/2,0,1/2,1/2,0)$ and $(0,1/2,1/2,1/2,0,0)$. The formula (19) implies that

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{vmatrix} = \frac{-1}{[2]} = -\varepsilon,$$

$$\begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \end{vmatrix} = \frac{\sqrt{-1}^{-1}}{[2]^{1/2}} = \frac{-\sqrt{-1}}{\varepsilon^{1/2}} = \begin{cases} -\sqrt{-1}, & \text{if } \varepsilon = 1, \\ -1, & \text{if } \varepsilon = -1. \end{cases}$$

Thus we have 4 initial data depending on the choice of $\varepsilon = \pm 1$ and $w = \pm\sqrt{2}$. In the case $\varepsilon = +1$ we have

$$w_0 = 1, \quad w_{1/2} = \sqrt{-1}, \quad \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{vmatrix} = -1,$$

$$\begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \end{vmatrix} = -\sqrt{-1}.$$

and in the case $\varepsilon = -1$ we have

$$w_0 = 1, \quad w_{1/2} = -1, \quad \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{vmatrix} = 1,$$

$$\begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \end{vmatrix} = -1.$$

9.2 Interpretation. The state sum invariants of simple 2-polyhedra corresponding to the initial data described in Section 9.1 admit the following interpretation in a more traditional spirit.

Let X be a simple 2-polyhedron with boundary and α an admissible coloring of ∂X. Since the only admissible triples (up to permutations) are $(0,0,0)$ and $(0,1/2,1/2)$, the closures of 1-strata of ∂X whose α-color equals $1/2$ form a closed 1-dimensional manifold lying in ∂X. Denote this 1-manifold by $S(\alpha)$. Note that closed 1-dimensional submanifolds of ∂X bijectively corresponds to elements of $H_1(\partial X; \mathbb{Z}_2)$. Similarly, with each admissible coloring φ of X we associate the surface $S(\varphi)$ formed by the closures of 2-strata of X with φ-color $1/2$. It is obvious that

$$\partial S(\varphi) = S(\partial\varphi).$$

It is easy to see that the formula $\varphi \to S(\varphi)$ establishes a bijective correspondence between the admissible colorings of X extending $\alpha \in adm(\partial X)$ on the one hand and the surfaces imbedded into X formed by (closures of) 2-strata and bounded by $S(\alpha)$ on the other hand. The latter surfaces correspond bijectively to elements $s \in H_2(X, \partial X; \mathbb{Z}_2)$ with $\partial s \in H_1(\partial X; \mathbb{Z}_2)$ being the class of $S(\alpha)$.

9.2.A. *Let $\varphi \in adm(X)$. If $\varepsilon = -1$ then*

$$|X|_\varphi = w^{\chi(\partial X)-2\chi(X)},$$

if $\varepsilon = 1$ then

$$|X|_\varphi = (-1)^{\chi(S(\varphi))} w^{\chi(\partial X)-2\chi(X)}$$

where $\chi(S)$ is the Euler characteristic of S.

Proof. The vertices of X with respect to the surface $S = S(\varphi)$ are of the following four types:
(1) the vertices not lying on S (the corresponding 6-tuple is $(0, 0, 0, 0, 0, 0)$);
(2) the vertices adjacent to four germs of the 2-strata contained in S (the corresponding 6-tuple is $(1/2, 1/2, 0, 1/2, 1/2, 0)$);
(3) the vertices adjacent to three germs of the 2-strata contained in S (the corresponding 6-tuple is $(0, 1/2, 1/2, 1/2, 0, 0)$);
(4) the vertices lying in ∂S.

Let us denote the numbers of the vertices of these four types by n_1, n_2, n_3 and n_4 respectively.

Denote the number of the 1-strata of X homeomorphic to $\mathbb{R}^1$ and contained in S by e. The obvious relation

$$n_4 + 3n_3 + 4n_2 = 2e$$

implies that $n_4 + n_3$ is even.

Let $\varepsilon = -1$. The formula (6) implies that

$$|X|_\varphi = w^{\chi(\partial X)-2\chi(X)}(-1)^{n_3+n_4} = w^{\chi(\partial X)-2\chi(X)}.$$

Let $\varepsilon = 1$. Then

$$|X|_\varphi = w^{\chi(\partial X)-2\chi(X)}\sqrt{-1}^{2\chi-u+2n_2}(-\sqrt{-1})^{n_3}$$

where χ is the Euler characteristic of the union of the 2-strata contained in S and u is the number of 1-strata of ∂X contained in ∂S. Obviously $u = n_4$ and

$$X(S) = \chi - e + n_2 + n_3 = \chi - n_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}n_4.$$

Therefore

$$\sqrt{-1}^{2\chi-u+2n_2}(-\sqrt{-1})^{n_3} = (-1)^{\chi(S)}.$$

This implies our claim in the case $\varepsilon = 1$. $\square$

9.2.B COROLLARY. *If $\varepsilon = -1$ then*

$$\Omega_\chi(\alpha) = 2^b w^{\chi(\partial X)-2\chi(X)}$$

where b is the dimension of the $\mathbb{Z}_2$-vector space $H_2(X; \mathbb{Z}_2)$. If $\varepsilon = 1$ then

$$\Omega_\chi(\alpha) = w^{\chi(\partial X)-2\chi(X)} \sum_{\substack{S\in H_2(X,\partial X;\mathbb{Z}_2),\\ \partial(S)-[S(\alpha)]}} (-1)^{\chi(s)}$$

where $[S(\alpha)]$ is the class of $S(\alpha)$ in $H_1(\partial X; \mathbb{Z}_2)$ and $\chi(s)$ is the Euler characteristic of the unique relative cycle realizing s.

9.3 The case of closed 3-manifolds. For a space Y we denote $dim\, H_i(Y;\mathbb{Z}_2)$ by $b_i(Y)$.

9.3.A. *Let M be a closed 3-manifold. If $\varepsilon = -1$ then*

$$|M| = 2^{b_2(M)-b_0(M)}.$$

If $\varepsilon = 1$ then

$$|M| = 2^{-b_0(M)} \sum_{t\in H^1(M;\mathbb{Z}_2)} (-1)^{\langle t^3+w_1^2 t[M]\rangle} \tag{28}$$

where $w_1 \in H^1(M;\mathbb{Z}_2)$ is the first Stiefel–Whitney class of M.

Proof. The first claim follows from Corollary 9.2.B applied to the 2-skeleton of the dual cell subdivision of any triangulation of M and Lemma 5.1.A.

The second claim is proven similarly using the fact that the Euler characteristic of an embedded closed surface $S \subset M$ is congruent modulo 2 to $\langle t^3 + w_1^2 t, [M]\rangle$ where t is the cohomology class dual to $[S] \in H_2(M;\mathbb{Z}_2)$. $\square$

9.3.B Remark. If M is orientable and $t^3 = 0$ for all $t \in H^1(M;\mathbb{Z}_2)$ then the right hand side of (20) obviously equals $2^{b_1(M)-b_0(M)}$. If M is orientable and there exists $t \in H^1(M;\mathbb{Z}_2)$ with $t^3 \neq 0$ then the right hand side of (20) is equal to zero. This follows from the fact that for orientable M the mapping

$$t \mapsto \langle t^3, [M]\rangle : H^1(M;\mathbb{Z}_2) \to \mathbb{Z}_2$$

is a linear homomorphism. Indeed if $u, t \in H^1(M;\mathbb{Z}_2)$ then

$$u^2 t + u t^2 = Sq^1(ut) = w_1 ut = 0.$$

9.4 The case of 3*-manifold with boundary.* Let M be a compact 3-manifold with triangulated boundary and α an admissible coloring of ∂M. Let $S(\alpha)$ be the 1-cycle in ∂M formed by barycentric stars of the edges of ∂M with α-color 1/2.

9.4.A. *Let $\varepsilon = -1$. If the cycle $S(\alpha)$ presents a non-trivial element of $H_1((M;\mathbb{Z}_2)$ then $\Omega_M(\alpha) = 0$. If $S(\alpha)$ is null-homologous in M then $\Omega_M(\alpha)$ does not depend on the choice of α and equals $w^{-c} 2^{b_2(M)-b_3(M)}$ where c is the number of vertices of ∂M.*

Proof. The proof is similar to that of 9.3.A. $\square$

9.4.B. *Let $\varepsilon = 1$. If the cycle $S(\alpha)$ presents a non-trivial element of $H_1(M;\mathbb{Z}_2)$ then $\Omega_M(\alpha) = 0$. If M is orientable and there exists $t \in H^1(M,\partial M;\mathbb{Z}_2)$ with $t^3 \neq 0$ then $\Omega_M(\alpha) = 0$ for any α. If M is orientable and $t^3 = 0$ for all $t \in H^1(M,\partial M;\mathbb{Z}_2)$ and $S(\alpha)$ is null-homologous in M then*

$$\Omega_M(\alpha) = w^{-c} 2^{b_2(M)-b_0(M)} (-1)^{\chi}$$

where χ is the residue modulo 2 of the Euler characteristic of any compact surface embedded in M and bounded by $S(\alpha)$. (Under our assumptions χ does not depend on the choice of the surface.)

Proof. The proof is similar to that of 9.3.A, cf. also Remark 9.3.B. □

9.5 The topological quantum field theory. Let F be a closed surface. It is easy to compute the vector space $Q(F)$ defined in Section 2.3. For any $\varepsilon = \pm 1$ the space $Q(F)$ is $\mathbb{C}[H_1(F;\mathbb{Z}_2)]$ (i.e. the linear space over $\mathbb{C}$ freely generated by elements of $H_1(F;\mathbb{Z}_2)$). This follows from 9.2.A and the fact that the Euler characteristic of an annulus equals zero. The morphisms induced by cobordisms are easily computable via theorems of Section 9.4.

9.6 Remarks on possible generalizations. The results of 9.2 suggest to consider state sums for which colorings are 2-cycles of simple 2-polyhedra. It is not difficult to prove that the state sums of Section 9.2 are the only (up to linear combinations) state sums based on initial data with two colors such that the colorings of simple 2-polyhedra are 2-cycles over $\mathbb{Z}_2$. For cycles with $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ coefficients, all state sums of such kind (with 4 colors) are linear combinations of the state sums corresponding to $\mathbb{Z}_2$ coefficients. One can consider also $\mathbb{Z}_3$-cycles (this urges to involve orientations of 2-strata and thus to reduce the class of 2-polyhedra). In this case there is essentially only one state sum, and its value equals the number of elements of $H_2(X;\mathbb{Z}_3)$ (for the information on state sums related to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ and $\mathbb{Z}_3$ we are indebted to G. Mikhalkin).

Acknowledgements—Parts of this work were done during our visits to University Paris-Sud at Orsay and Max-Planck-Institute für Mathematik in June and November 1989. We are indebted to L. Siebenmann and F. Hirzebruch for their invitations and support.

REFERENCES

1. J.W. ALEXANDER: The combinatorial theory of complexes, *Ann. Math.* **31** (1930), 294–322.
2. M. ATIYAH: Topological quantum field theories, *Publ. Math. IHES* **68** (1989), 175–186.
3. R. ASKKY and J. WILSON: *SIAM J. Math. Anal.* **10** (1979).
4. B.G. CASLER: An embedding theorem fur connected 3-manifolds with boundary, *Proc. Amer. Math. Soc.* **16** (1965), 559–566.
5. V.G. DRINFELD: Quantum groups, in "*Proc. Int. Congress of Mathematicians*", Acad. Press, Berkeley, California, 1986, 798–820.
6. L.D. FADDEEV, N.Yu. RESHETIKHIN and L.A. TAKHTAJAN: Quantization of Lie groups and Lie algebras, in "Aluebraic Analysis", pp. 129–139, Acad. Press, (1988).
7. M. JIMBO: Quantum R-matrix for the generalized Toda system, *Commun. Math. Phys.* **102** (1986), 537–547.
8. L.H. KAUFFMAN: State models and the Jones polynomial, *Topology* **26** (1987), 395–407.
9. A.N. KIRILLOV and N.Yu. RESHETIKHIN, Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants for links, in *Infinite dimensional Lie algebras and groups*, Adv. Ser. in Math. Phys., 1988, pp. 285–339. V. G. Kac.
10. S.V. MATVEEV: Special spines of piecewise linear manifolds, *Mat. sbornik* **92** (134) (1973), 282–293.
11. S.V. MATVEEV: Transformations of special spines and the Zeeman conjecture, *Math. USSR Izvestia* **31** (1988), 423–434.
12. M.H.A. NEWMAN: Combinatorial topology of convex regions, *Proc. Nat. Ac. Sci USA* **16** (1930), 240–242.
13. N.Yu. RESHETIKHIN and V.G. TURAEV: Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* (to appear).
14. H. SEIFERT and W. THRELFALL: *Lehrbuch der Topologie*, Teubner, Leipzig, (1934).
15. M.L. STARETS: Heegaard diagrams and colored polyhedra, *J. Soviet Math.* **12** (1979), 180–188 English transl. in.
16. V.G. TURAEV: The Yang–Baxter equation and invariants of links, *Invent. Math.* **92** (1988), 527–553.
17. V.G. TURAEV: Shadow links and IRF-models of statistical mechanics., Preprint of the Louis Pasteur University, Strasbourg (1990).
18. E. WITTEN: Quantum field theory and Jones polynomial, *Commun. Math. Phys.* **121** (1989), 351–399.
19. U. PACHNER: Bistellare Äquivalenz kombinatoischer Mannigfaltigkeiten, *Arch. Math.* **30** (1978), 89–98.
20. U. PACHNER: Konstruktion methoden und das kombinatorische Homömorphieproblem für Triangulationen kompakter semilinearen Mannigfaltigkeiten, *Abh. Math. Sem. Univ. Hamburg* **57** (1987), 69–86.
21. R. PIERGALLINI: Standard moves for standard polyhedra and spines. III Convegno Nazionale di Topologia Trieste, 9–12 Giugno **1986** (1988), 391–414.

URA-CNRS, Department of Mathematics,
University L.-Pasteur,
7 rue Descartes,
67084, Strasbourg, France

Department of Mathematics,
University of California,
Riverside, CA 92521, USA
&
POMI, Fontanka 27,
St. Petersburg 191011, Russia

APPENDIX 1
RELATIVE ALEXANDER THEOREM

Here we prove Theorem 3.2.A above. Recall its statement.

3.2.A THEOREM. *Let P be a dimensionally homogeneous polyhedron and Q its subpolyhedron. Any two triangulations of P coinciding on Q can be transformed one to another by a sequence of Alexander moves and transformations inverse to Alexander moves, which do not change the triangulation of Q.*

First, remind some results of Alexander [1] and Newman [12].

Recall that the boundary ∂P of a dimensionally homogeneous polyhedron P of dimension n is defined to be the union of the closed $(n-1)$-simplexes of P which are faces of an odd number of n-simplexes. For manifolds this notion coincides with the usual notion of boundary. A star subdivision of P along a simplex E is said to be *internal* if E does not lie in ∂P.

A triangulated polyhedron is called a *formal cell* if one can transform it into a simplex by a series of star subdivisions and inverse operations.

A.1.A NEWMAN'S THEOREM. *Any linearly triangulated convex compact subset of an Euclidean space is a formal cell.*

A short proof of A.1.A see in [1, Section VI].

A.1.B ALEXANDER'S THEOREM [1, Section [13:2]]. *Any formal cell can be transformed into the cone over its* (*triangulated*) *boundary by a series of internal star subdivisions and inverse operations.*

Now let us proceed to prove Theorem 3.2.A. It suffices to consider the case when one of the two triangulations of P is a subdivision of the other one. Indeed, for any two combinatorially equivalent triangulations of P coinciding on Q we can consider the cell subdivision of P formed by intersections of their simplexes. Inductively on $m = 0, 1, \dots$ we replace each m-cell not lying in Q by the cone over its boundary. This produces a triangulation of P which is finer than the two initial ones and coincides with them on Q.

We use induction on $dim P$. For $dim P = 0$ the claim is obvious. Assume that for $dim P < n$ the claim holds true. Let us prove it for $dim P = n$. Let X and Y be two triangulations of P coinciding on Q and let Y be finer than X. For each closed n-simplex A of X the simplexes of Y lying in A form a triangulation of A. By A.1.A this triangulation makes A a formal cell. Using A.l.B we transform this triangulation of A into the cone over ∂A. Thus the triangulation Y is transformed to a triangulation Z such that it is finer than X and on each n-simplex A of X it is the cone over the triangulation of ∂A induced by Z.

Consider the triangulated pair $(X_{n-1}, X_{n-1} \cap Q)$, where X_{n-1} is the $(n-1)$-skeleton of X. The triangulation Z induces a subdivision of X_{n-1} identical on $X_{n-1} \cap Q$. By the inductive assumption this subdivision can be "Alexander" transformed identically on $X_{n-1} \cap Q$ to the triangulation induced by X. Because of the cone structure of Z on n-simplexes of X, these transformations can be extended to a chain of Alexander transformations identical on Q and converting Z to X.

Remark. In fact Alexander [1] refined Theorem 3.1.A. He proved that it is sufficient to use the Alexander transformations along edges (and their inverses) only. This refinement can not be directly extended to the relative case. For example, let P be a simplex and $Q = \partial P$. The canonical triangulation of P can be converted into any of its non-trivial subdivisions identical on Q by no chain of internal Alexander moves along edges and their inverses. This follows from the fact that the canonical triangulation of P has no internal edges.

Under certain assumptions on Q one may use Alexander's arguments to show that any two combinatorially equivalent triangulations of a polyhedron $P \supset Q$ coinciding on Q can be related by a sequence of Alexander transformations along edges not lying in Q and inverses of such transformations. Here is an example of such an assumption on Q: any 3 edges of Q forming a triangle bound a 2-simplex of Q. This assumption is not restrictive since each triangulation has a subdivision satisfying it. For example one may take the first barycentric subdivision.

APPENDIX 2
STRATIFICATIONS, SPINES AND PRESENTATIONS OF MANIFOLDS

A.2.1 Stratifications. By a *stratification* of a (piecewise linear) manifold X we mean a partition of X on disjoint parts (which are called *strata*) with the following properties:

(1) each stratum is a submanifold of either $IntX$ or ∂X;
(2) as a manifold each stratum has empty boundary;
(3) the closure of each stratum is a subpolyhedron of X which is the union of a finite number of strata.

We consider only stratifications which satisfy an additional property of "local triviality" along strata. An arbitrary stratification can be canonically subdivided to satisfy this property. A stratification of a manifold X is said to be *locally trivial* if each point of any stratum S has a neighborhood U in X such that there exists a homeomorphism $U \to V \times \mathbb{R}^4$, where V is $\mathbb{R}^p$ or $\mathbb{R}^p_+$, mapping $U \cap S$ onto $pt \times \mathbb{R}^4$ and the intersection of U with any stratum onto $C \times \mathbb{R}^4$, where C is a submanifold of V. These C's constitute a stratification of V.

A locally trivial stratification of X is said to be *simple*, if for each point of X the stratification of V mentioned above is homeomorphic to the cone over the standard stratification (by faces) of the boundary of the p-dimensional simplex, in the case $V = \mathbb{R}^p$, and, in the case $V = \mathbb{R}^p_+$, to the same stratification, but with one p-dimensional stratum removed.

If each stratum is homeomorphic to an Euclidean space, then the stratification is called a *cellular* stratification.

The most classical stratifications of manifolds are triangulations. They are locally trivial cellular stratifications, but in general they are not simple (in the sense specified above).

Another classical set of locally trivial cellular stratifications are stratifications dual to triangulations of manifolds without boundary, i.e. partitions of manifolds on barycentric stars of simplexes of triangulations. These are simple stratifications.

In the case of a manifold with non-empty boundary the corresponding simple stratifications consist of the intersections of the barycentric stars with the interior and the boundary of the manifold.

Remark. There is another version of the theory, in some sense dual to the version above, but coinciding with it in the case of empty boundary. In this variant strata are allowed to have boundary, but forbidden to lie in ∂X. Then a triangulation of a manifold with non-empty boundary is not a stratification, but the barycentric stars form a stratification in this sense.

A.2.2 Simple polyhedra. Let Π^n be the n-th skeleton of the standard triangulation of the boundary of the standard $(n+2)$-dimensional simplex. In particular, Π^0 consists of three points, Π^1 is the graph homeomorphic to a circle with 3 radii, $\Pi^{-1} = \varnothing$.

Let Σ^n_q with $0 \leqslant q \leqslant n+1$ be the q-fold suspension over Π^{n-q}. In particular, $\Sigma^1_0 = \Pi^1$, Σ^1_1 is homeomorphic to a circle with diameter, Σ^1_2 is the 2-fold suspension of the empty set, i.e. a circle.

A polyhedron X is called a *simple polyhedron* of dimension n if the link of each its point is homeomorphic to Σ_q^{n-1} for some $q,\ 0 \leqslant q \leqslant n$. (This condition can be reformulated as follows: any point of X has a neighborhood homeomorphic to the cone $(\Pi^{n-2-q}) \times \mathbb{R}^q$.) In particular, for $n = 2$ this definition is equivalent to the one given in Section 4.1.

Simple polyhedra appear naturally as skeletons of codimension one of simple stratifications of manifolds. In particular, simple 2-skeletons of 3-manifolds introduced in Section 6.1 are skeletons of codimension one of the corresponding simple stratifications of the manifolds.

It is obvious that a simple n-dimensional polyhedron has a natural stratification in which the strata of dimension q consist of points with links homeomorphic to Σ_q^{n-1}.

Simple polyhedron is said to be *cellular* if each stratum of the natural stratification is homeomorphic to Euclidean space. The codimension 1 skeleton of simple cellular stratification of a manifold is a simple cellular polyhedron.

An $(n-1)$-dimensional spine of an n-manifold, which is (as a polyhedron) simple (respectively simple cellular), is called a simple (respectively simple cellular) spine.

The following generalizes Theorem 6.1.A.

A.2.2.A THEOREM. *Any compact manifold has a simple cellular spine.*

This theorem, as well as the next one, was proved for 3-dimensional manifolds by Casler [4] and in the general case by Matveev [10].

A.2.2.B THEOREM. *If two compact manifolds have the same simple cellular spine and either both are closed or both have non-empty boundary, then these manifolds are homeomorphic.*

Theorem A.2.2.B can not be extended straightforwardly to the case of simple non-cellular spines. Indeed, the Klein bottle with a disk adjoined along a circle which is a fiber of the fibration of the Klein bottle over S^1, is a simple spine of both

$$S^2 \times S^1 - (\text{open ball})$$

and

$$\big(\text{non orientable } D^2\text{-bundle over } S^1\big) - (\text{open ball}).$$

But this effect is due to existence 2-stratum with more than one end only. Here is an appropriate generalization of Theorem A.2.2.B:

A.2.2.C THEOREM. *If two compact manifolds have the same simple spine, each* 2*-stratum of which is a surface with at most one end, and either both are closed or both have nonempty boundary, then these manifolds are homeomorphic.*

The proof of A.2.2.C is not difficult. We omit it since we do not use this theorem.

Simple spines of an n-manifold M and simple spines of M with several open balls removed will be called *simple* $(n-1)$*-skeletons of* M. For example, for any compact n-manifold M the union of the barycentric stars of all r-simplexes of $\mathit{Int}\, M$ with $r > 0$ is a simple $(n-1)$-skeleton of M. It is clear that each oriented compact connected n-manifold M is restored (up to homeomorphism), if one knows its simple $(n-1)$-skeleton and the number of spherical components of its boundary.

Theorem 6.1.B admits the following high-dimensional generalization.

A.2.2.D. *Any compact manifold M with non empty boundary is homeomorphic to the cylinder of a topological immersion of ∂M onto an arbitrary simple spine of M.*

Remark. Note that 6.1.B gives a convenient way of description of 3-manifolds. (It can be also generalized to higher dimensions, but in high-dimensional situation it does not give a visualization of manifolds.) Namely, consider the inverse image of the natural stratification of a simple spine of a 3-manifold under the topological immersion given by 6.1.B. It is a stratification with 1-skeleton being a simple graph. Each 0-stratum of it consists of 4 points, and if the simple spine is in fact *cellular* then each 1-stratum consists of 3 segments and each 2-stratum consists of 2 disks. The components of strata are provided with natural identifications. This picture can be used to describe compact 3-manifolds: any stratification of the boundary of a compact 3-manifold with the properties listed above and with identifications of components of strata agreed in the obvious sense on adjacent strata determines a special 2-polyhedron (as the quotient space) and a topological immersion (the factorization map) and thus the 3-manifold (as the cylinder of this immersion). The natural version of this construction for the case of *closed* 3-manifolds is essentially the classical presentation via a polyhedron with pairwise identification of faces, see [14] and Starets [15]. The restrictions on identifications of vertices (4-fold identification as above) and edges (3-fold identification) are not involved in these classical presentations of closed 3-manifolds. These restrictions guarantee that the quotient space is a 3-manifold, but in the closed case it is sufficient to put a weaker restriction in terms of Euler characteristic, see [14]. The same relaxation of restrictions can be done in the case of 3-manifolds with boundary. It leads to the following type of descriptions.

Fix a cell decomposition of a closed surface F, fix a division of the set of 2-cells (faces) on pairs, and for each of these pairs fix a homeomorphism between the faces involved. The homeomorphisms are assumed to preserve the natural partitions of the boundaries of the 2-cells. Denote by M the quotient space of $F \times [0,1]$ by identification of each point $(x, 1)$ of $F \times 1$ with $(h(x), 1)$ for all the fixed homeomorphisms h of faces. Suppose that $\chi(M) = \chi(F)/2$. Then M is a manifold with boundary F.

A.2.3 Remarks on Matveev moves. Matveev and Piergallini considered (together with $\mathcal{M}$ and $\mathcal{L}$) the move shown in Fig. 17. We will denote it by $\mathcal{F}$. It is clear that $\mathcal{F}$ can be considered as a special case of $\mathcal{L}$ and that $\mathcal{L}$ is simpler than $\mathcal{F}$. However from some point of view $\mathcal{F}$ is better. Indeed, applications of $\mathcal{F}^{\pm 1}$ and $\mathcal{M}^{\pm 1}$ to a simple cellular spine of a compact 3-manifold can be realized inside this 3-manifold and thus give simple cellular spines of the same 3-manifold. Moreover Matveev and Piergallini [11,21] proved the following theorem which is more general than Theorem 6.2.B.

A.2.3.A. *Two simple cellular* 2*-dimensional polyhedra, one of which is a simple cellular spine of some* 3*-manifold, are simple cellular spines of the same* 3*-manifold, if and only if one of the polyhedra can be obtained from the other one by a sequence of transformations* $\mathcal{F}^{\pm 1}$ *and* $\mathcal{M}^{\pm 1}$.

Contrary to $\mathcal{T}^{\pm 1}$ and $\mathcal{M}^{\pm 1}$, an application of $\mathcal{L}$ to a simple cellular spine of a 3-manifold can give a simple cellular polyhedron which is a spine of no 3-manifold. Thus one can not just replace $\mathcal{T}^{\pm 1}$ by $\mathcal{L}^{\pm 1}$ in A.2.3.A.

If an application of $\mathcal{L}$ to a spine gives a spine of some 3-manifold, then this 3-manifold is homeomorphic to the initial one. Further, an application of $\mathcal{L}^{-1}$ to a simple cellular polyhedron can give a simple polyhedron, which is not cellular. But any application of $\mathcal{L}^{-1}$ to some simple spine of a 3-manifold gives a simple spine of the same 3-manifold.

A.2.4 Moves of high-dimensional simple cellular polyhedra. The system of moves $\mathcal{L}$, M, B can be generalized straightforwardly to the case of arbitrary simple n-polyhedra in such a way

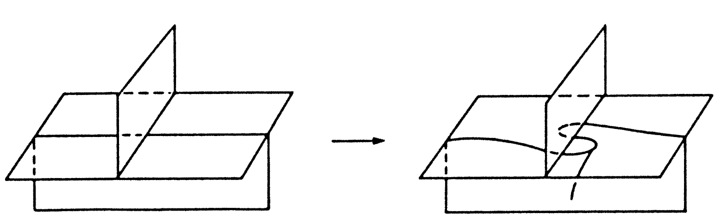

Fig. 17.

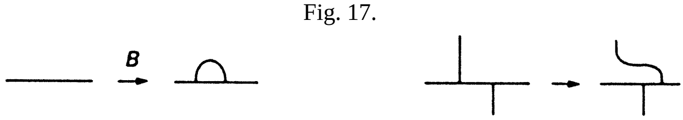

Fig. 18.

that a generalization of Theorem 6.2.A holds true. Except the analogue of $\mathcal{B}$, these moves can be described as follows. One of the highest-dimensional strata, say R, is moving via isotopy of its boundary in the complement of R so that the intersection of ∂R with another stratum undergoes a Morse modification of some index. In Fig. 18 this system of moves is shown for $n = 2$. For arbitrary odd n it consists of $(n^2 + 3)/4$ transformations, and for even n of $(n^2 + 4)/4$ transformations.

In high-dimensional situation combinatorics of a simple stratification of a manifold is not as rich as in dimensions $\leqslant 3$, since it does not contain an important part of the topological information on the manifold. This information can be hidden in the topology of strata. Thus it is more natural to consider simple *cellular* polyhedra and heir moves.

The system of moves for simple cellular n-polyhedra consists of moves $\mathcal{M}_1, \dots, \mathcal{M}_{n+1}$. The simplest (however slightly implicit) description of $\mathcal{M}_i$ is the following: a simple cellular n-polyhedron Y is obtained from a simple cellular n-polyhedron X by $\mathcal{M}_i$, if there exist a simple $(n+1)$-polyhedron Z and a PL-function $f : Z \to \mathbb{R}$ such that

(1) $X = f^{-1}(0)$
(2) $Y = f^{-1}(1)$,
(3) a restriction of f to each stratum of Z intersects $f^{-1}[0, 1]$ has no critical point,
(4) $f^{-1}[0, 1]$ contains only one vertex c of Z,
(5) on i of $n + 2$ edges of Z adjacent to this vertex c the function f takes values $< f(c)$ and on the others $n + 2 - i$ edges it takes values $> f(c)$.

(These conditions mean that in the sense of the Goresky–MacPherson stratified Morse theory $f|_{f^{-1}[0,1]}$ is a Morse function with only one critical point, which is c, and in c it has index i in the sense of Khovansky. Thus one can consider our move $\mathcal{M}_i$, as a kind of stratified Morse modification of index i.)

It is easy to see that M_i is inverse to $\mathcal{M}_{n+2-i}$. If $i \neq 1, n + 1$ it is a replacement of one $(i - 1)$-stratum of X with the closure homeomorphic to simplex by a new $(n + 2 - i)$-stratum with the closure homeomorphic to simplex. If $i = 1$ then M_i is an inserting the boundary of the $(n + 1)$-simplex with the canonical stratification instead of a vertex of X.

In the 2-dimensional case considered in the main text of this paper, $\mathcal{M}_2$ is just the Matveev move $\mathcal{M}$ and $\mathcal{M}_3 = \mathcal{M}^{-1}$. In Fig. 19 we show the moves $\mathcal{M}_1$ and $\mathcal{M}_2$ for $n = 1$ and $\mathcal{M}_1$ for $n = 2$.

The transformations of simple cellular stratifications of n-manifolds inducing $\mathcal{M}_i$ on the $(n - 1)$-skeletons will be called also $\mathcal{M}_i$.

One can show that any two simple cellular stratifications of a closed n-manifold can be transformed one to another by a series of moves $\mathcal{M}_i$. Furthermore one can modify the theory to the relative case as in Section 6.

A.2.5 Singular triangulations and their dualization. An important property of triangulations is that each triangulation can be completely described (up to homeomorphism) in a discrete combinatorial way. Therefore triangulations provide a method of combinatorial description of

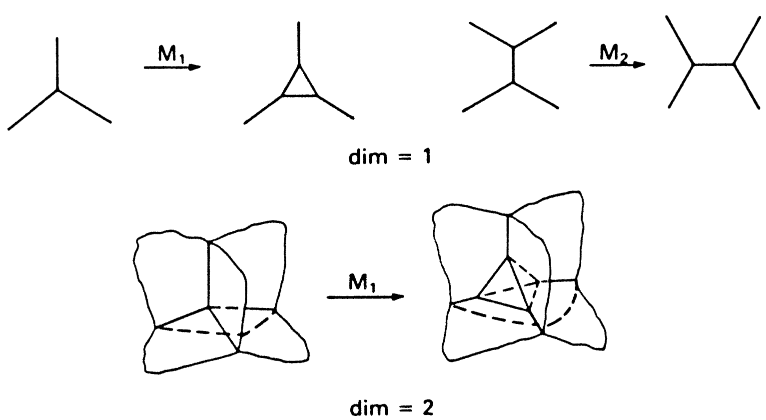

Fig. 19.

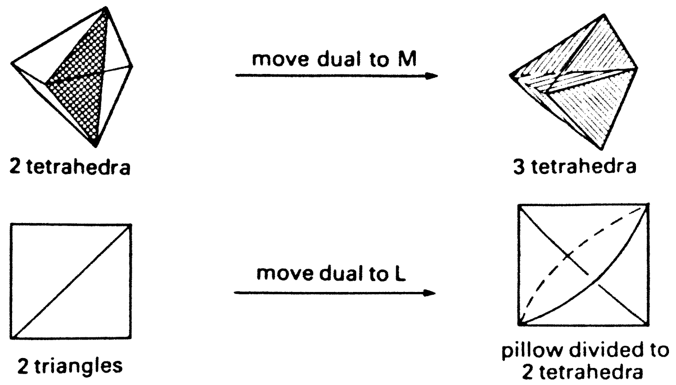

Fig. 20.

manifolds. But usually triangulations have so many simplexes that it is hard to use them in practice when some manifold is to be described. The natural way to avoid this difficulty is to generalize triangulations. The most usual generalization is the notion of CW-complex, but it leads to a loss of the combinatorial character of the theory. The following notion lies between the notions of triangulation and CW-complex.

Let X be a space. A family of continuous maps $\varphi_\alpha : T^{d_\alpha} \to X_{d_\alpha}$, $\alpha \in A$, of standard simplexes T^{d_α}, $d_\alpha \in \mathbb{Z}_+$, is called a *singular triangulation* of X, if

(1) all $\varphi_\alpha|_{Int\,T^{d_\alpha}}$ are embeddings,
(2) $\varphi_\alpha(IntT^{d_\alpha})$ are open cells of some CW-decomposition of X,
(3) for any face F of T^{d_α} the restriction $\varphi_\alpha|_F$ can be obtained from some φ_β by composition with a linear isomorphism $F \to T^{d_\beta}$.

Note that replacing $\varphi_\alpha|_{intT^{d_\alpha}}$ by φ_α in (1) and incorporating the condition that the intersection of any two simplexes is their common face convert this definition into a definition of triangulation.

The construction of the barycentric star stratification can be generalized in an obvious way to a construction which assigns to any cellular stratification of a manifold a dual stratification defined up to ambient isotopy. An application of this construction to a singular triangulation of a PL-manifold gives a simple cellular stratification. Conversely an application of this construction to any simple cellular stratification gives a singular triangulation. Thus the construction yields a 1–1 correspondence between singular triangulations and simple cellular stratifications.

Usually one can find for a given manifold singular triangulations and simple cellular stratifications which are considerably smaller than triangulations. For example it is easy to prove that any closed connected manifold has a singular triangulation with only one vertex. A closed orientable surface of genus g has singular triangulations with $4g - 2$ triangles.

The moves $\mathcal{M}, \mathcal{L}$ of simple 2-polyhedra introduced in Section 4.3, being applied to a simple cellular 2-skeleton of a 3-manifold, induce transformations of the corresponding simple cellular stratification of the 3-manifold. The corresponding transformations of the dual singular triangulation are shown in Fig. 20.

A.2.5.A COROLLARY OF 6.2.B. *Any two singular triangulations of a compact* 3*-manifold with the same number of vertices can be transformed one to another by a sequence of transformations dual to* $\mathcal{M}^{\pm 1}$ *and* $\mathcal{L}^{\pm 1}$.

Since the bubble move transforms a simple cellular stratification of a 3-manifold into a non-cellular one, it can not have a dual move. The bubble move can be replaced in this situation by the Alexander move along 3-simplex. It follows from A.2.5.A that this move together with moves shown in Fig. 20 (and their inverses) enable one to relate any two singular triangulations of a compact 3-manifold.

Let us describe now the move which is dual to the move $\mathcal{M}_i$ of Section A.2.4. Note first that the move dual to $\mathcal{M}_1$, is just the Alexander move along a simplex of dimension n, where n is the dimension of the manifold under consideration. Consider now the case of $\mathcal{M}_i$ with $i > 1$.

Suppose for a moment for the sake of simplicity that the triangulation is non-singular. Let the link of one of the $(i-1)$-simplexes is strata preserving homeomorphic to the canonically triangulated boundary of $(n-i+1)$-simplex. Then the star of this $(i-1)$-simplex is triangulated as a join of it with the boundary of $(n-i)$-simplex and the boundary of the star is triangulated as a join of boundaries of $(i-1)$- and $(n-i)$-simplexes. It can be span by the join of $(n-i)$-simplex with the boundary of $(i-1)$-simplex. The transformation dual to $\mathcal{M}_i$ replaces the star of the $(i-1)$-simplex by this join of $(n-i)$-simplex with the boundary of $(i-1)$-simplex.*

Consider now the case of *singular* triangulations. The move dual to $\mathcal{M}_i$ can be applied iff the closure of the barycentric star of one of the $(i-1)$-simplexes is strata preserving homeomorphic to the canonically triangulated $(n-i+1)$-simplex. Note that for any simplex S there is a natural strata preserving map of the join of the closure of S and the boundary of its barycentric star onto the closed star of S, this map is identity on S and 1-1 on the complement of the boundary of the barycentric star. In the situation under consideration the move dual to $\mathcal{M}_i$ replaces the stratification of the star of the $(i-1)$-simplex by the image under this map of the triangulation of the join above presented as the join of the barycentric star with the boundary of the $(i-1)$-simplex.

Since any two simple cellular stratifications of a closed manifold can be transformed one to another by a series of moves $\mathcal{M}_i$, any two singular triangulations of a closed manifold can be transformed one to another by a series of the moves dual to $\mathcal{M}_i$.**

* Added in proof: Transformations dual to $\mathcal{M}_i$ for triangulations were introduced by Udo Pachner in [19]. Pachner called them *bistellar transformations*. In [20] he proved that any two triangulations of the same PL-manifold can be obtained from each other by a sequence of bistellar transformations.

** Added in proof: It follows from Pachner's results [20].