Stony Brook University
Mathematics Department
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Seminar in Mathematics
Teaching/Learning
MAT 519
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Comments on Test 1

1. Simplify $(\sqrt{x})^2$.

Correct answer. $(\sqrt{x})^2 = x$ for $x \geq 0$ (for $x < 0$ the expression $\sqrt{x}$ is not defined). It follows from the definition of $\sqrt{x}$, as $\sqrt{x}$ is defined for non-negative real x as the non-negative real number such that $(\sqrt{x})^2 = x$.

Comment. A square root of a number x (real, complex or of any other nature) is a number y such that $y^2 = x$. For a complex number $x \neq 0$ two different square roots exist. A negative real number has two distinct complex square roots, neither of which is real. A positive real number has also two distinct complex square roots, and both of them are real. They are $\sqrt{x}$ and $-\sqrt{x}$. You see that only one of them is denoted by $\sqrt{x}$, the positive one.

For any square root y of x the definition of square root implies that $y^2 = x$.

Comments on student's solutions. Inspired (or, rather, confused) by the problem about $\sqrt{x^2}$ from the questionnaire, 4 students gave the answer $(\sqrt{x})^2 = |x|$. Why is this answer wrong? Because $(\sqrt{x})^2$ is defined only for $x \geq 0$, while $|x|$ is defined for any real x. Yes, for a non-negative x these two expressions are equal, but for two functions to be considered equal they must be defined for the same values of the argument.

Twelve students gave the answer $(\sqrt{x})^2 = x$ without mentioning the restriction $x \geq 0$. It is as wrong as the answer $(\sqrt{x})^2 = |x|$: the left hand side is defined only if $x \geq 0$, while the right hand side is defined for any real x.

In fact, $x = |x|$ for those values of x, for which $(\sqrt{x})^2$ is defined. If we accepted one of these two answers, we would have to accept the other one. However, then formally from these two equalities (namely, from $(\sqrt{x})^2 = |x|$ and $(\sqrt{x})^2 = x$) it would follow $x = |x|$.

The formulas considered above are equalities between functions of the variable x. The notion of function includes the set of arguments for which the function is considered. The function $(\sqrt{x})^2$ is considered only for $x \geq 0$, therefore it cannot be equal to a function considered for all real values of the argument x. To keep things sane, one has to specify explicitly the restriction on x under which the formula is considered.

On the other hand, both answers "$(\sqrt{x})^2 = x$ for $x \geq 0$" and "$(\sqrt{x})^2 = |x|$ for $x \geq 0$" are correct, because $|x| = x$ for $x \geq 0$.

Informally, the answer "$(\sqrt{x})^2 = x$" without mentioning the restriction on x is less dangerous, because the expression $|x|$ implicitly suggests that negative values of x are allowed (otherwise, why not to write just x instead of $|x|$?)

A correct answer was given by two students.

Three students gave this answer too, but their arguments contain speculations about the meaning of the equality $(\sqrt{x})^2 = x$ for negative x. For negative x they wrongly claimed that $(\sqrt{x})^2$ is imaginary, imaginary undefined, or even unreal complex. Let me repeat again that for real negative x the symbol $\sqrt{x}$ is not defined, period. What would it be if it was defined is a question which does not make sense. The square of any square root y of a negative real number x is definitely x, that is a negative real number. However, in this situation none of such y is denoted by $\sqrt{x}$. If you like, because there is no uniform way to choose one from the two square roots.

Two students gave the answer $(\sqrt{x})^2 = \pm x$. No comment.

2. What is the set of real numbers satisfying the inequality $\dfrac{x}{x+1} \geq x$*?*

Correct solution.
The inequality $\dfrac{x}{x+1} \geq x$ is equivalent to $\dfrac{x}{x+1} - x \geq 0$. Further,

$$\frac{x}{x+1} - x = \frac{x}{x+1} - \frac{x(x+1)}{x+1} = \frac{-x^2}{x+1}.$$

Hence, the original inequality is equivalent to $\dfrac{-x^2}{x+1} \geq 0$. According to a general theorem formulated in my comments to the questionnaire, this inequality holds true if and only if

$$\text{either} \quad \begin{cases} -x^2 \geq 0 \\ x+1 > 0 \end{cases} \qquad \text{or} \quad \begin{cases} -x^2 \leq 0 \\ x+1 < 0 \end{cases}$$

The system

$$\begin{cases} -x^2 \geq 0 \\ x+1 > 0 \end{cases}$$

holds true only for $x = 0$. In the system

$$\begin{cases} -x^2 \leq 0 \\ x+1 < 0 \end{cases}$$

the inequality $-x^2 \leq 0$ holds true for any x, therefore this system holds true for $x < -1$.

The final answer: the inequality $\dfrac{x}{x+1} \geq x$ is satisfied by a number x if and only if either $x < -1$ or $x = 0$. In other words, the set of solutions of this inequality is $(-\infty, -1) \cup \{0\}$.

Comments on students' solutions.
Most of the answers (16 out of 25) were wrong.

9 works contain a correct answer. However the solutions were either not presented at all or were too messy to follow.

Many of the attempts to solve the inequality were based on evaluating left and right hand sides at few points and guess the answer. This method is not sane in general, but for this specific inequality it works especially bad.

It may work well in a situation of multiple choice, when you have to choose the correct solution from a finite collection of candidates.

Multiple choice tests direct study of mathematics wrongly!

3. What is wrong in the following solution of equation $x^2 = 9$?

$x^2 = 9 \implies x = \sqrt{9} \implies x = \pm 3.$

Correct answer. Both implications are wrong. The implication $x^2 = 9 \implies x = \sqrt{9}$ is wrong because it may happen that $x^2 = 9$, but $x = -\sqrt{9}$. The implication $x = \sqrt{9} \implies x = \pm 3$ is wrong, because $\sqrt{9} = 3$.

The whole proof can be corrected by inserting $\pm$ in front of $\sqrt{9}$ in the middle formula: $x^2 = 9 \implies x = \pm\sqrt{9} \implies x = \pm 3.$

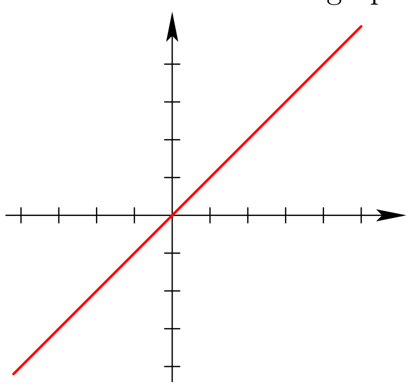
4. Draw the graph of the following functions:

(a) $y = |x - 1|$

(b) $y = |x - 1| - 1$

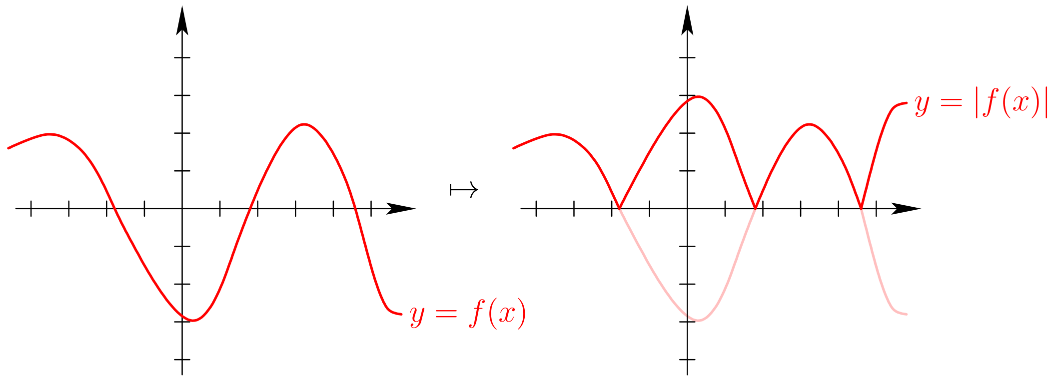
(c) $y = ||x - 1| - 1|$

Correct solution. Let us start with the well-known graph of $y = x$.

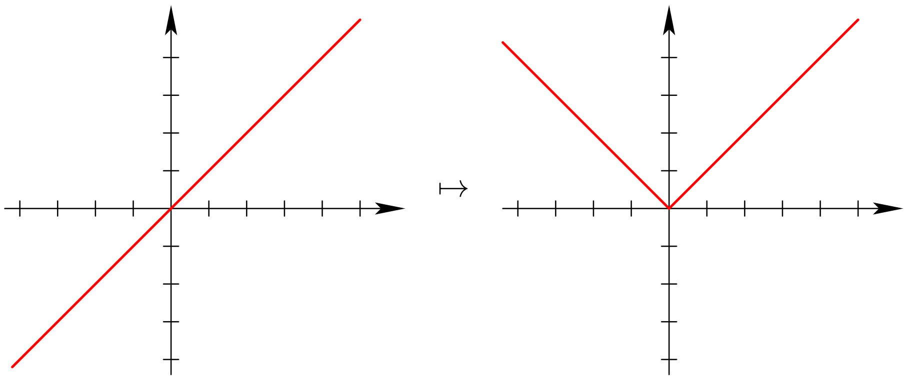


The next function whose graph we draw is $y = |x|$. We will use the following:

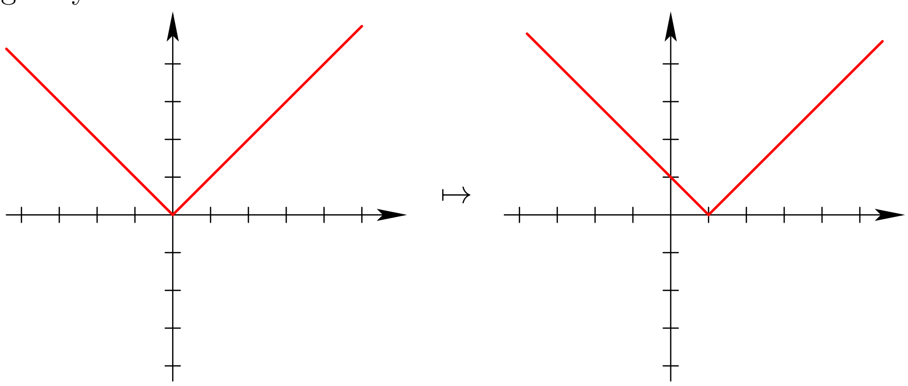
Graph Rule 1. *The graph of the function $y = |f(x)|$ can be obtained from the graph of the function $y = f(x)$ by flipping its bottom (everything below the x-axis) about the x-axis, keeping the upper part where it was.*



Applying this rule to $y = x$, we get the graph of $y = |x|$, which is hopefully familiar to you:

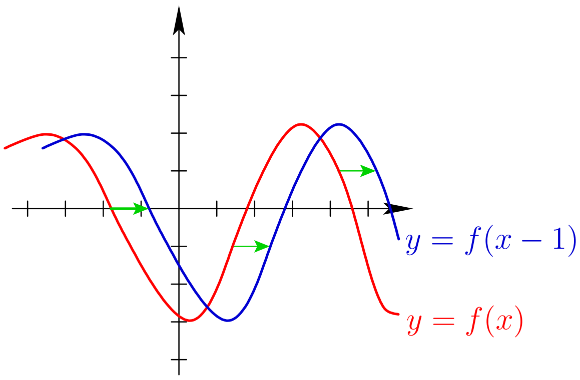


The graph of $y = |x - 1|$ can be obtained from the graph of $y = |x|$ by translation to the right by 1:

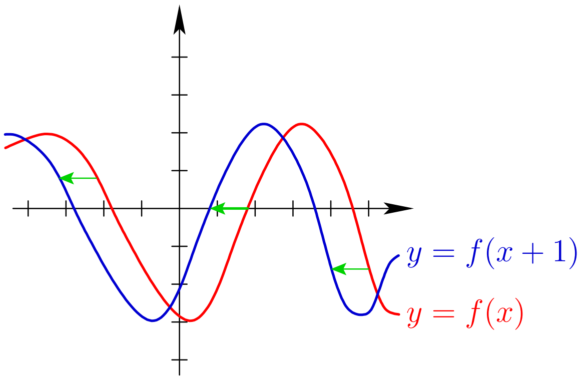


This is an application of the following:

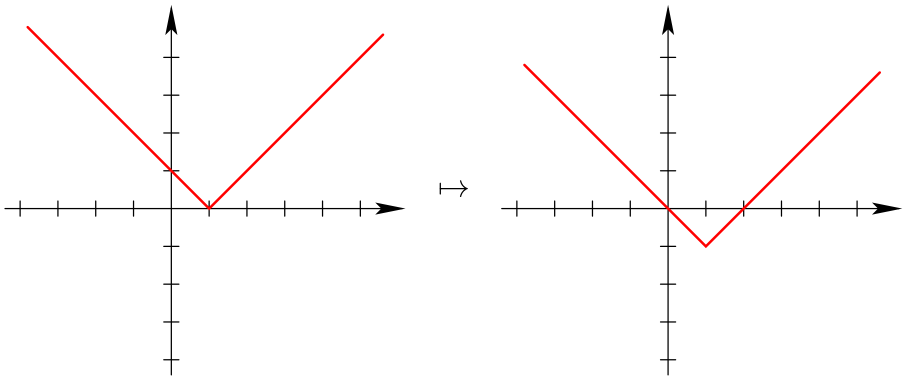
Graph Rule 2. *The graph of the function $y = f(x - a)$ can be obtained from the graph of the function $y = f(x)$ by translating it to the right by a.*



If $a > 0$, then Graph Rule 2 should be understood literally. If $a < 0$, then the translation should be made to the left by $-a$, which is interpreted as the translation to the right by a.

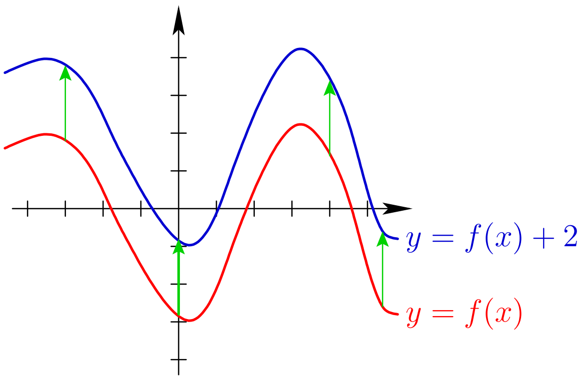


From the graph of function $y = |x-1|$ we obtain the graph of function $y = |x-1|-1$ by translating it downwards by 1:



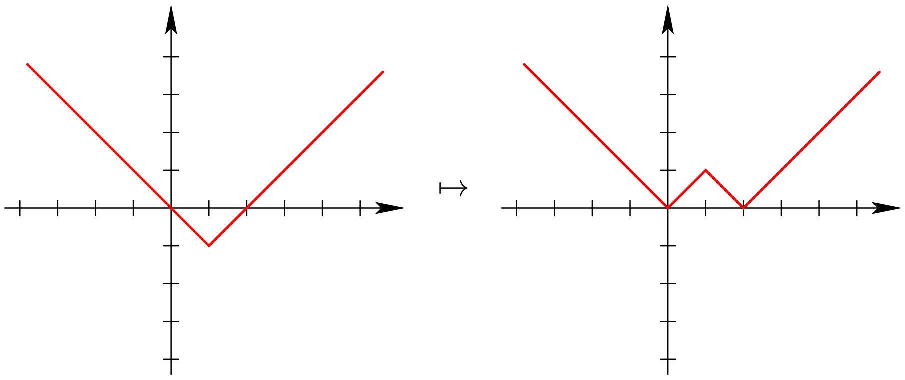
This is an application of the following:

Graph Rule 3. *The graph of the function* $y = f(x) + a$ *can be obtained from the graph of the function* $y = f(x)$ *by translating it upwards by* a*.*



Here, as in Graph Rule 2 above, a translation by a negative number a means a translation by the positive number $-a$ in the direction opposite to the one that was described.

Finally, the graph of $y = ||x-1|-1|$ is obtained from the graph of $y = |x-1|-1$ by applying Graph Rule 1:



Comments on students' solutions.
In most of the works (20 out of 25) the correct graphs are presented. In 6 of these works there is no attempt to justify the drawings. In the rest 14 attempts were made, however the justifications were nothing but evaluation of the functions in few points.

These attempts are far from being satisfactory. Through finitely many points one can draw infinitely many graphs. If you have no a priori idea about the shape of the graph, points that are found by evaluating the function do not help much.

In fact, in the five works where the correct answers were not found this method was the source of confusion and mistakes. One of the students expected that the graphs should look like parabola. Probably, this idea was suggested by my explanations about drawing of the graph of the function $y = (f(x))^2$ using the graph of $y = f(x)$. Points of the graph earned by honest calculations of the functions for 5 values of x did not help the student to reject this wrong idea.

The functions in the problem 4 are quite simple. If the functions were more complicated, then the number of wrong results would be much higher.

In the correct solution given above, the graph rules provide easy and complete proofs that the graphs are correct. No evaluations of the functions are needed.

In real life it may happen that you do not know a formula which defines the function, but instead you can find values of this function at some points. For example, you make experiments with the purpose to find a relation between two magnitudes. The magnitudes may be of different nature, coming from Physics, Sociology, or whatever else. You suspect that the magnitudes are related to each other and you want to find the relation explicitly.

Then by making experiments you can find many points on the graph of the function relating the magnitudes, and the problem is to recover the function from a finite number of points on its graph. The problem is engraved by possible mistakes in measurement or influence of factors that are not taken into consideration. Making a finite number of experiments gives you a finite number of points presumably near the graph.

Problems of this kind are treated by methods of statistics. A solution begins with making a priori assumptions about the type of the function, and then you find the graph and the function with some degree of certainty.

We face here a different problem. Our problem is to draw the graph of a function defined by a known formula. This is much easier problem and it has a uniquely defined answer. An evaluation of the function at a point gives you a point of the graph. It helps to find the graph, so it can be considered a tool and used, but its application alone cannot give a reliable result. You should know something more about the function. Fortunately, many other tools are available, they are more powerful and can give the complete result.

There is one important situation in which an evaluation at a finite number of points works pretty well alone. This is a situation of a multiple choice test: you have to choose between a finite number of graphs the graph of a given function. Then you possess for free a rich information about the function: you know that its graph is one of a few given graphs. Then by evaluating the function at a point where the graphs differ solves the problem. Problems of this kind are artificial and do not motivate a study of other valuable methods.

Multiple choice tests direct study of mathematics wrongly!