Geometry for Teachers
MAT515, Fall 2010,
Lecture 3

Oleg Viro

September 8, 2010

Dropping perpendicular

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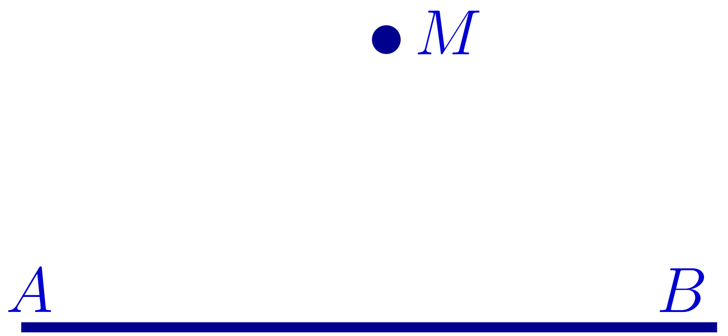
Let a line AB and an arbitrary point M outside the line be given.



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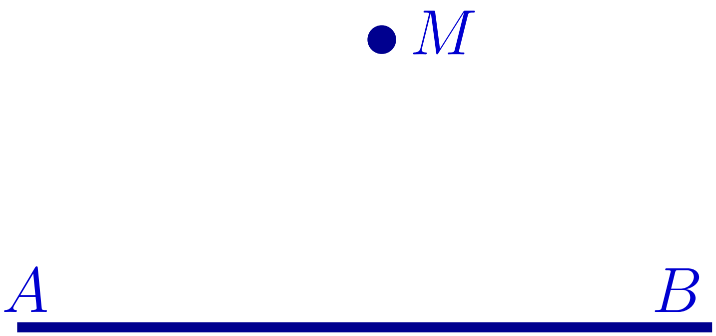
Theorem. *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Let a line AB and an arbitrary point M outside the line be given. Drop a perpendicular from M to AB.



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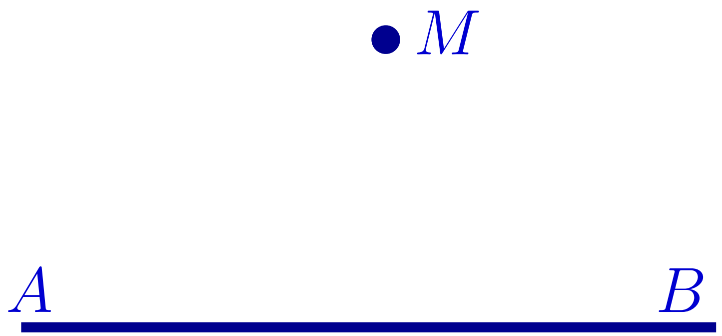
Apply the axial symmetry about AB.



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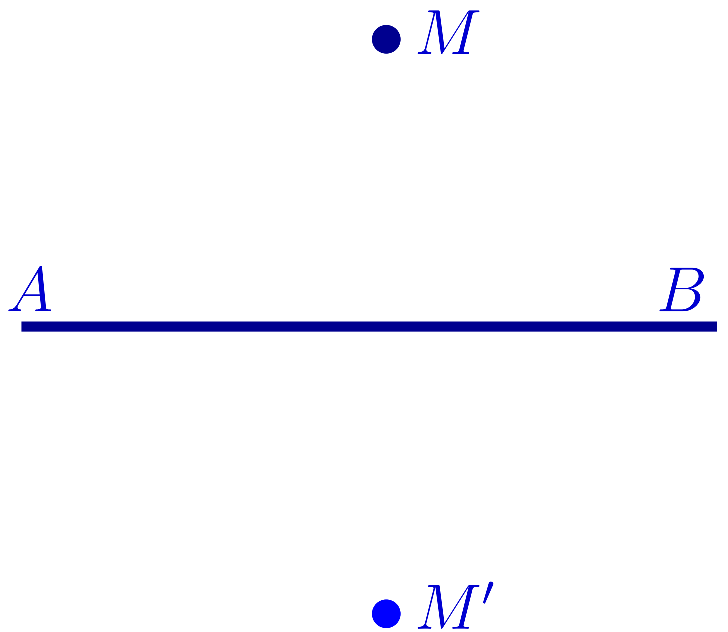

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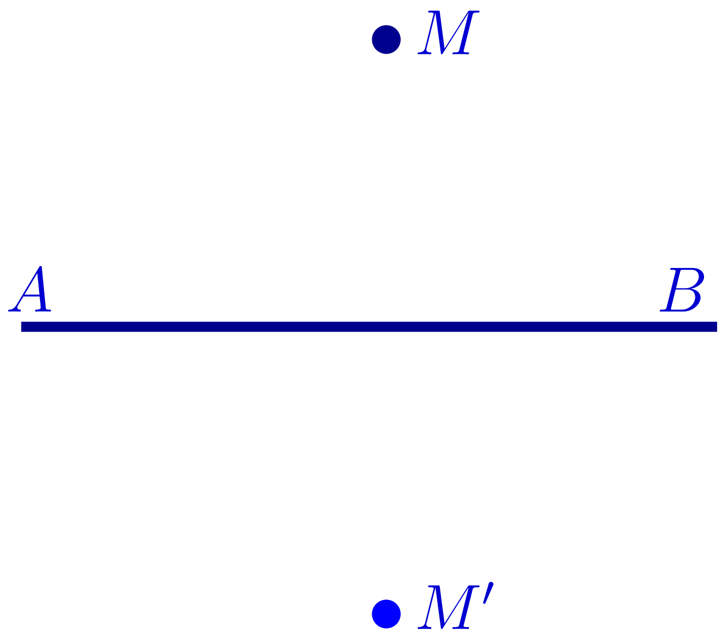
Apply the axial symmetry about AB.
Connect M and M' by a line.



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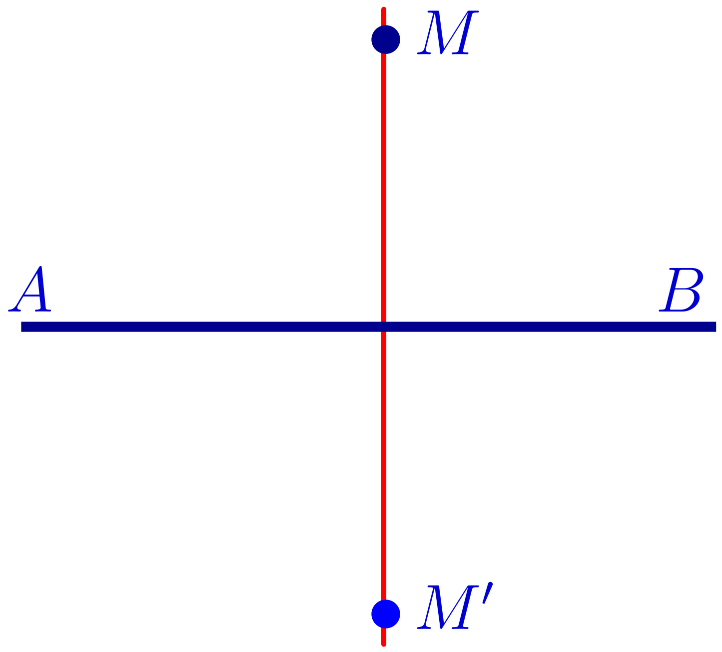

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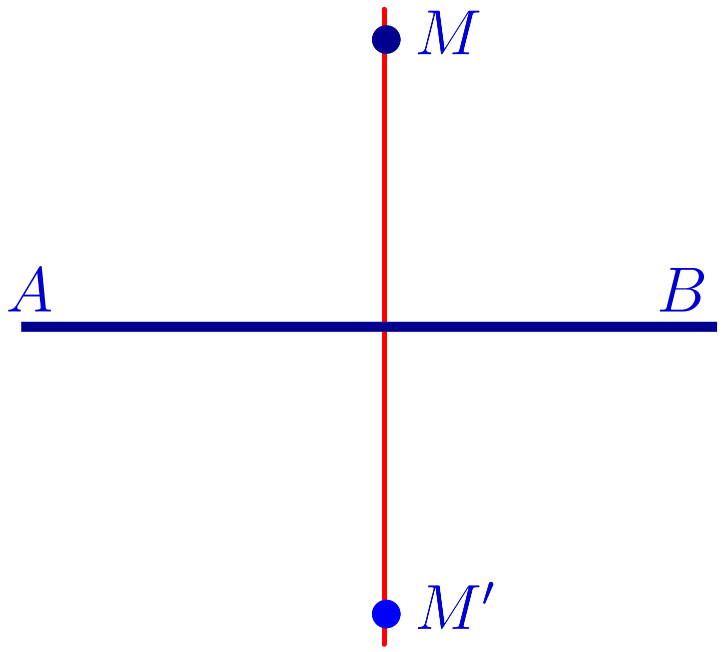
Prove that MM' is perpendicular to AB!



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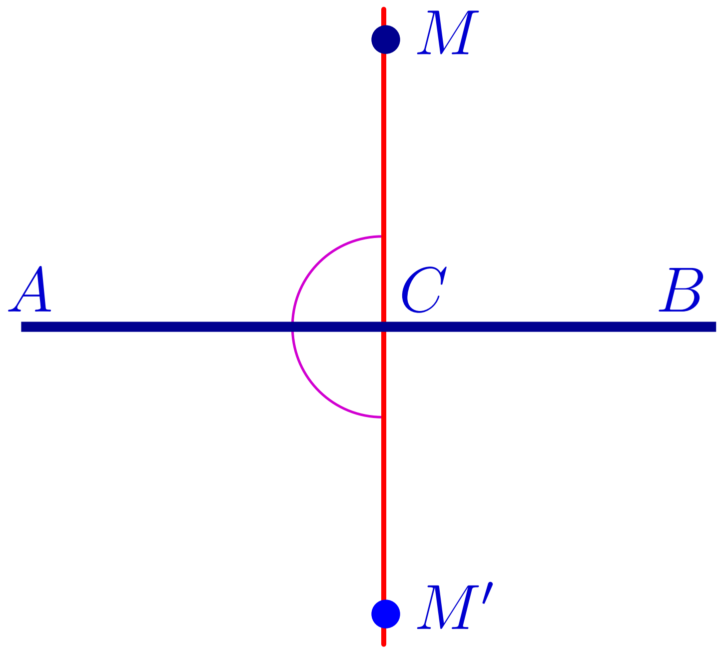
$\angle MCA = \angle ACM'$ as symmetric.



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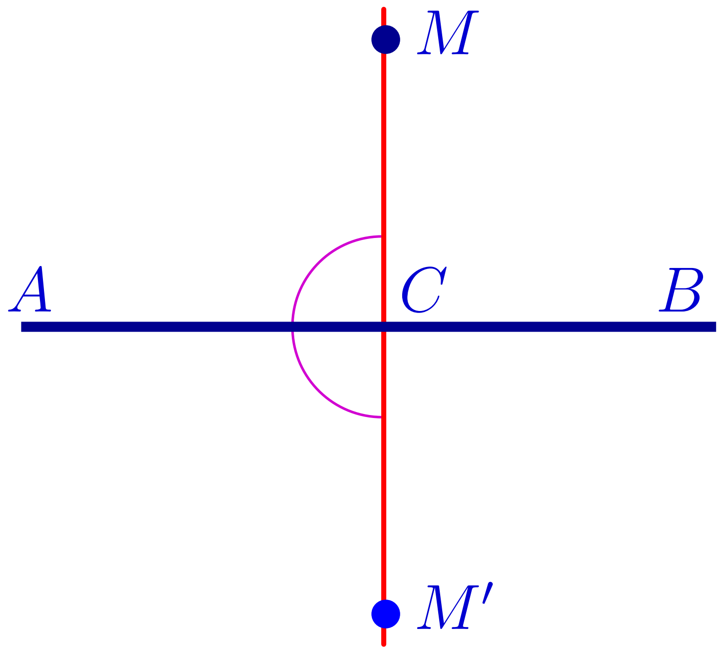
$\angle MCA = \angle ACM'$ as symmetric. $\angle MCA + \angle ACM' = 180°$ as $\angle MCA$ and $\angle ACM'$ are supplementary.



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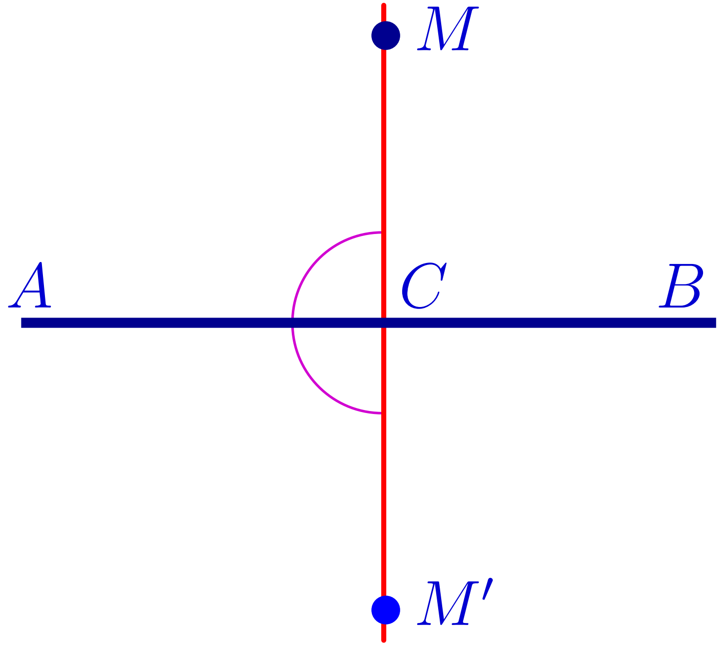
$\angle MCA = \angle ACM'$ as symmetric. $\angle MCA + \angle ACM' = 180°$ as $\angle MCA$ and $\angle ACM'$ are supplementary.

Hence $\angle MCA = \angle ACM' = 90°$ and $MM' \perp AB$.



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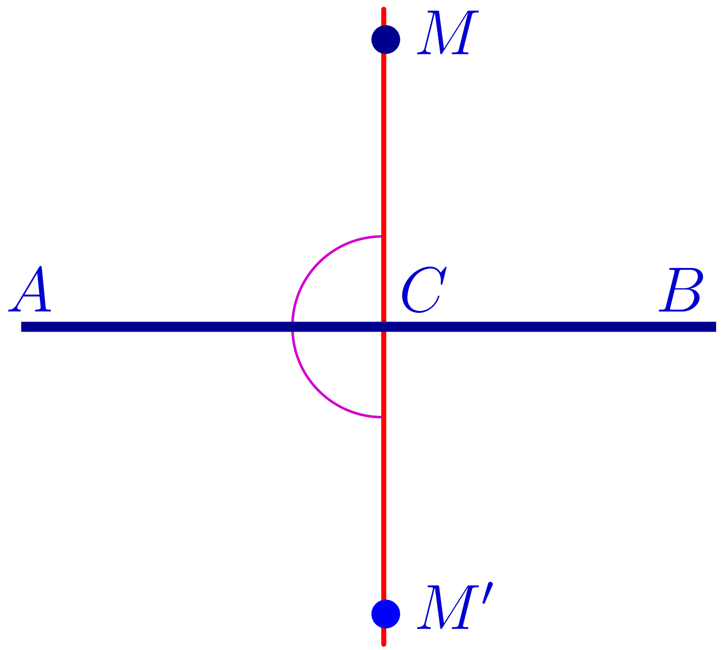
Uniqueness.



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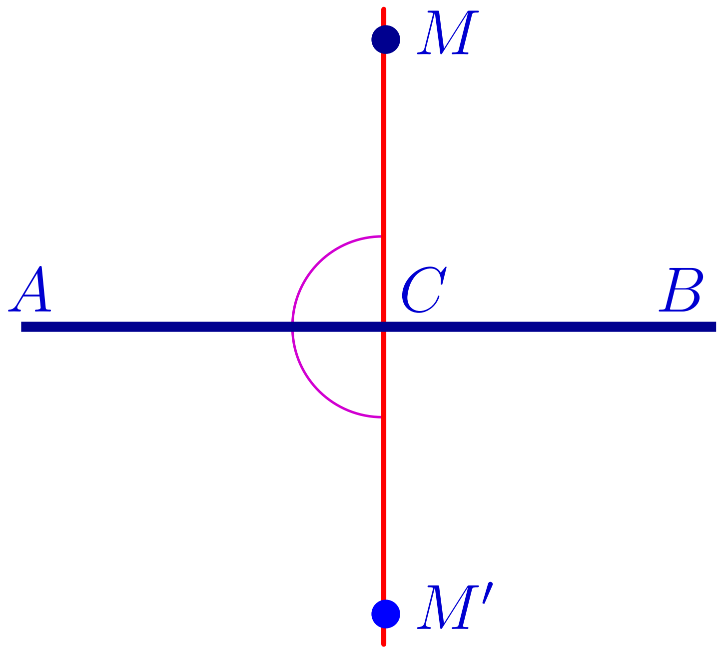
Theorem. *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Uniqueness. Assume there is another perpendicular MD.



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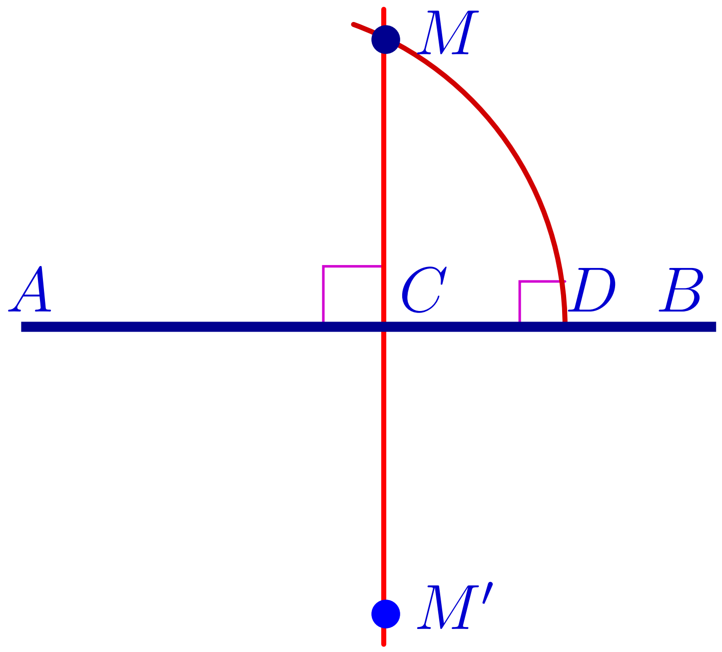

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Theorem. *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

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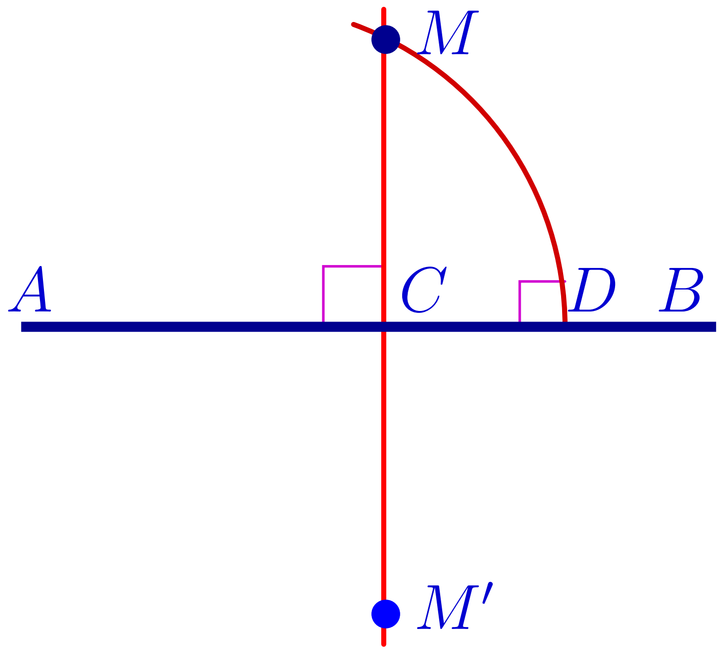

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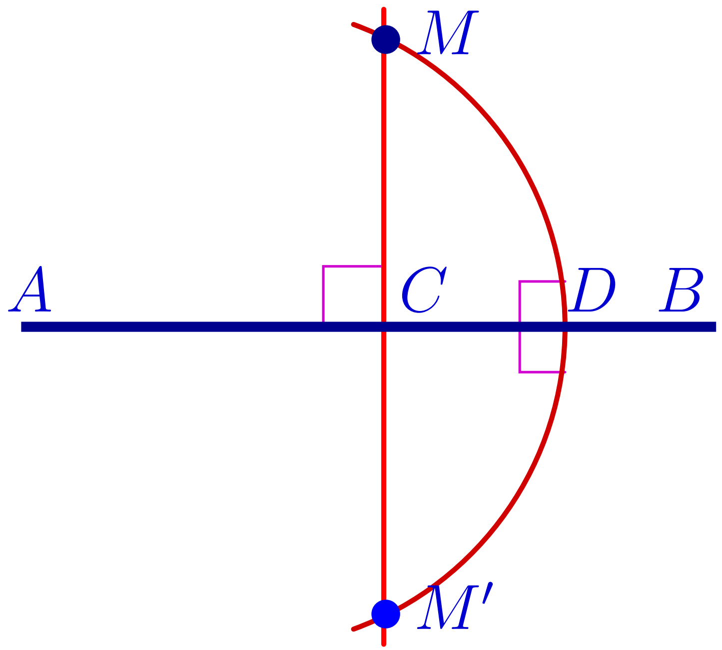

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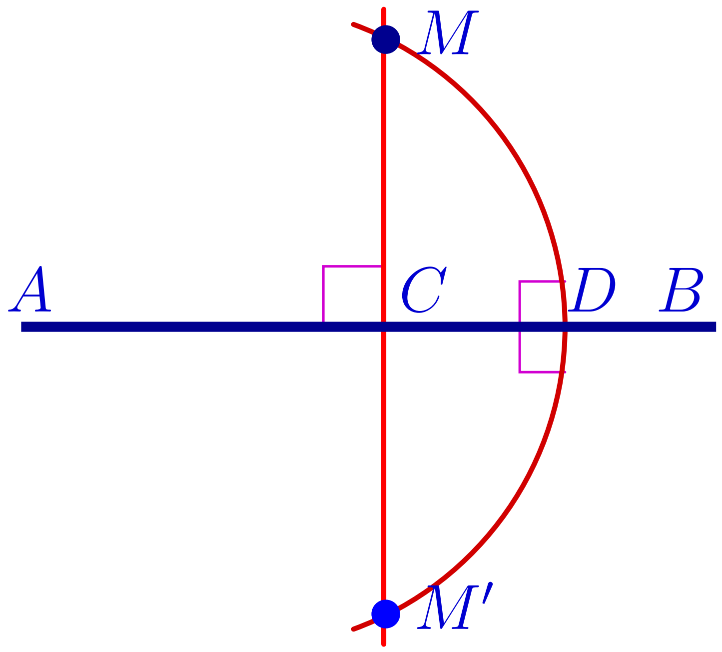
Angles $\angle MDA$ and $\angle ADM'$ are right, therefore $\angle MDM'$ is straight.



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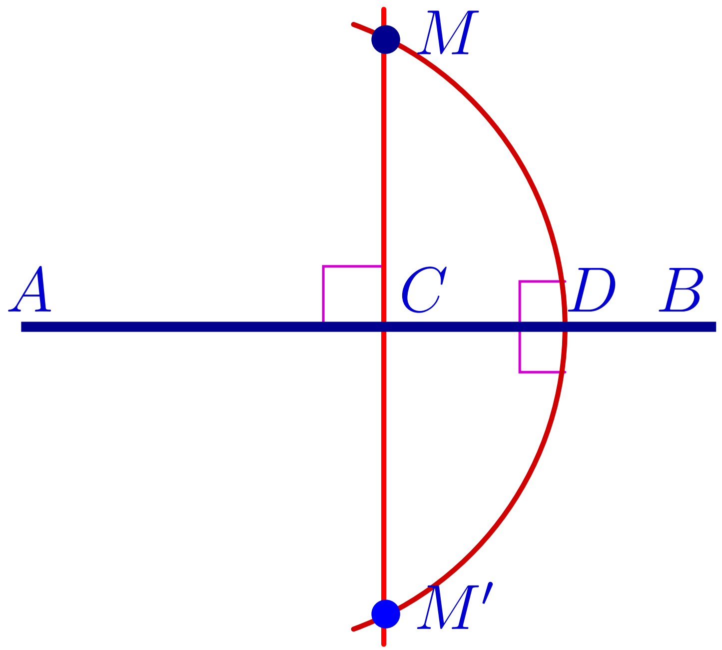

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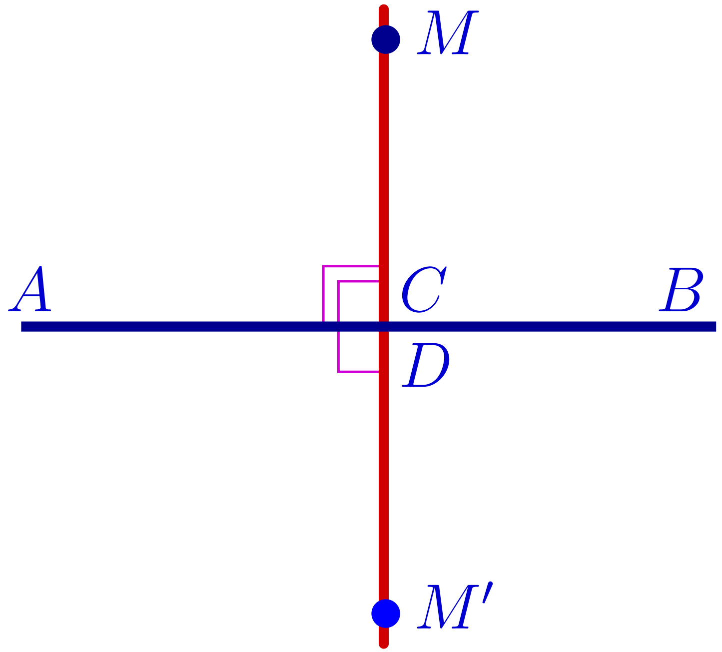
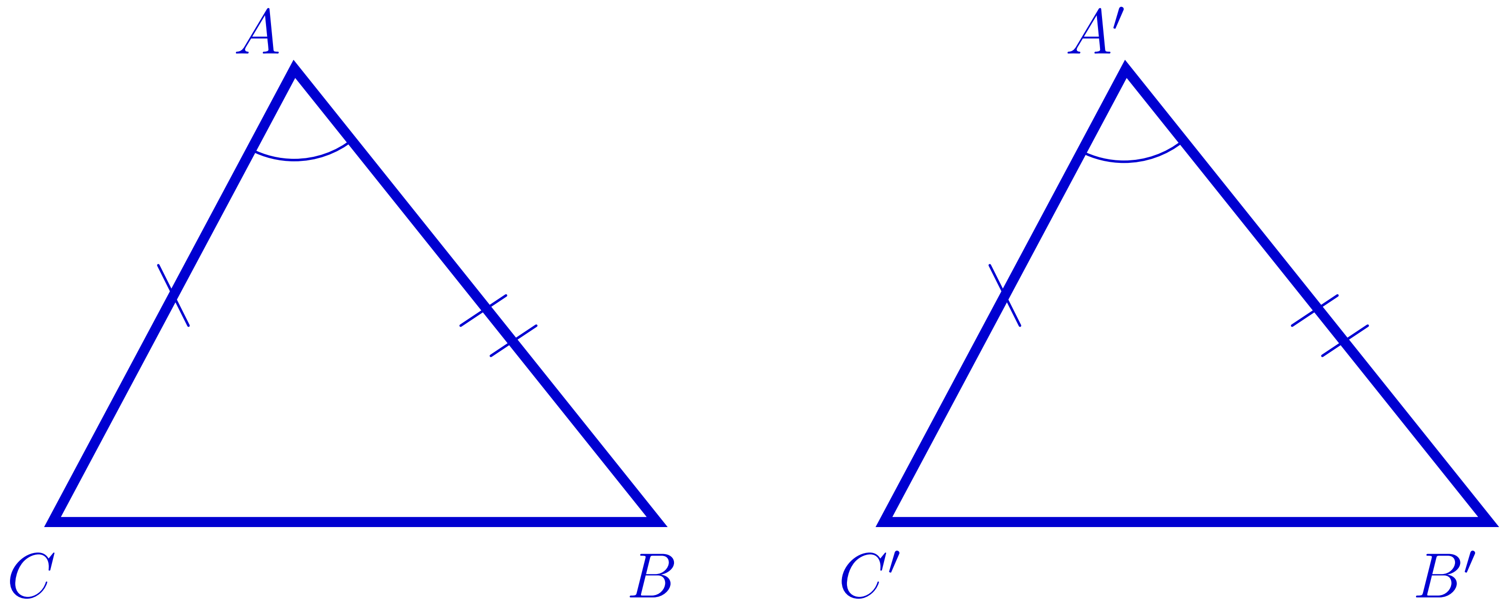


SAS-test

Theorem. *If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.*



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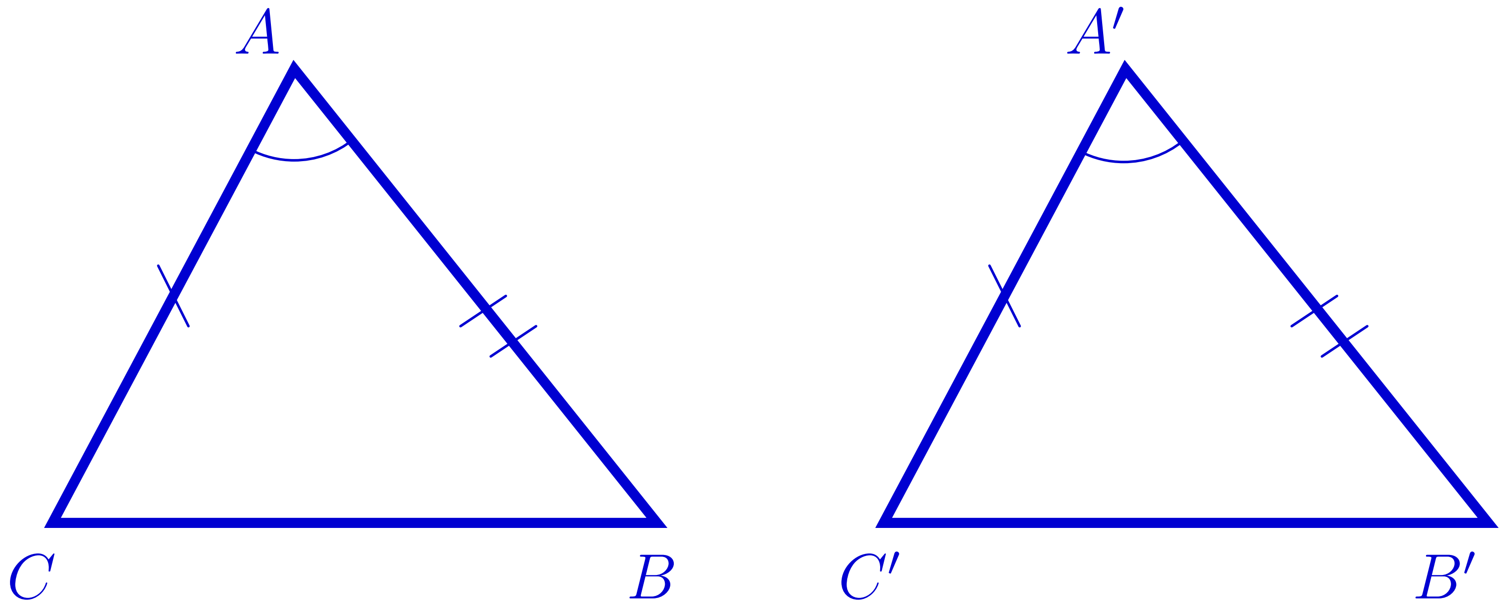
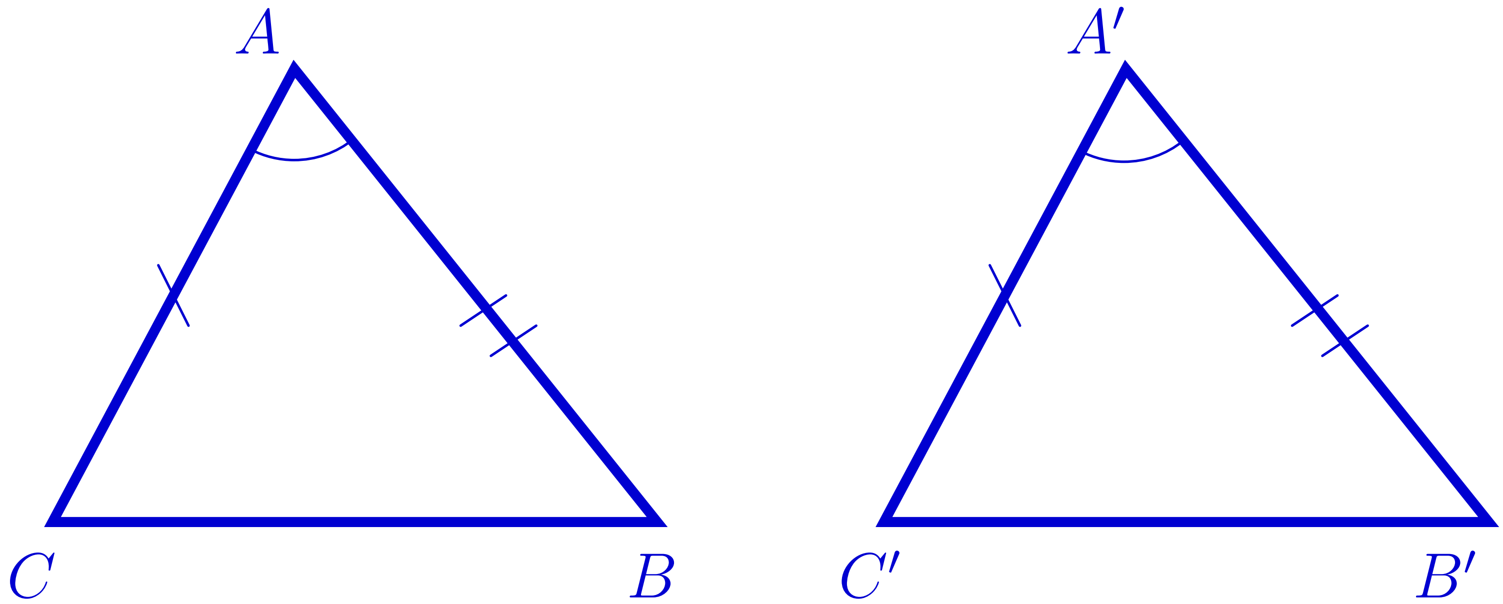
Theorem. *If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.*

Proof. Let ABC and $A'B'C'$ be triangles such that $AC = A'C', AB = A'B', \angle A = \angle A'$.



SAS-test

Superimpose $\triangle ABC$ onto $\triangle\ A'B'C'$ in such a way that A would coincide with A'



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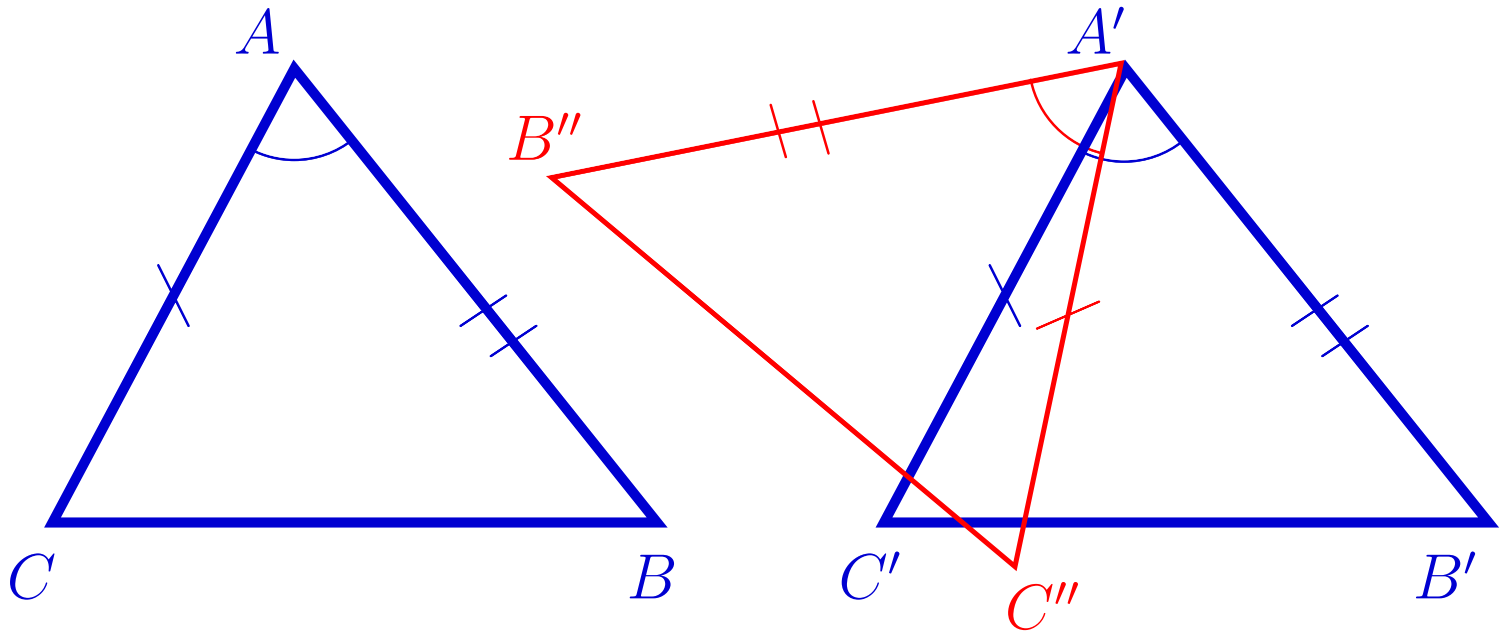

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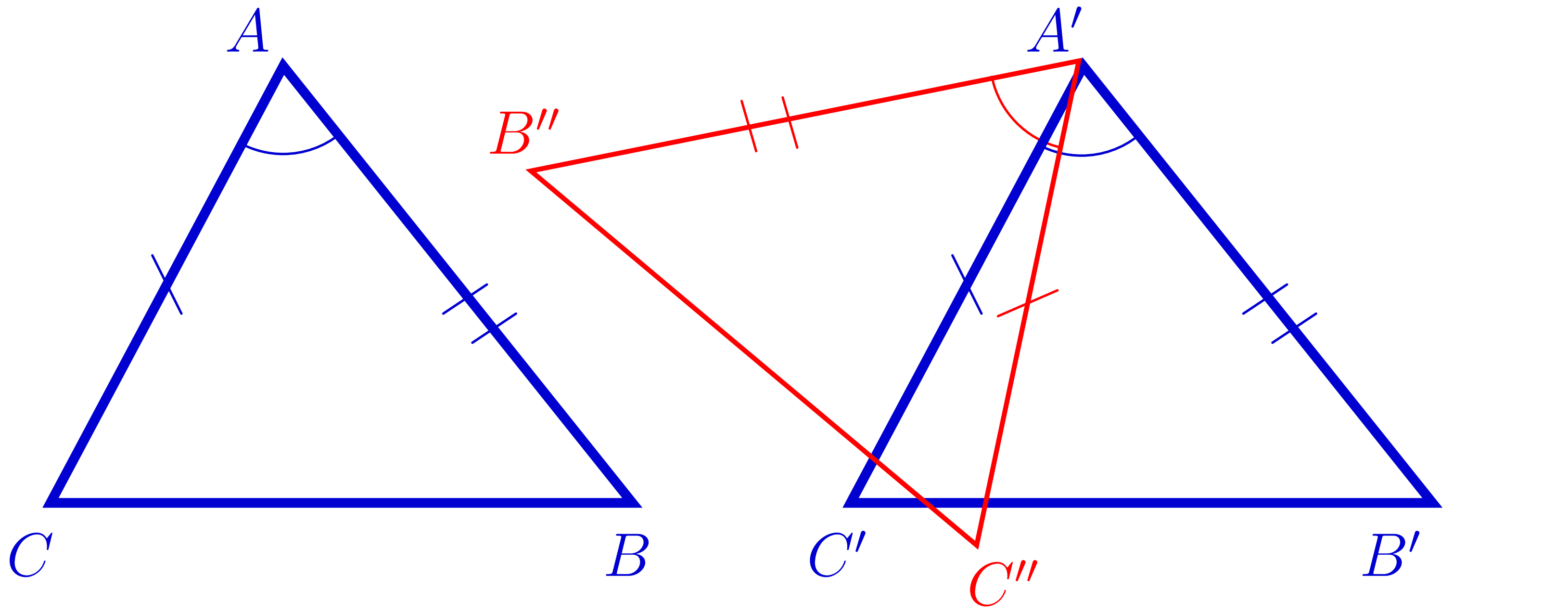
Superimpose $\triangle ABC$ onto $\triangle\ A'B'C'$ in such a way that A would coincide with A', the side AC would go along $A'C'$



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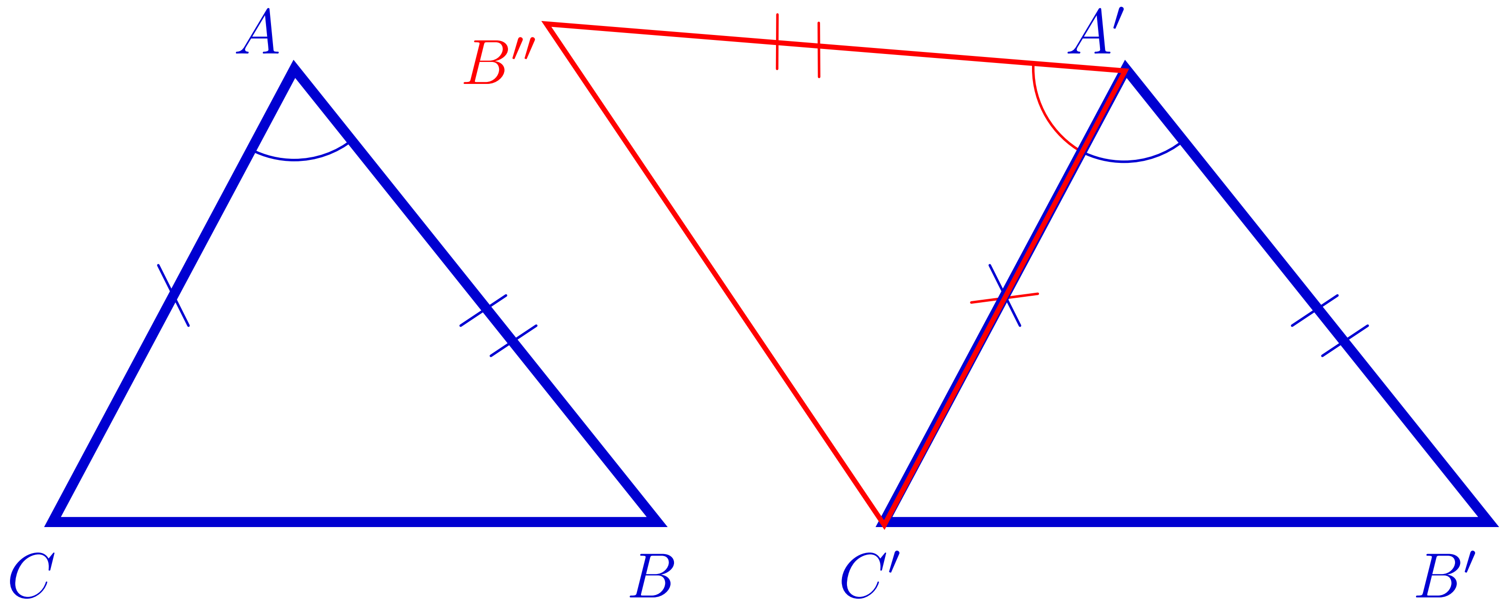

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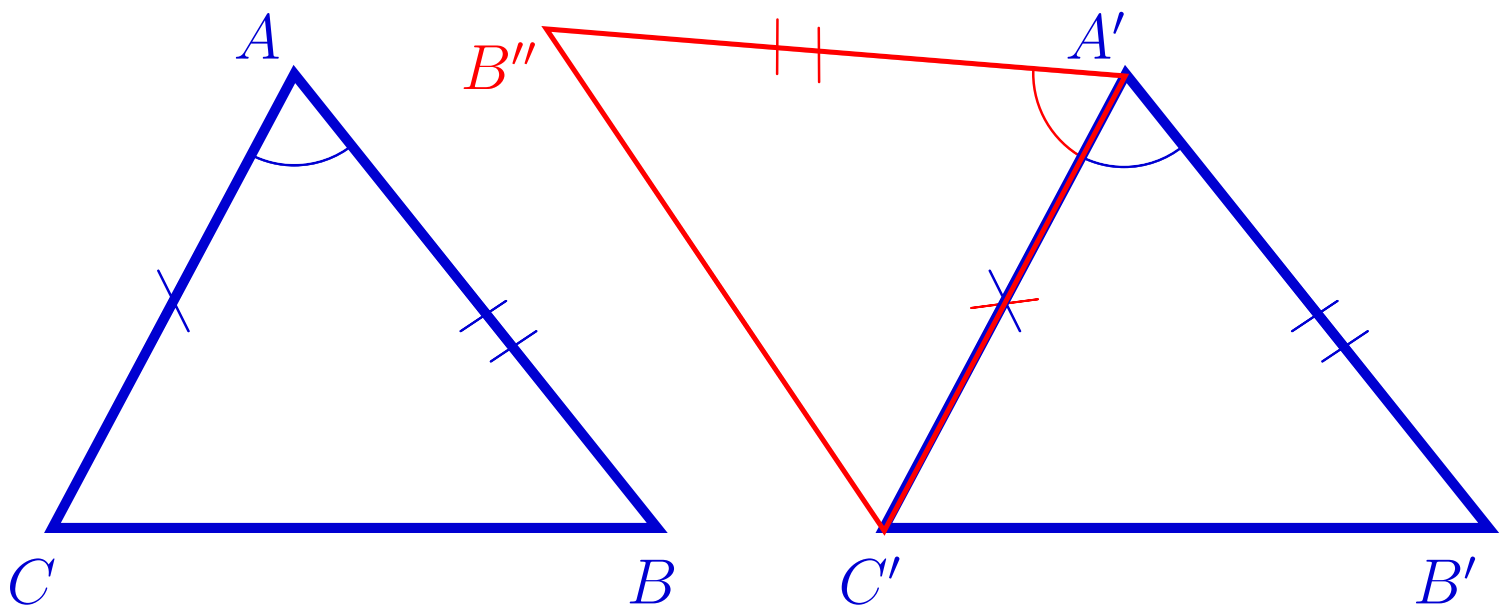
Superimpose $\triangle ABC$ onto $\triangle\ A'B'C'$ in such a way that A would coincide with A', the side AC would go along $A'C'$, and the side AB would lie on the same side of $A'C'$ as $A'B'$.



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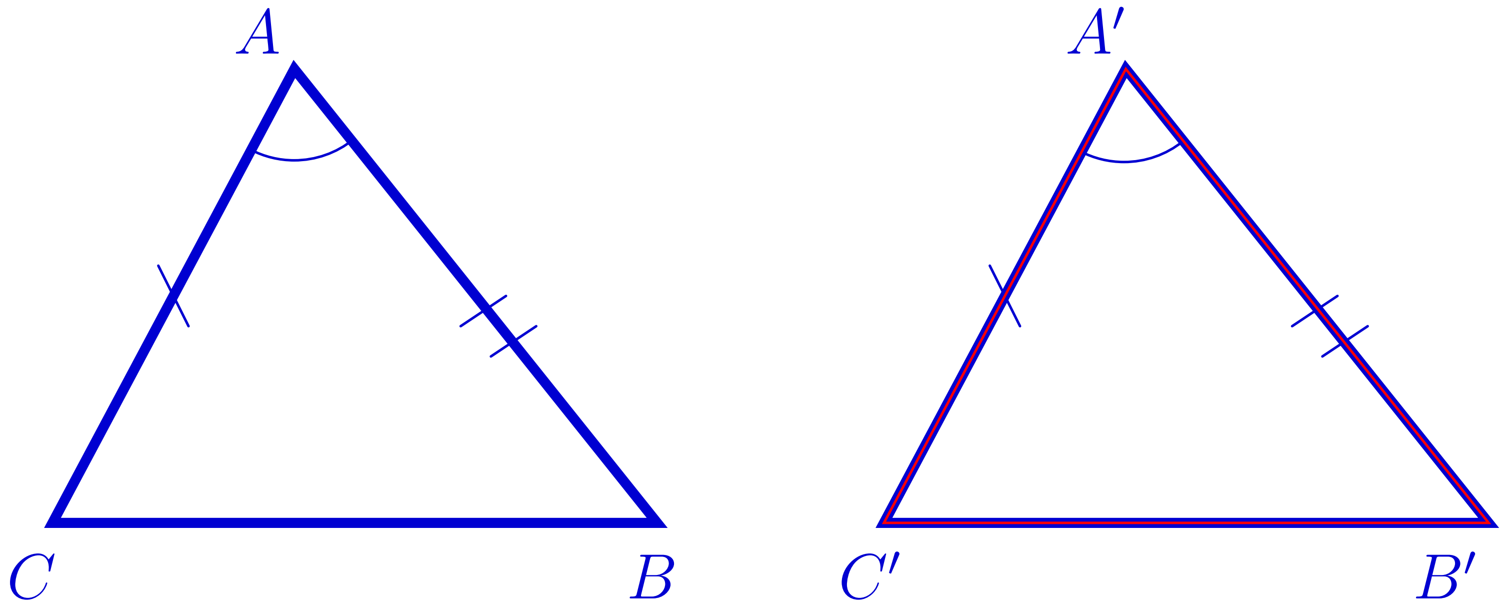

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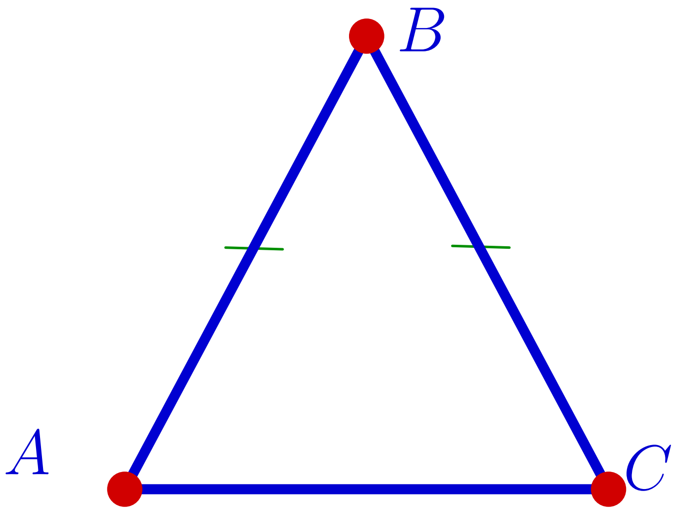

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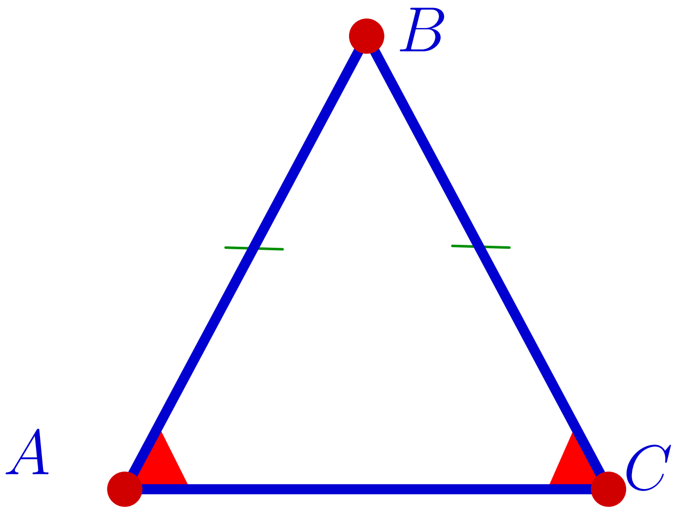

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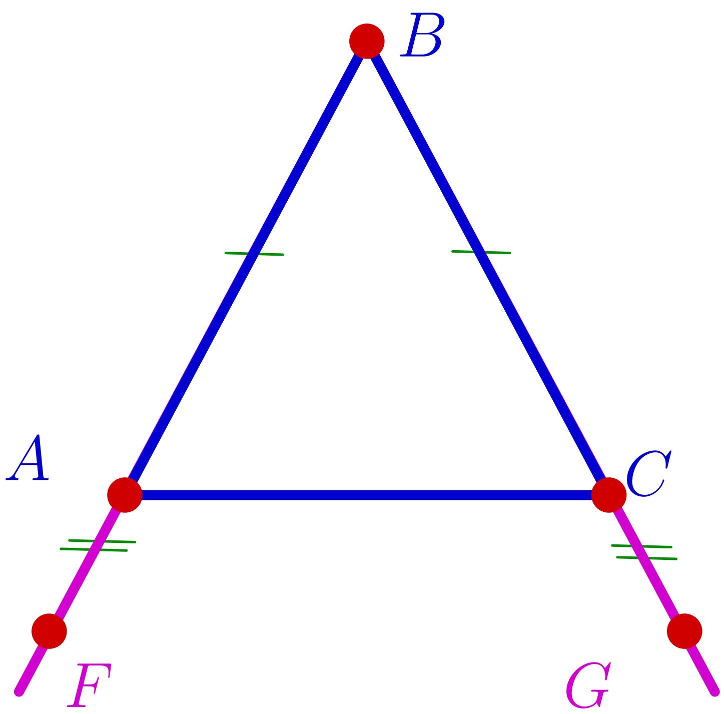

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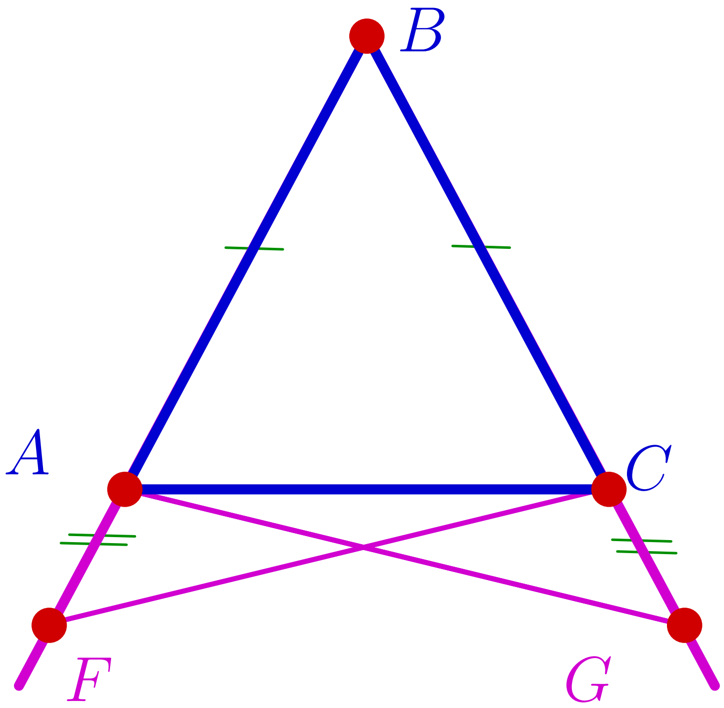

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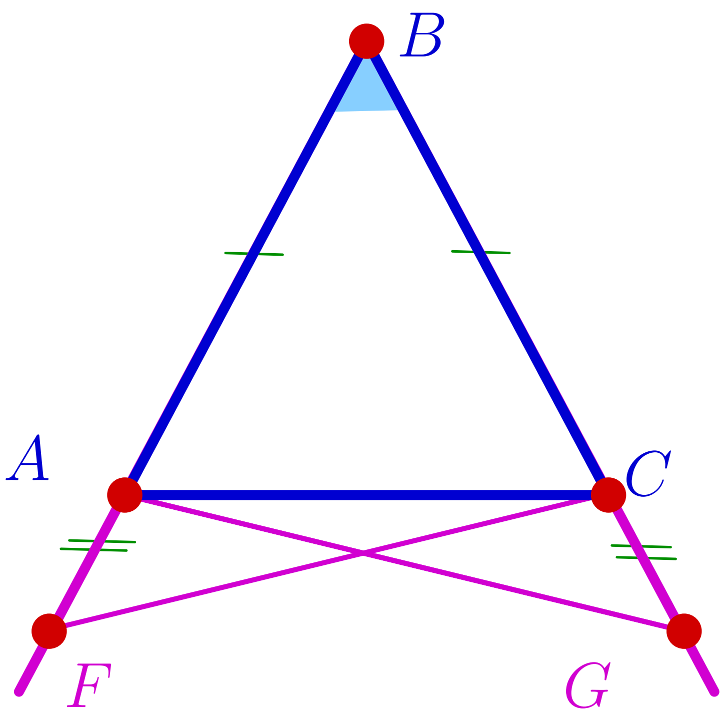

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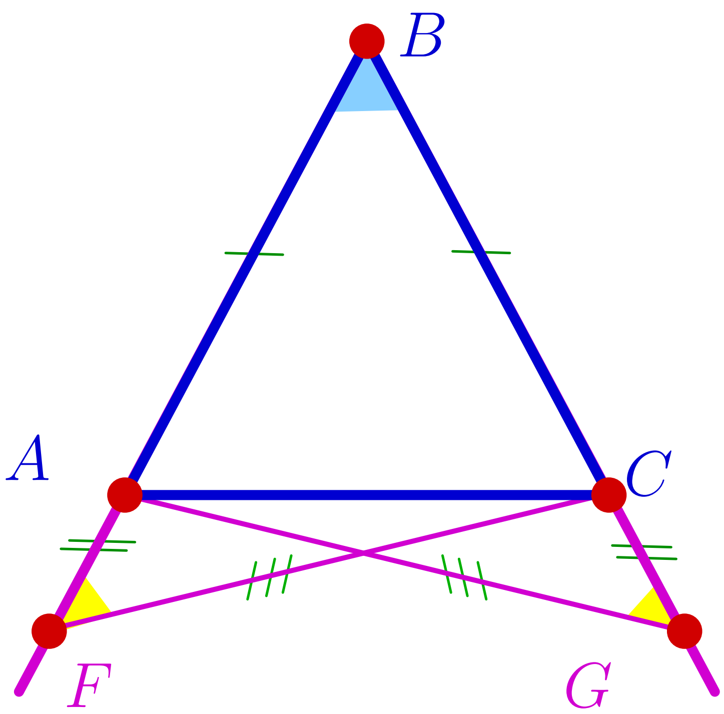

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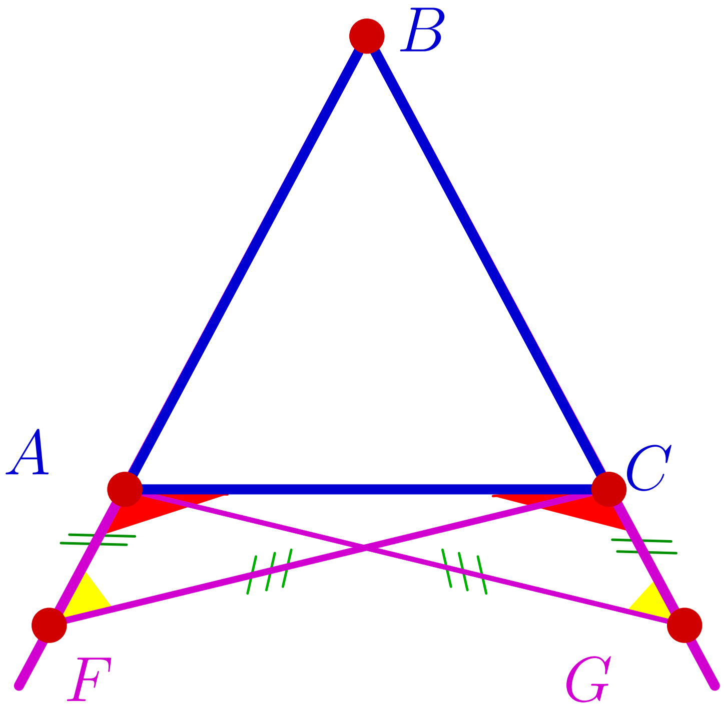

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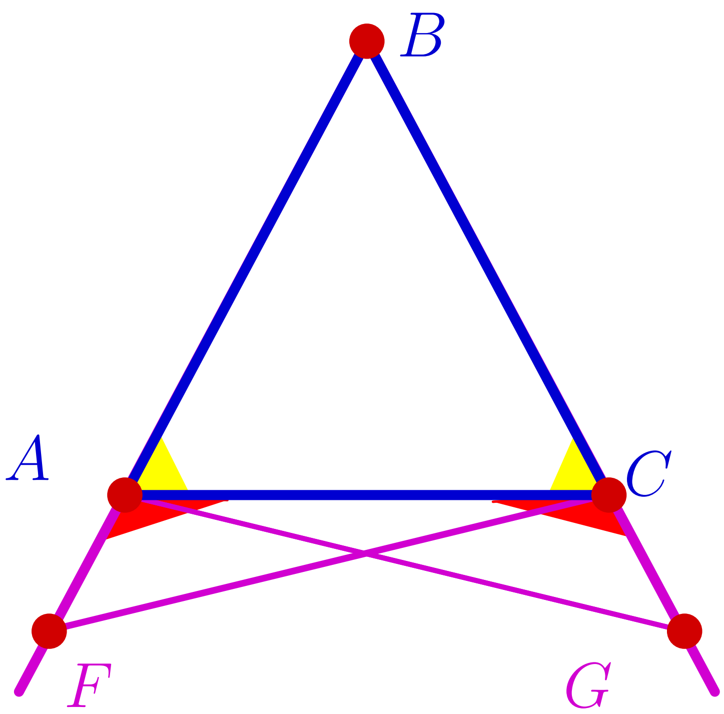
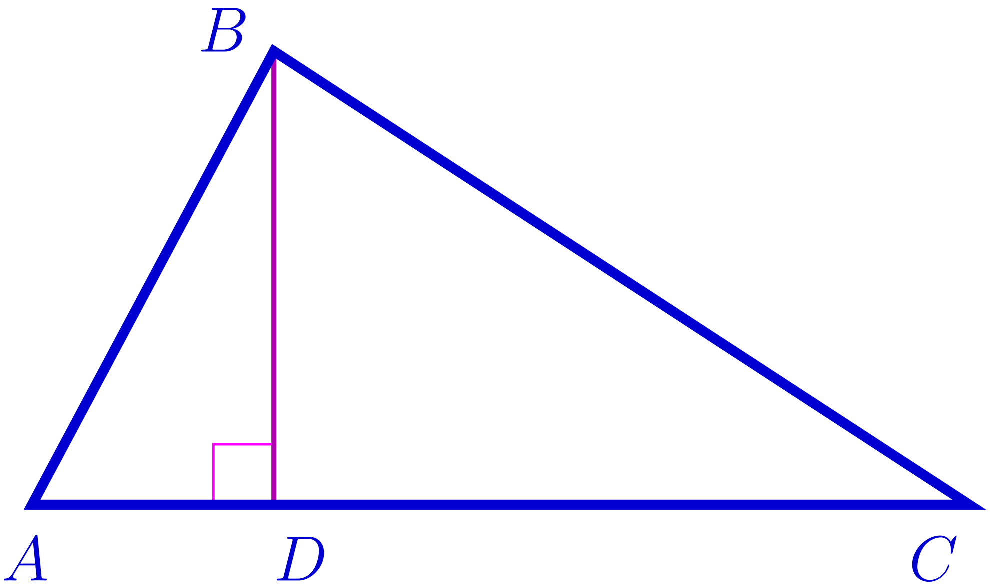
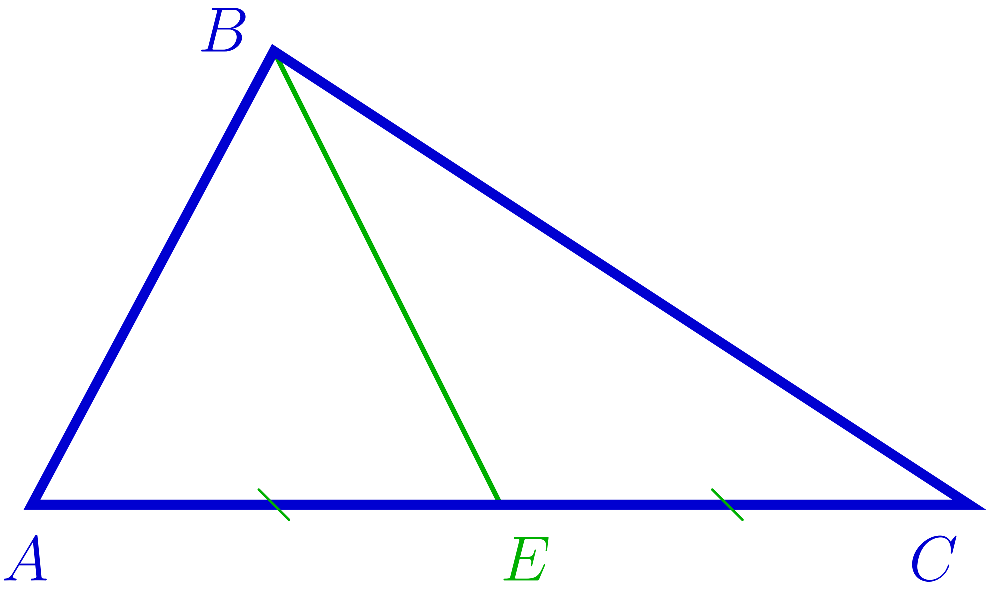


Lines in traingle



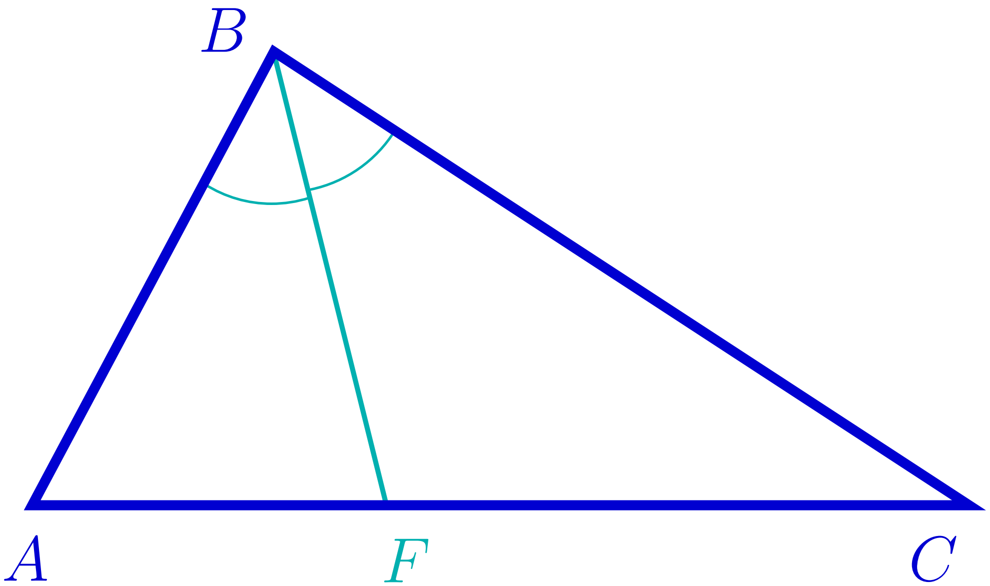
altitude BD

Lines in traingle



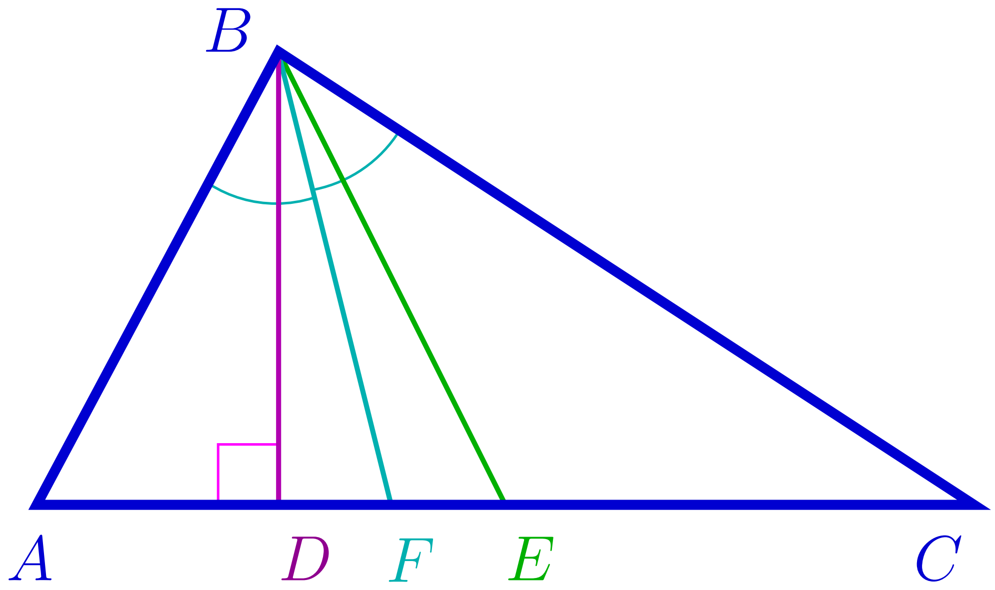
median BE

Lines in traingle



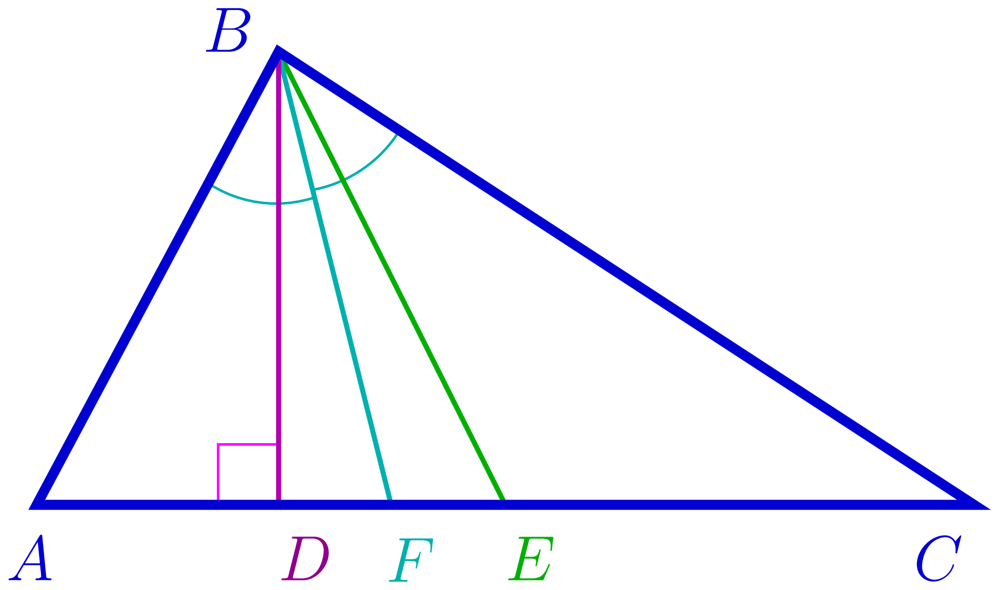
bisector BF

Lines in traingle



altitude BD, bisector BF, median BE

Lines in traingle



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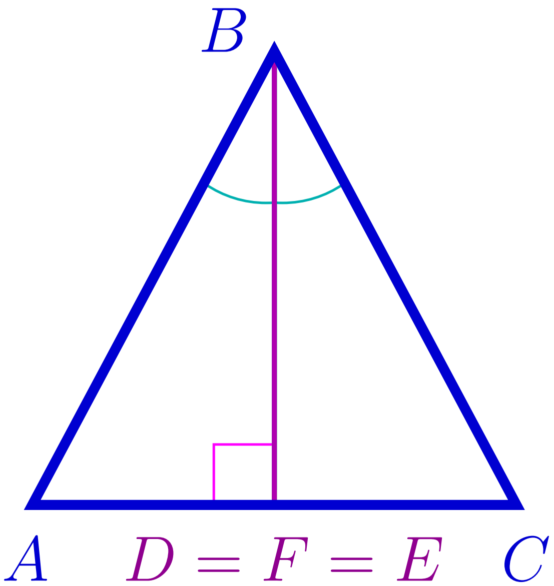
Theorem. *If the triangle is isosceles (i.e.,* AB *is congruent to* BC*), then* $D = F = E$ *and all three lines coincide.*

Lines in traingle

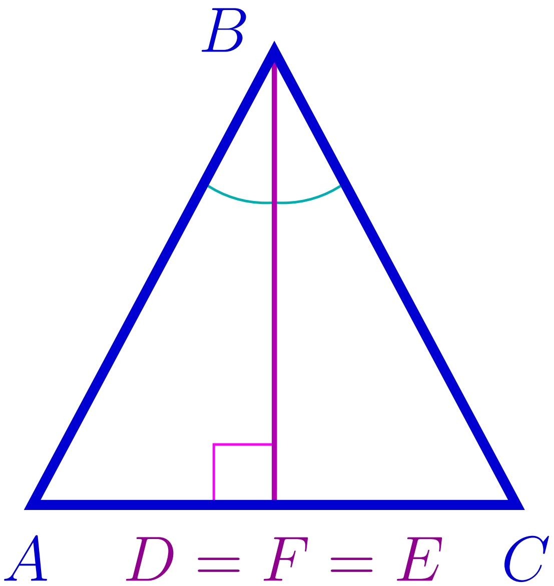


altitude = bisector = median

Theorem. *If the triangle is isosceles (i.e., AB is congruent to BC), then $D = F = E$ and all three lines coincide.*

Lemma. *If AB is congruent to BC, then the triangle ABC is symmetric about its bisector BF.*

Lines in traingle



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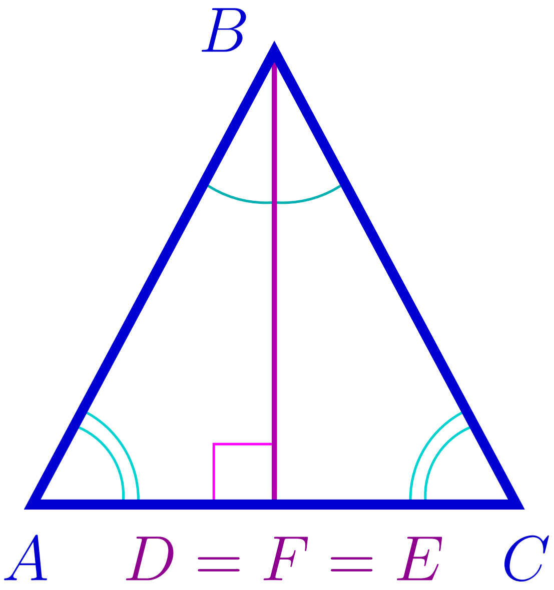

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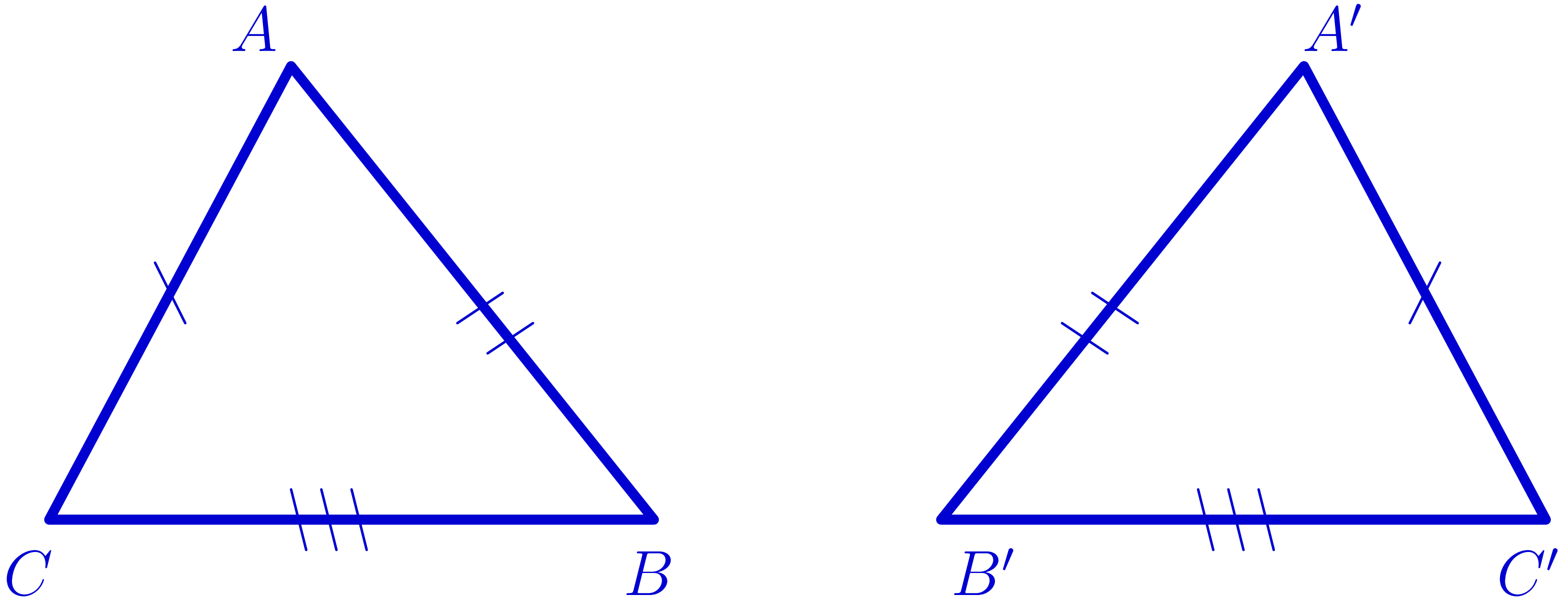
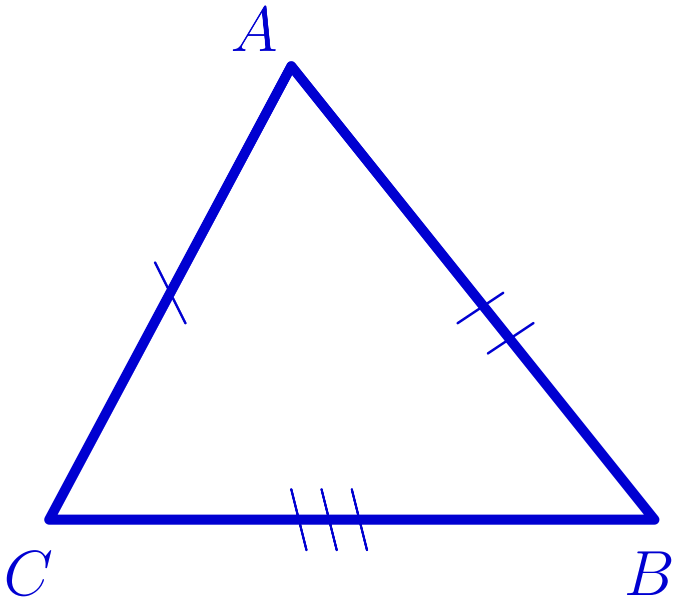
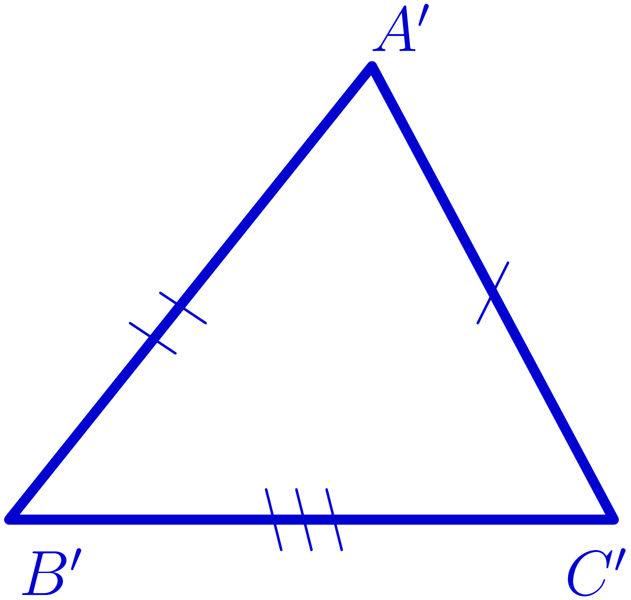


SSS-test

Juxtapose ABC and $A'B'C'$ in such a way that BC and $B'C'$ would coincide, and A and A' would lie on the opposite sides of $B'C'$.





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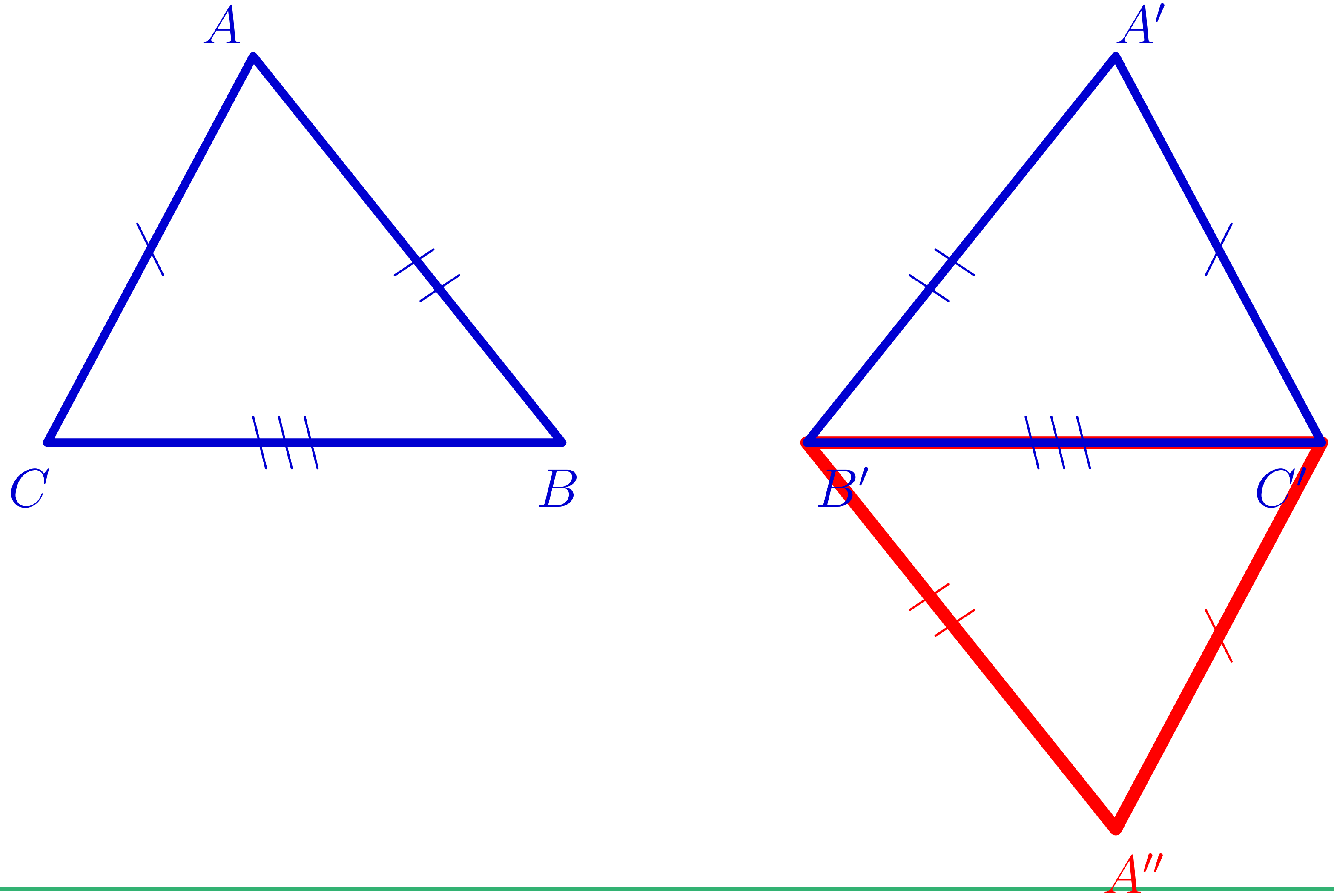
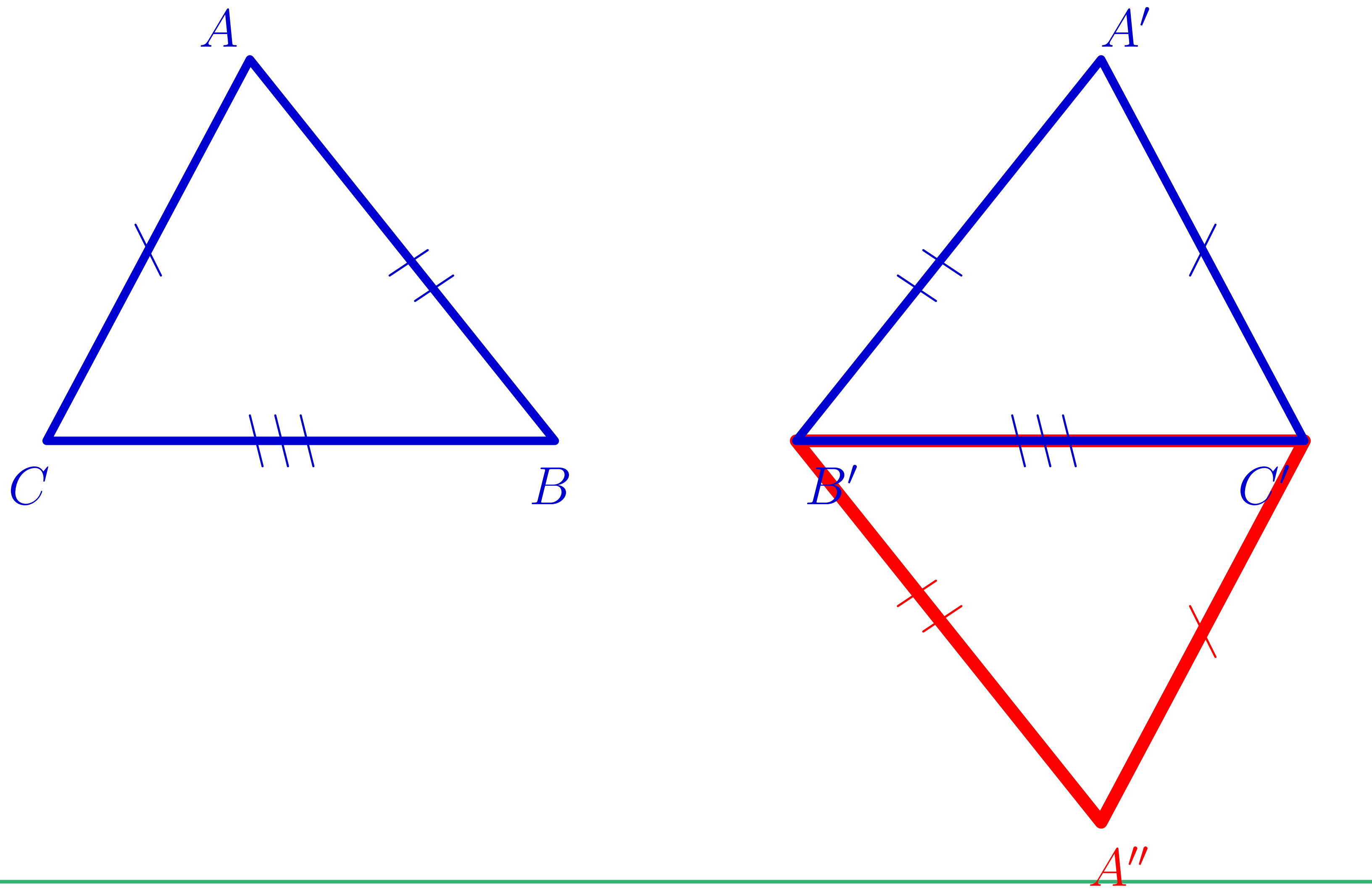


SSS-test

Joining A' and A'' we obtain isosceles triangles $A'B'A''$ and $A'C'A''$ with the common base $A'A''$.



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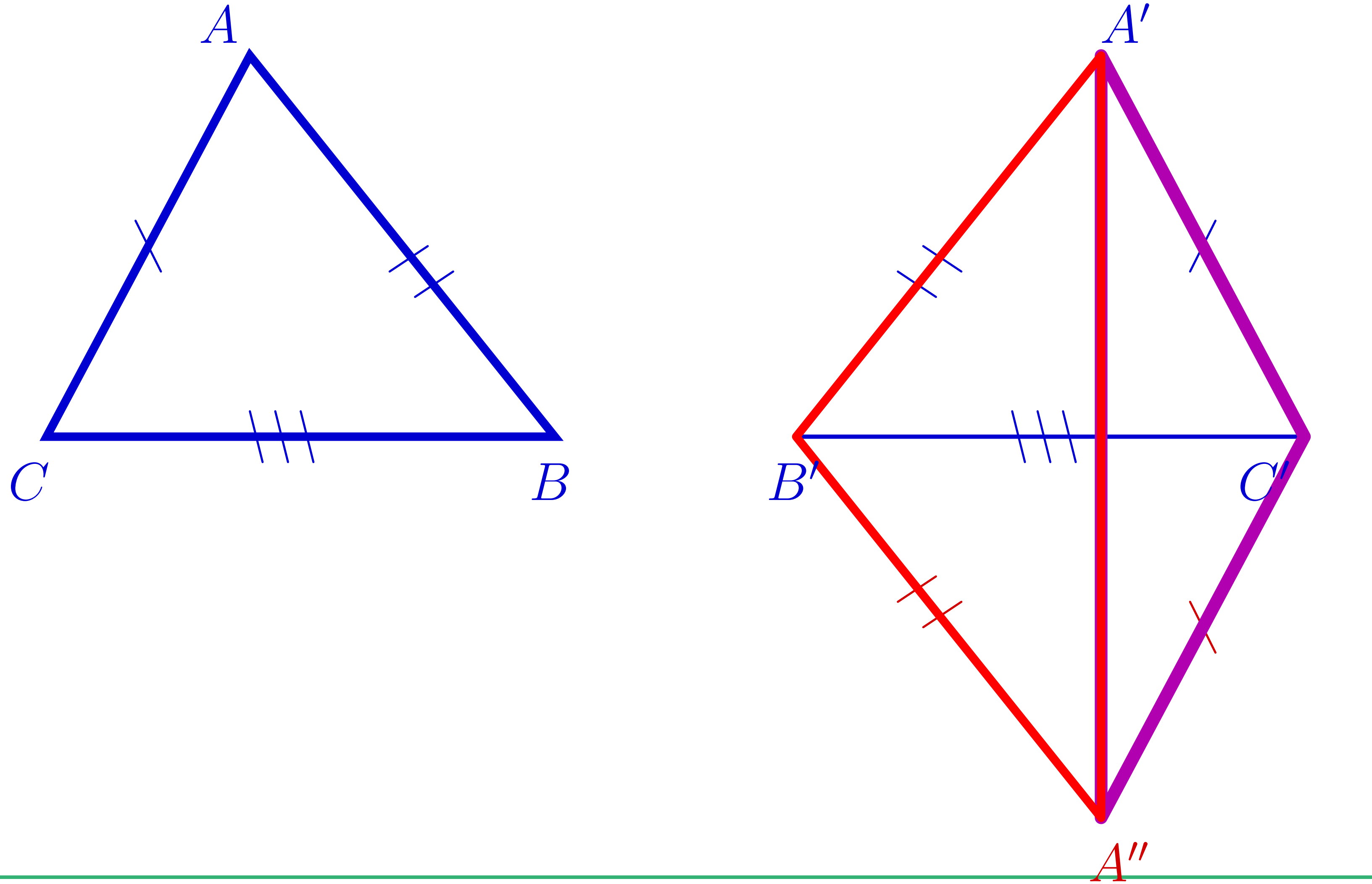
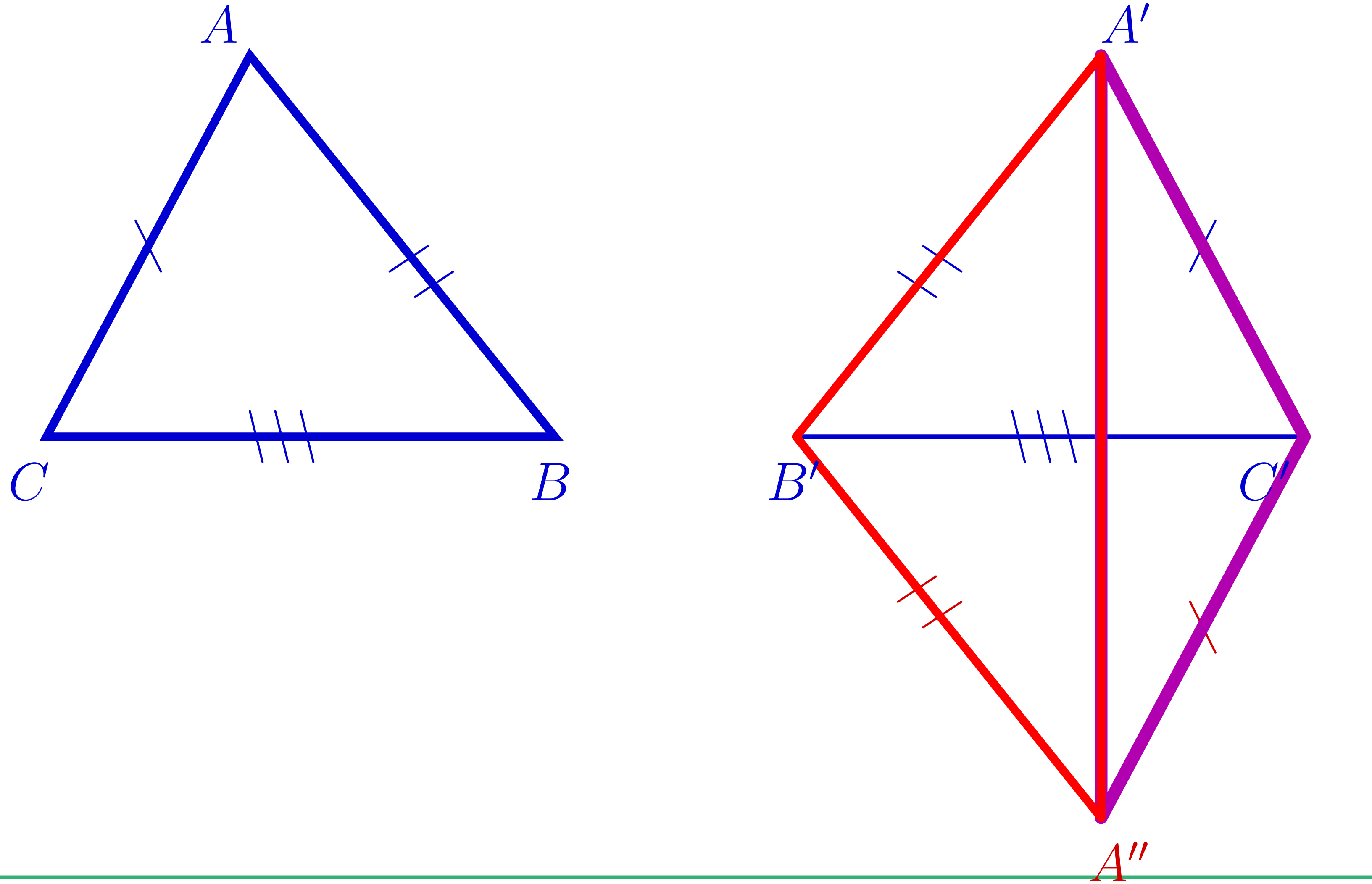


SSS-test

The angles at the base are congruent.



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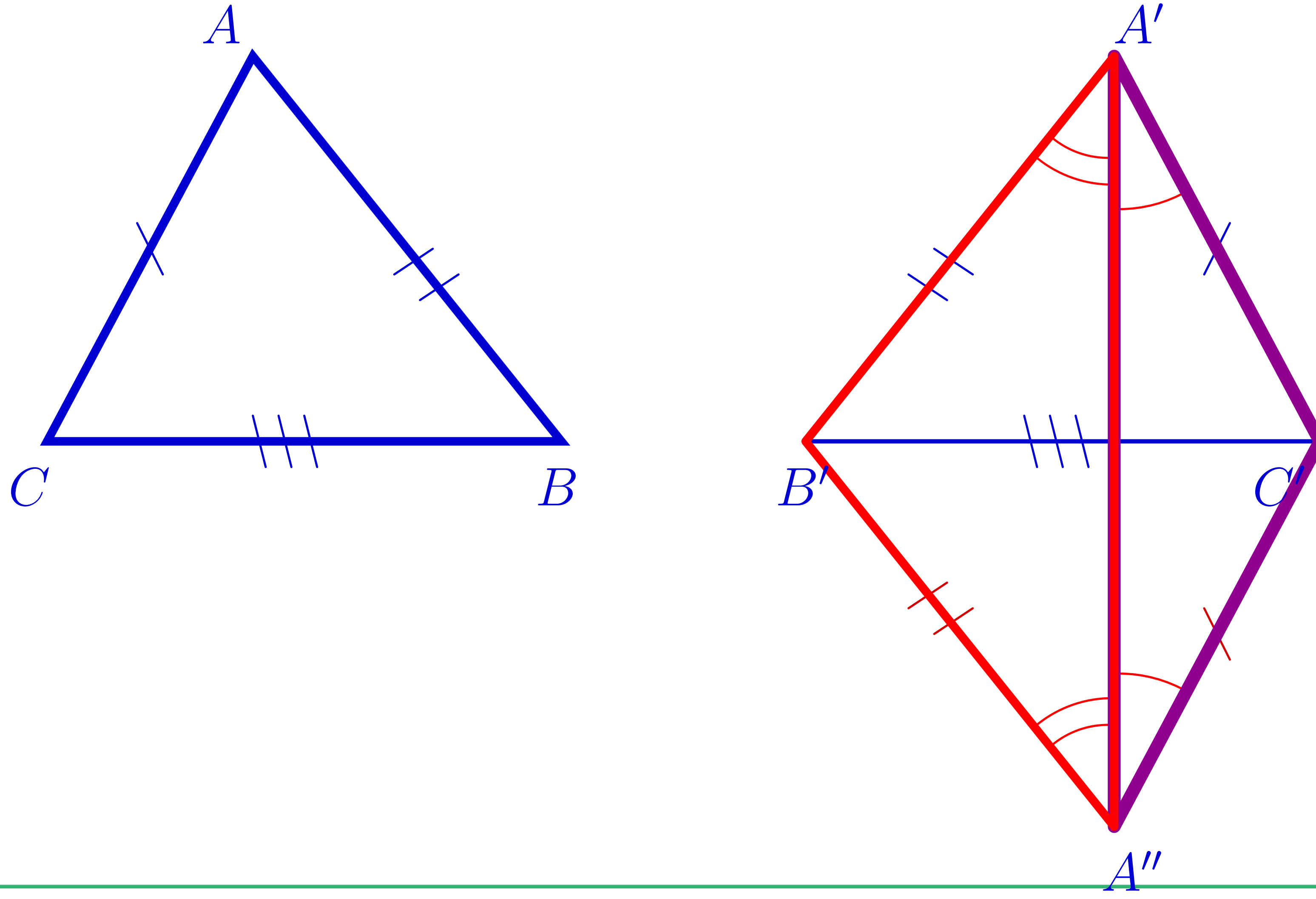
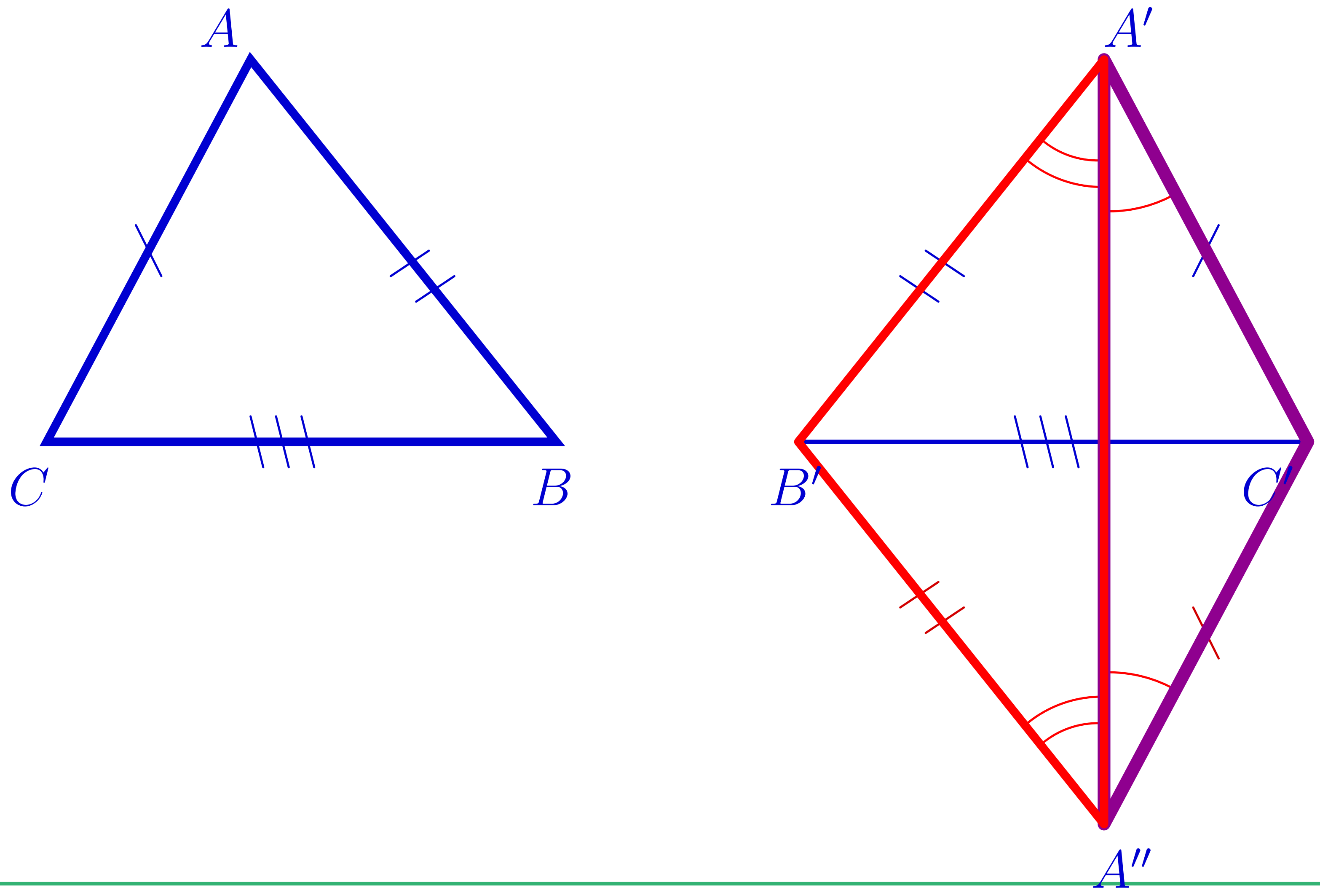


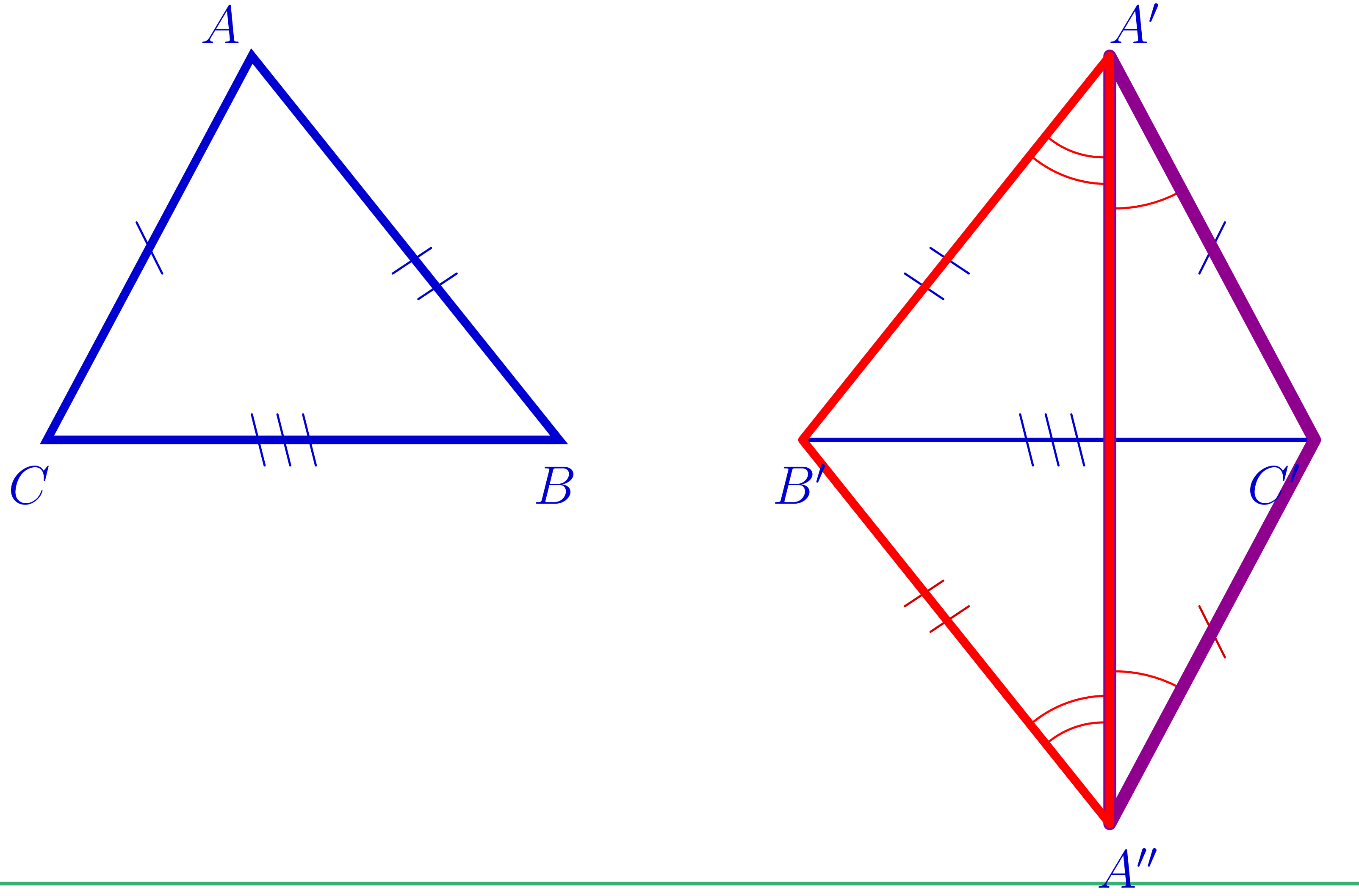
SSS-test

The angles at the base are congruent. Apply SAS-test.



SSS-test

Another case to consider?



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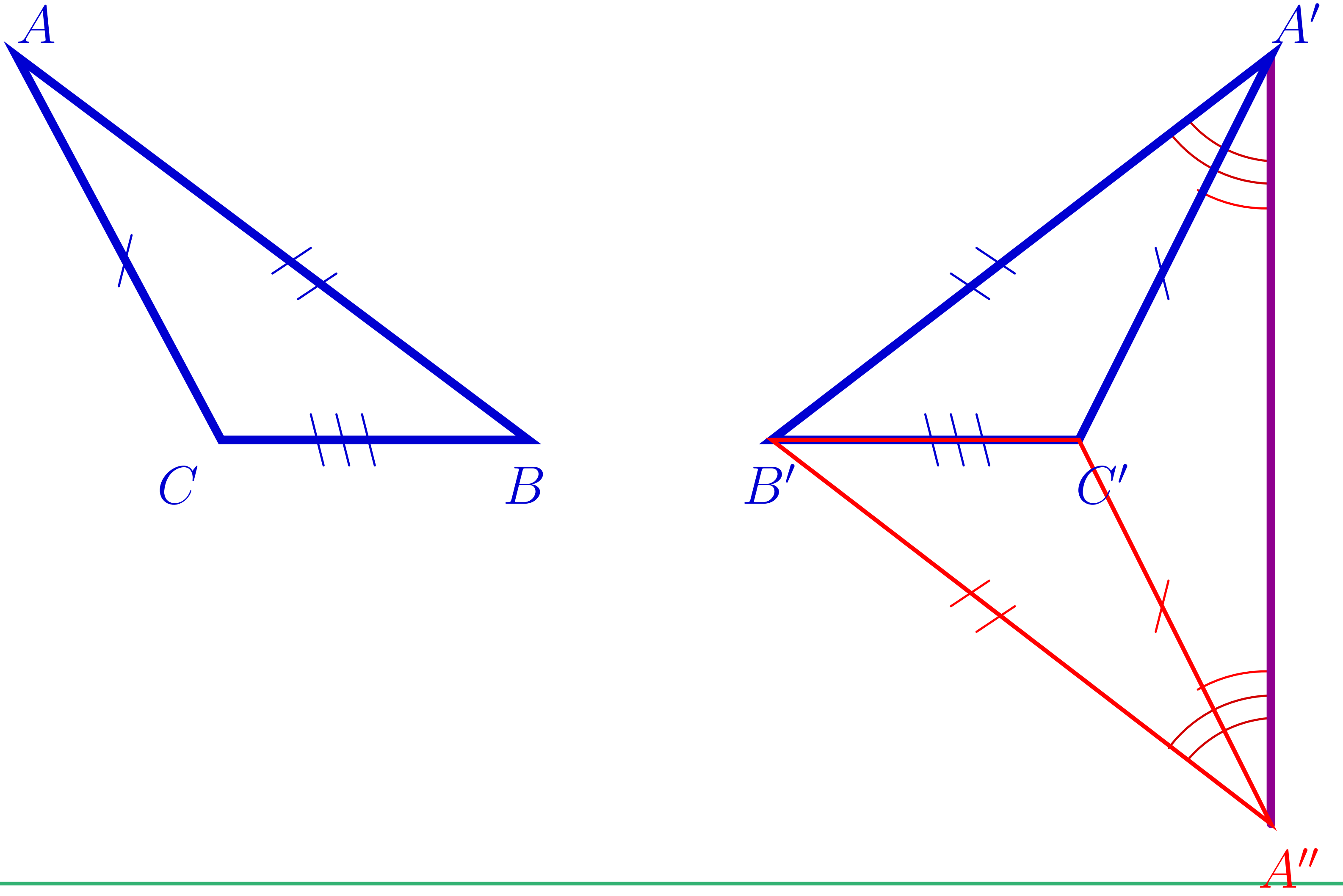

Exterior angle

Theorem. *An exterior angle of a triangle is greater than each interior angle not supplementary to it.*

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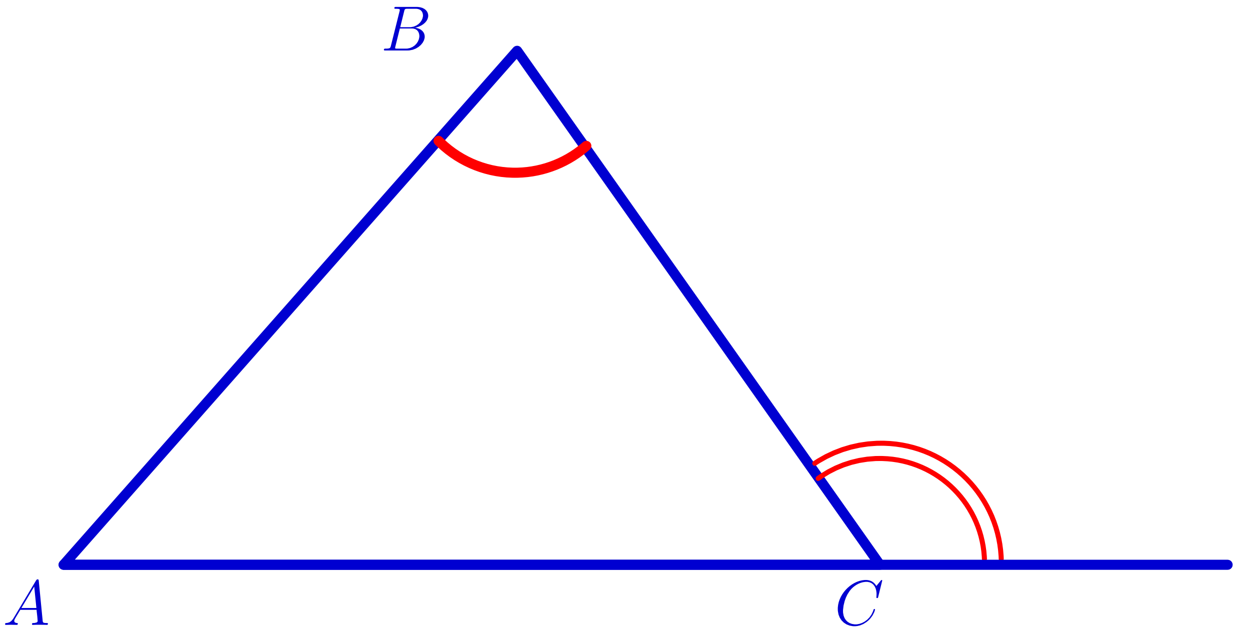

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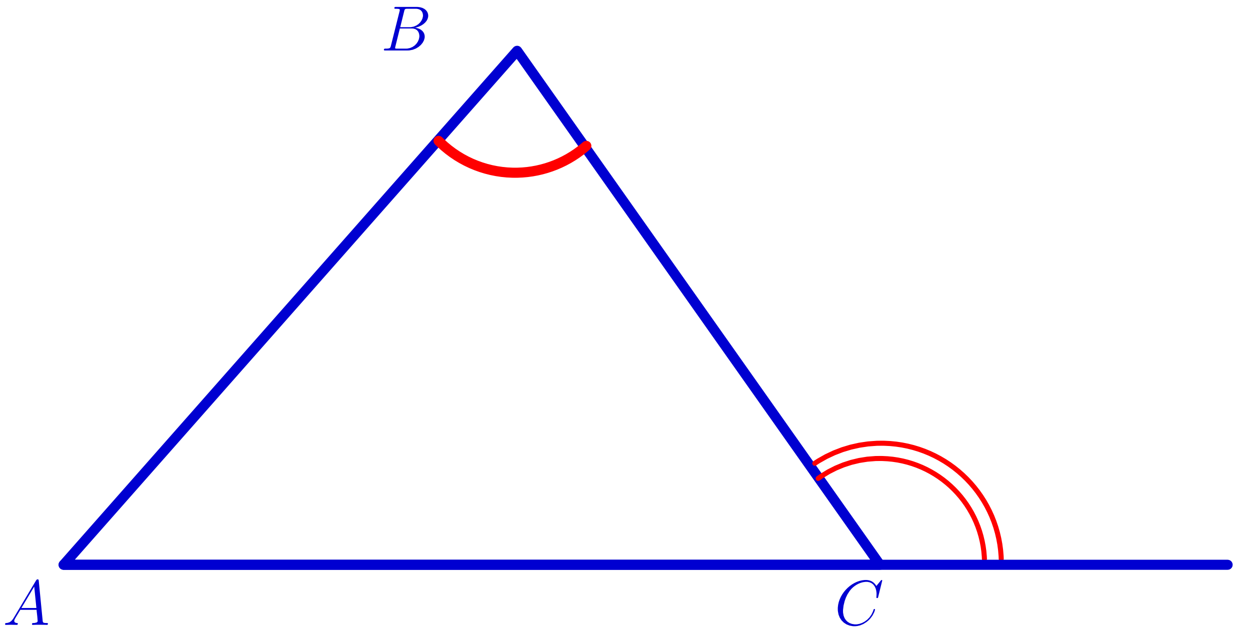

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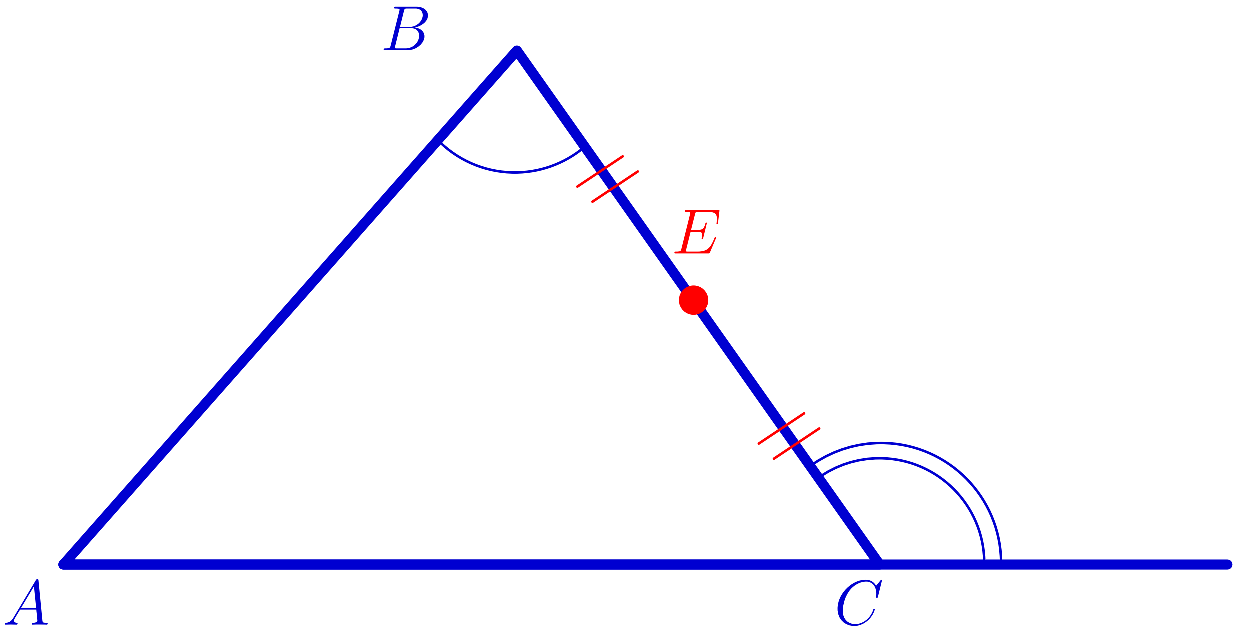

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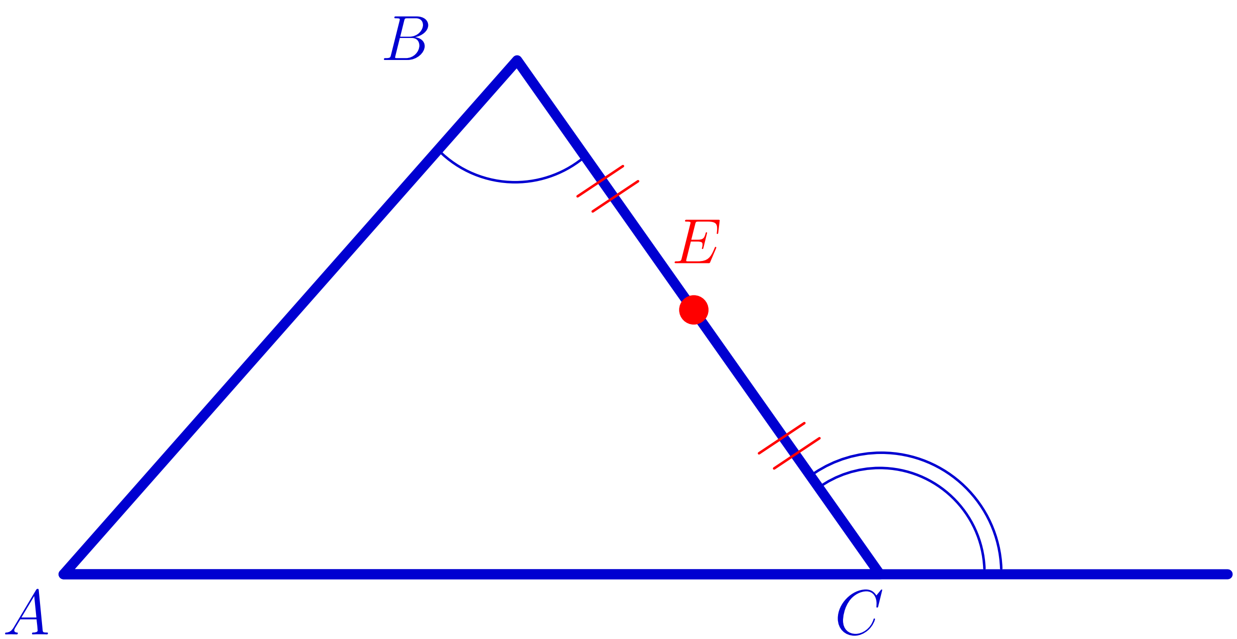

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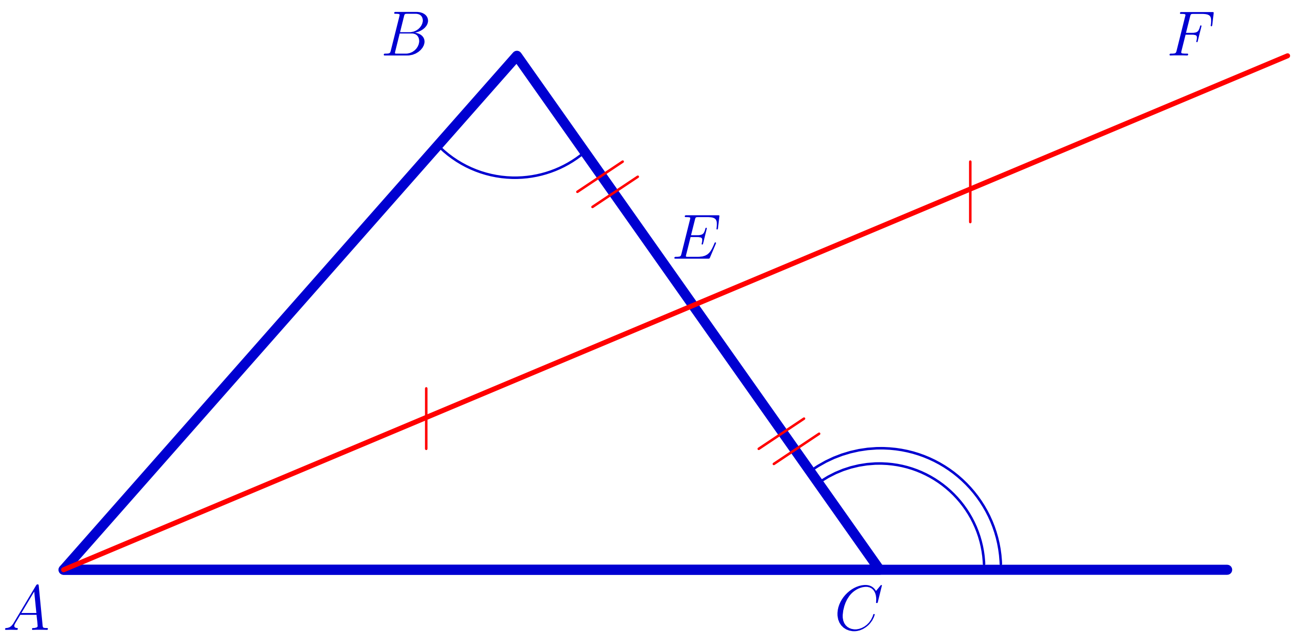

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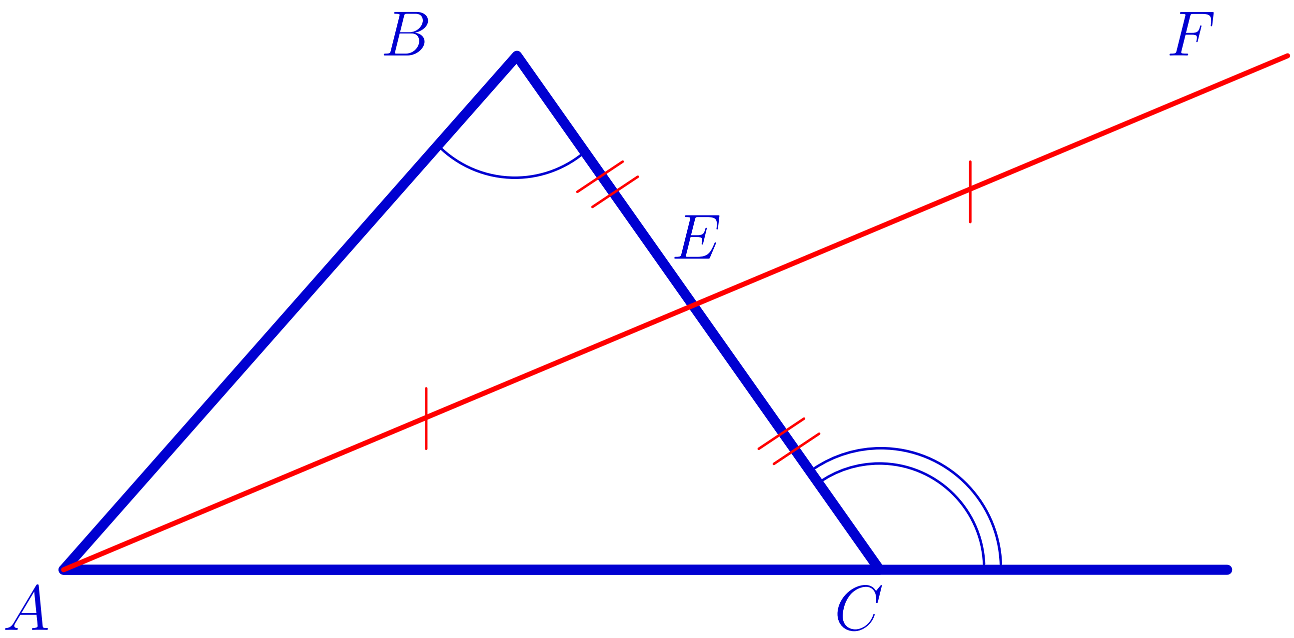

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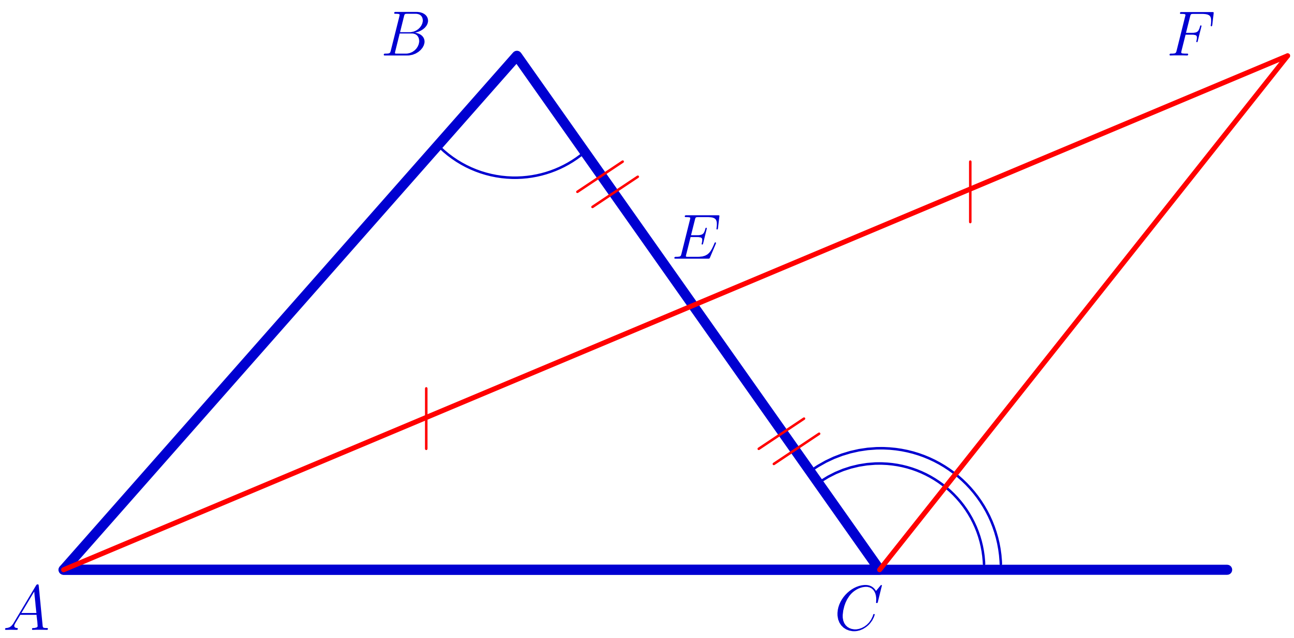

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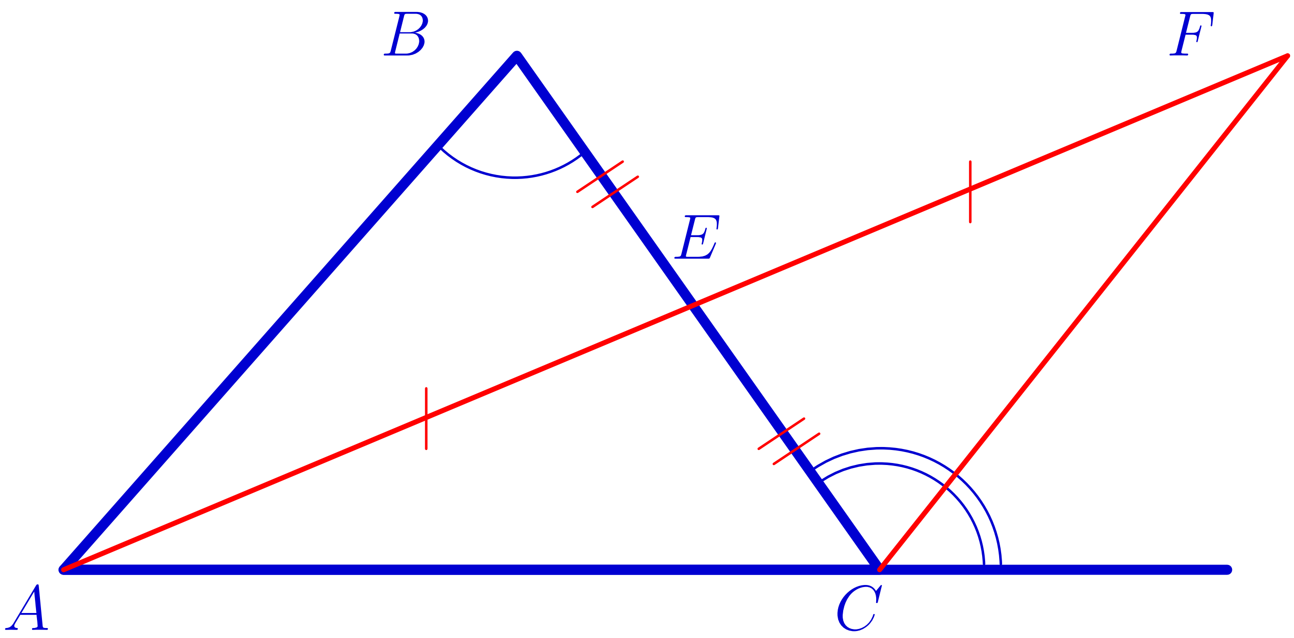

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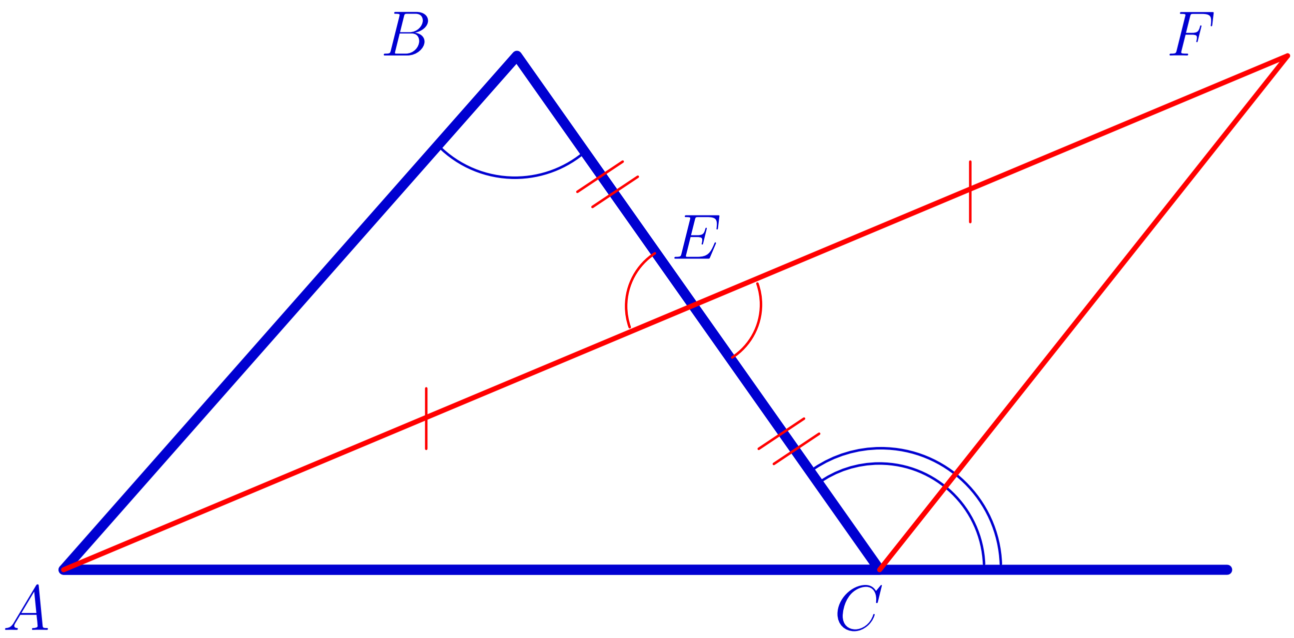

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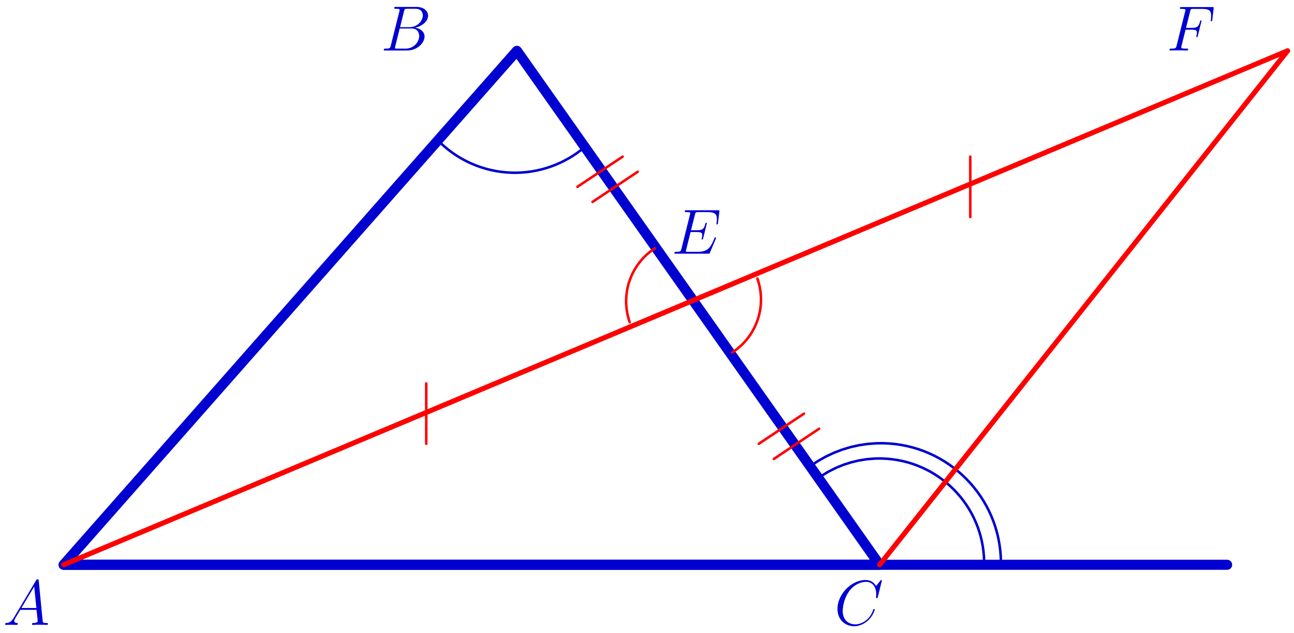

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Other interior angles are smaller.

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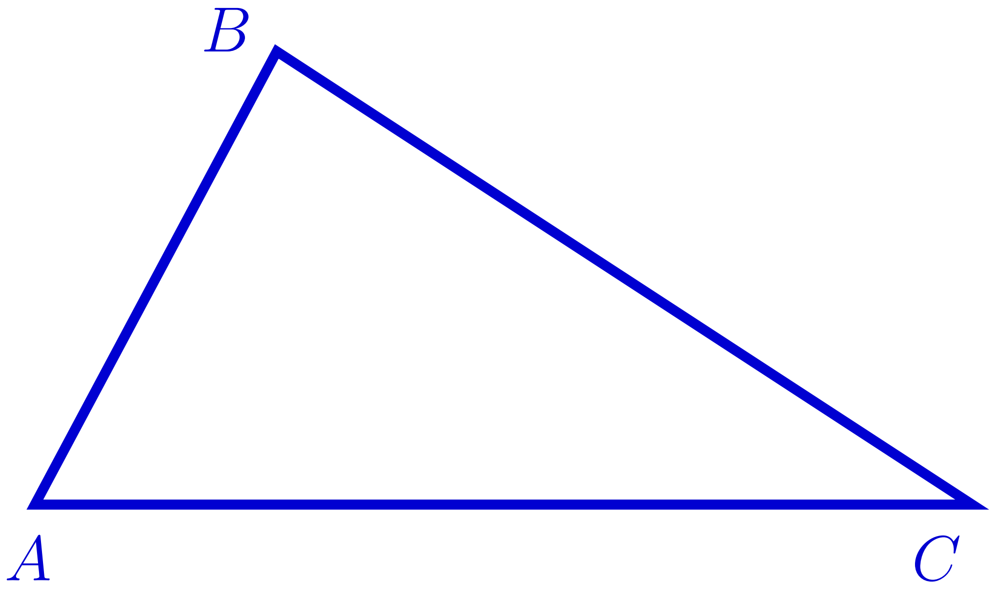
Proof. Let $AB < BC$.



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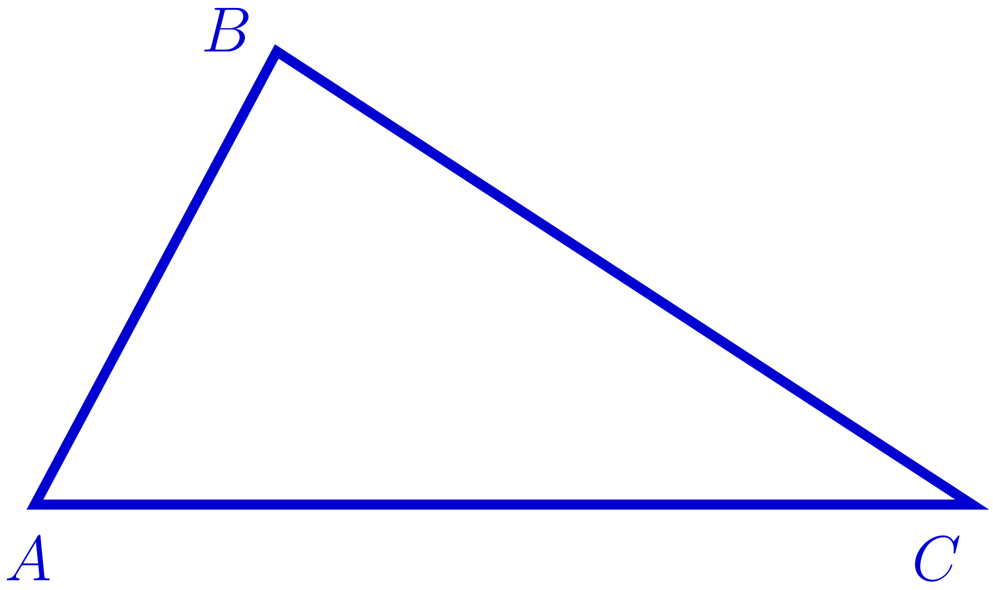

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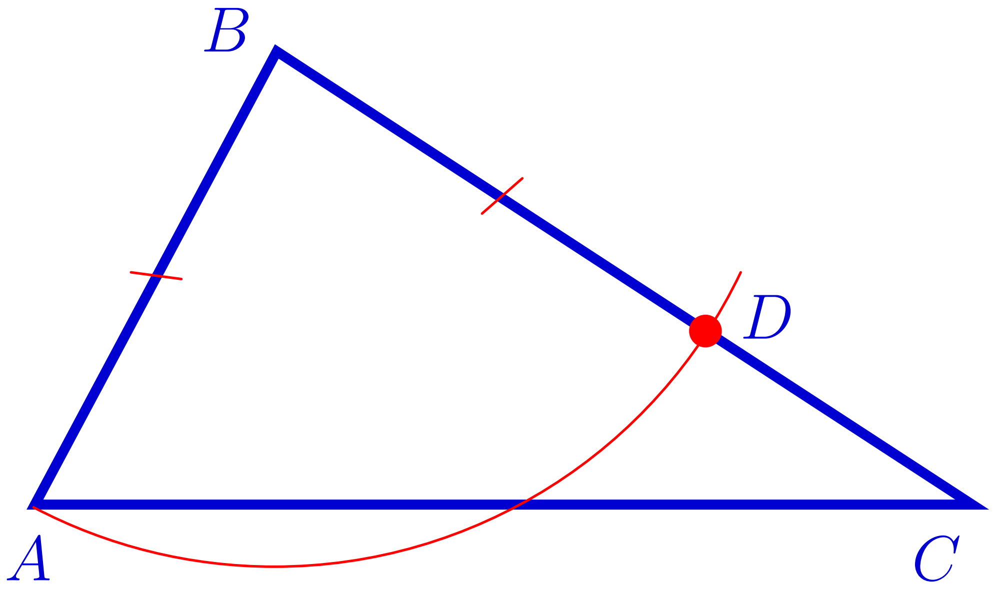

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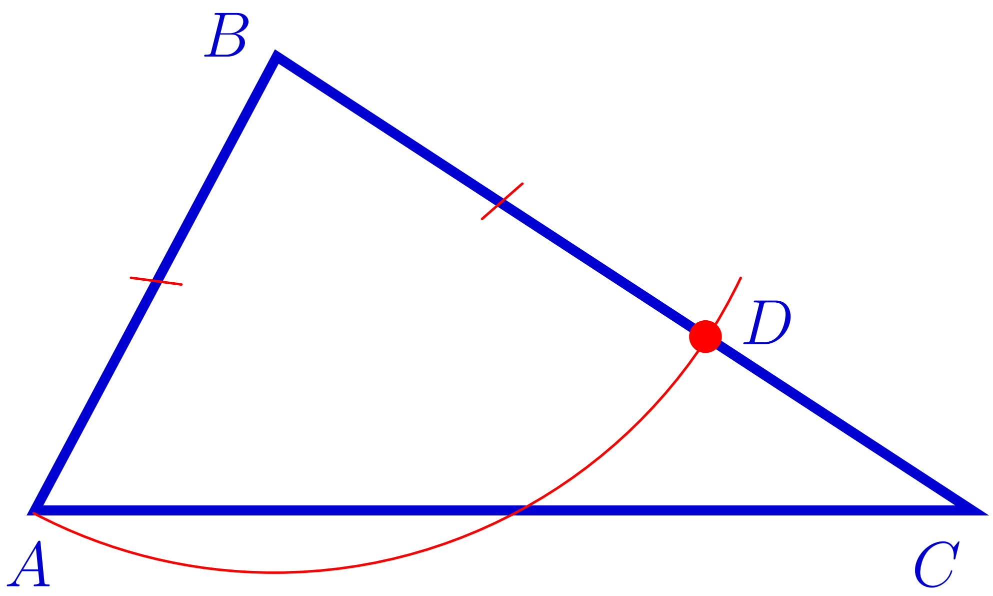

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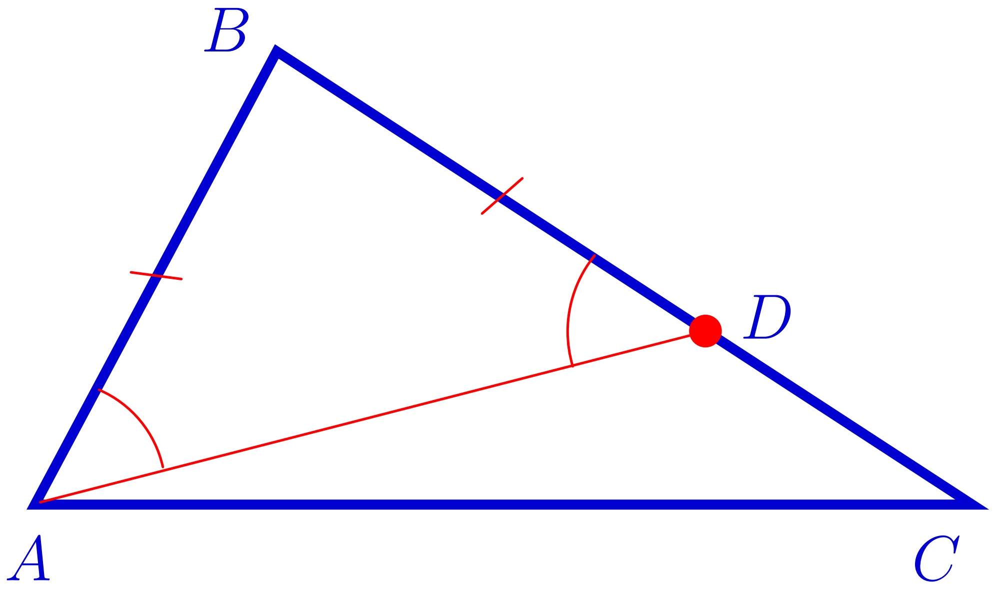

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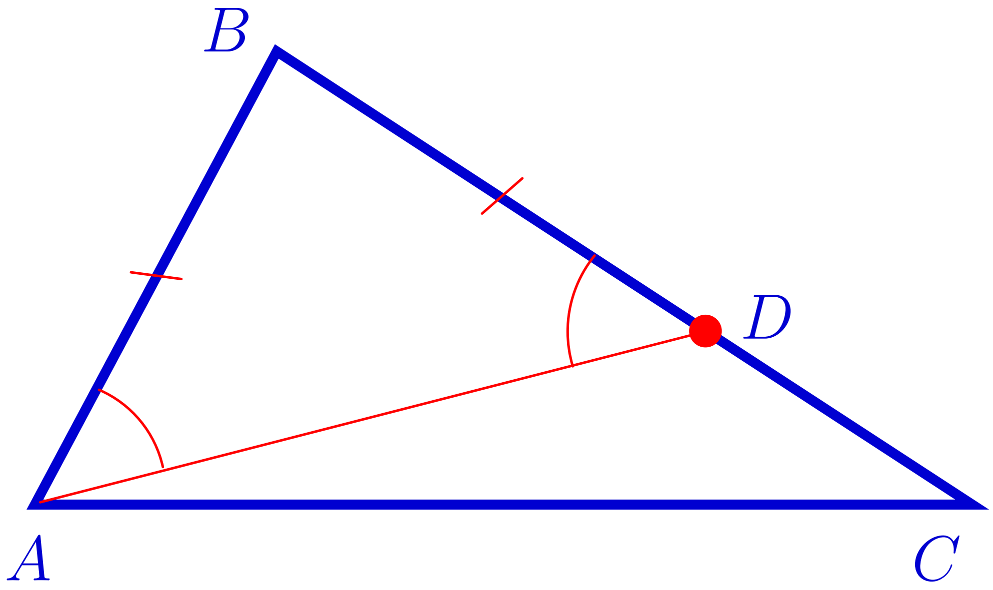
$\angle A > \angle BAD = \angle BDA > \angle C$.



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Proof by contradiction **reductio ad absurdum.**

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(1) the sides opposite to congruent angles are congruent;
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Corollary.
(1) In an equilateral triangle all angles are congruent.
(2) In an equiangular triangle all sides are congruent.

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Theorem. *In a triangle, each side is smaller than the sum of other two sides.*

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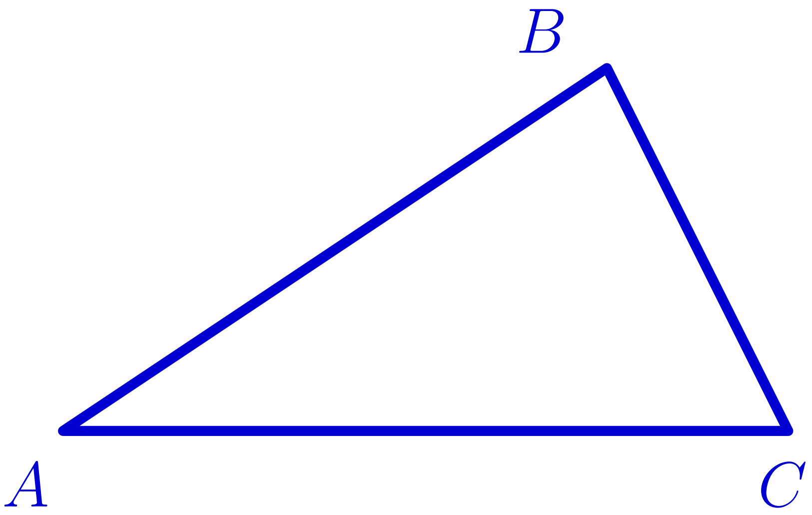

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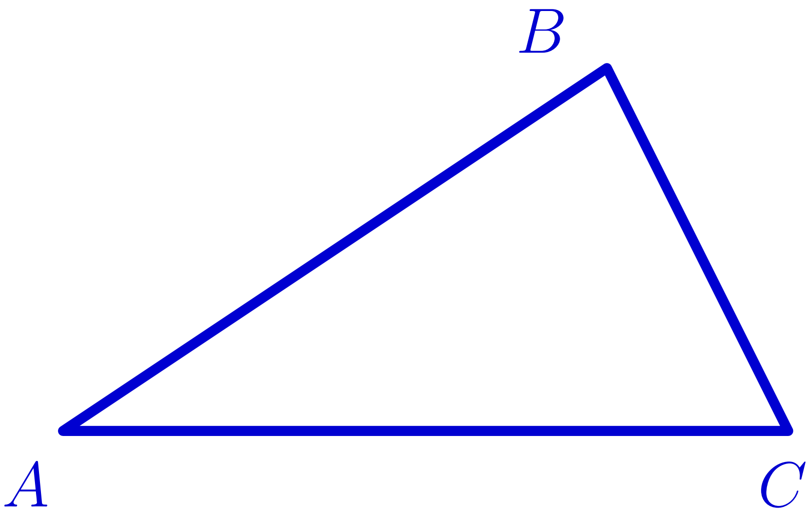
Theorem. *In a triangle, each side is smaller than the sum of other two sides.*

Proof. Let the greatest side be AC. Continuing the side AB past B mark on it the segment $BD = BC$.



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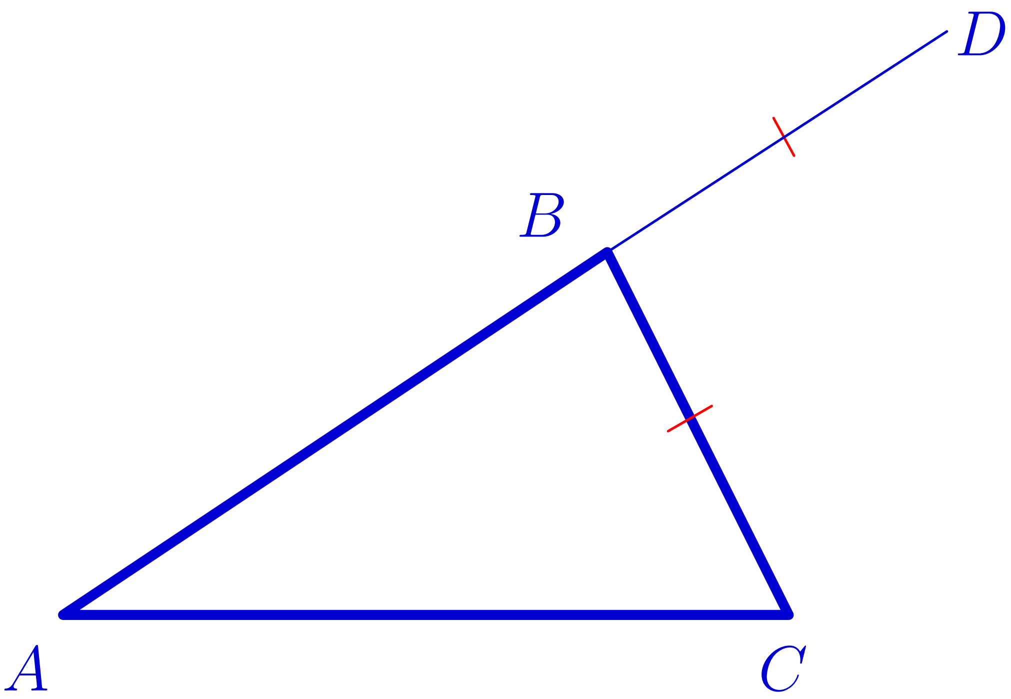

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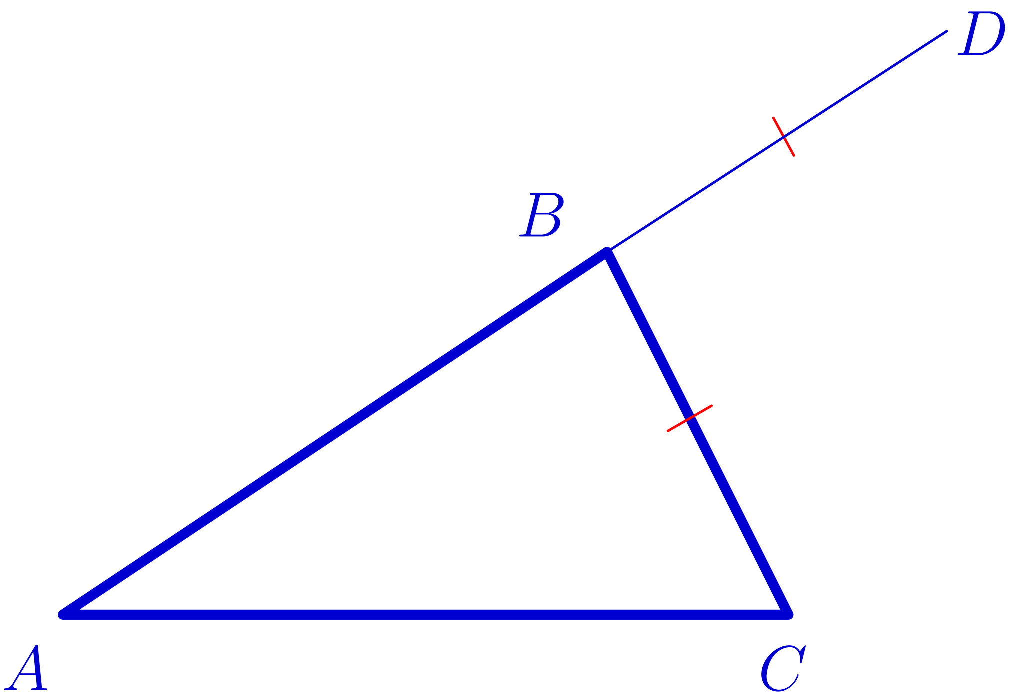
Proof. Let the greatest side be AC. Continuing the side AB past B mark on it the segment $BD = BC$. Draw DC.



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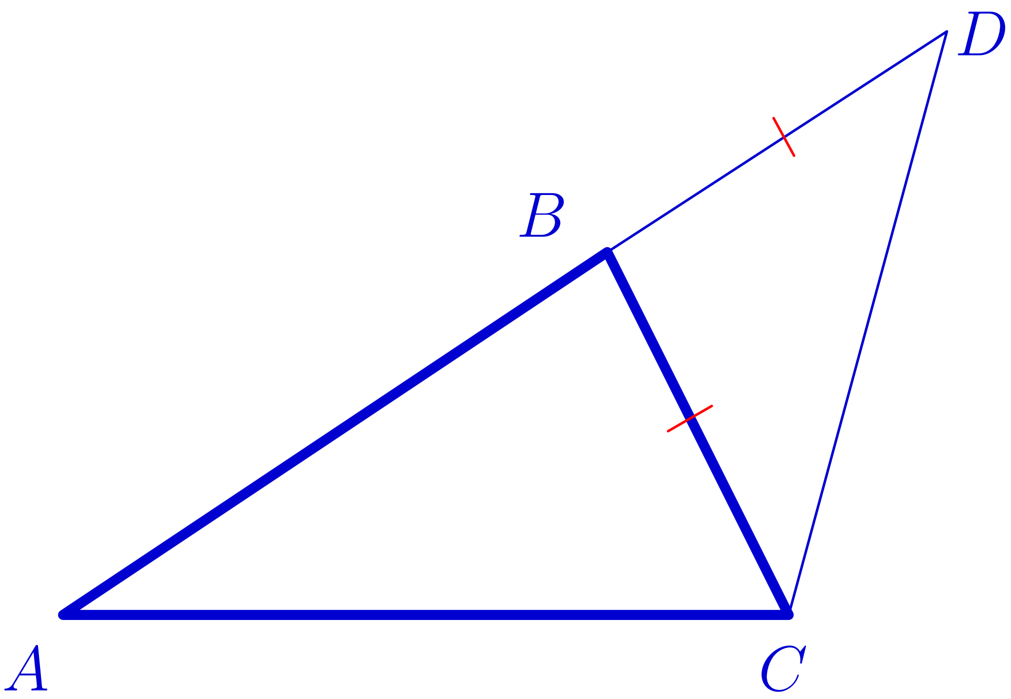

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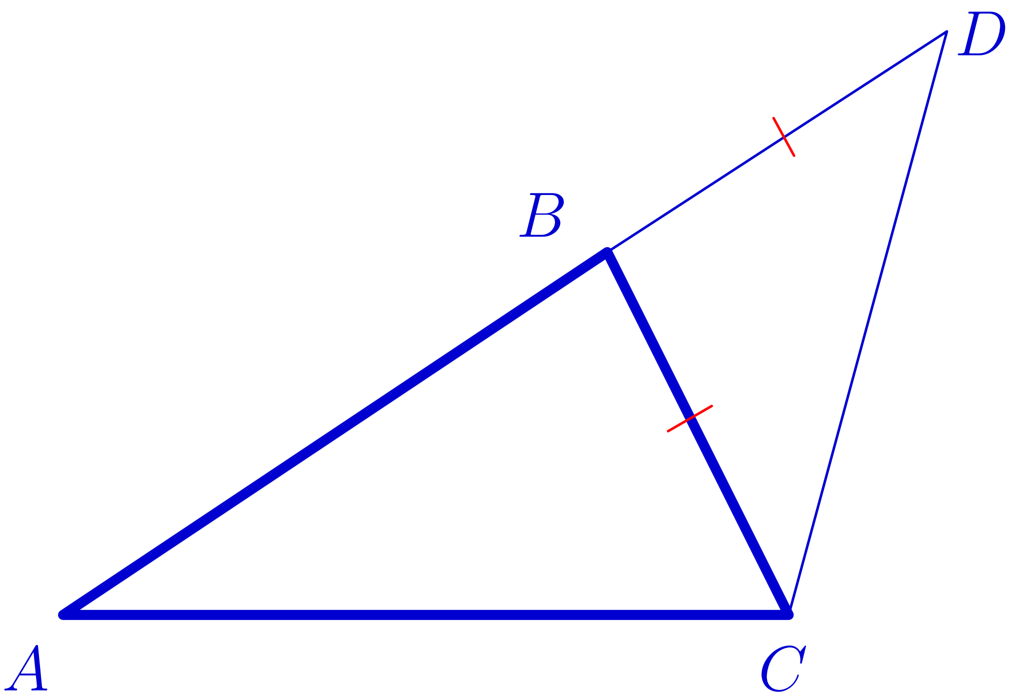

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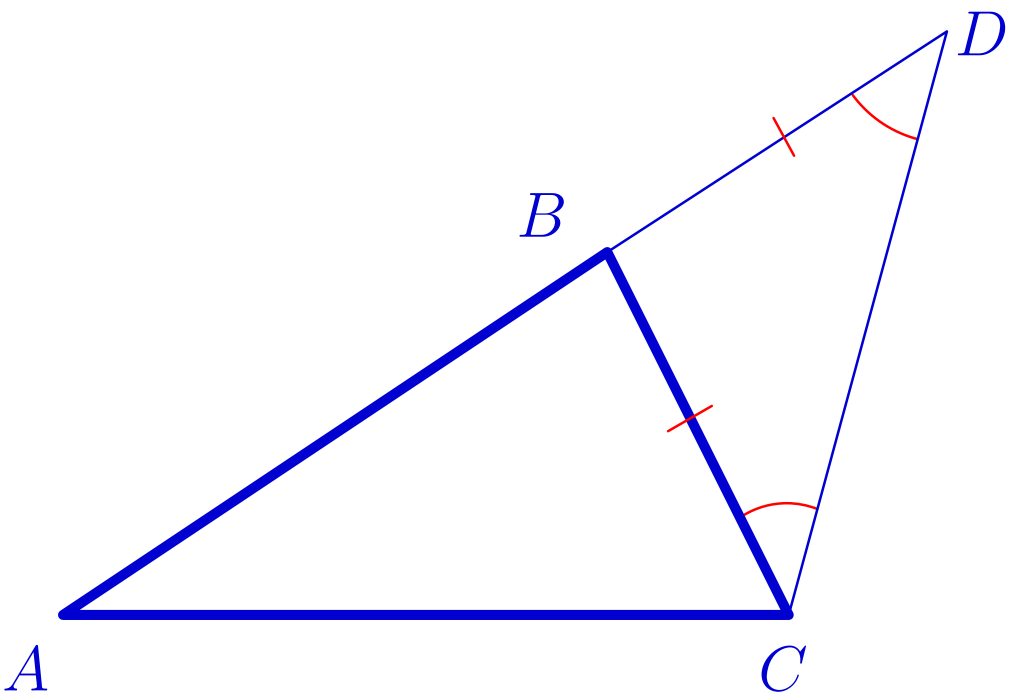

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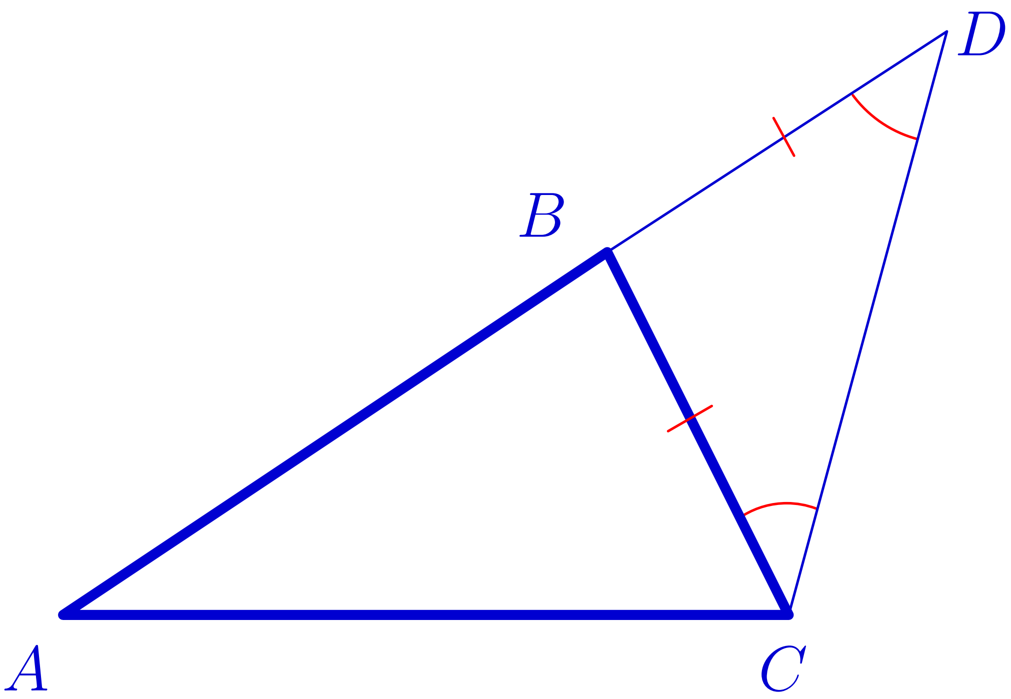

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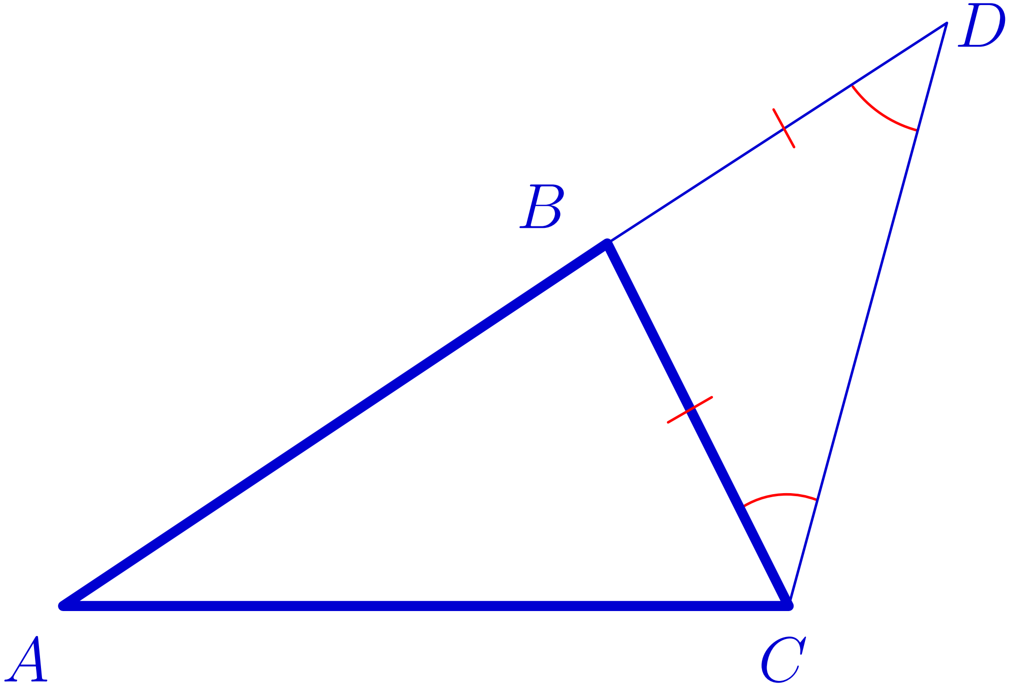
Therefore $\angle D < \angle DCA$. Hence $AC < AD = AB + BD = AB + BC$.



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