Dropping perpendicular

Theorem. *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Let a line $AB$ and an arbitrary point $M$ outside the line be given.
Theorem. From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

Let a line $AB$ and an arbitrary point $M$ outside the line be given. Drop a perpendicular from $M$ to $AB$. 

\[ A \underline{\quad \quad} B \]
Theorem. From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

Apply the axial symmetry about $AB$. 

\[ M \]

\[ A \quad B \]
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Apply the axial symmetry about $AB$. 

![Diagram](image-url)
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Apply the axial symmetry about $AB$. Connect $M$ and $M'$ by a line.
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Apply the axial symmetry about $AB$. Connect $M$ and $M'$ by a line.
Theorem. From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

Prove that $MM'$ is perpendicular to $AB$!
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Prove that $MM'$ is perpendicular to $AB$!

$\angle MCA = \angle ACM'$ as symmetric.
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

Prove that $MM'$ is perpendicular to $AB$!

$\angle MCA = \angle ACM'$ as symmetric. $\angle MCA + \angle ACM' = 180^\circ$ as $\angle MCA$ and $\angle ACM'$ are supplementary.
Theorem. From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

Prove that $MM'$ is perpendicular to $AB$!

$\angle MCA = \angle ACM'$ as symmetric. $\angle MCA + \angle ACM' = 180^\circ$ as $\angle MCA$ and $\angle ACM'$ are supplementary.

Hence $\angle MCA = \angle ACM' = 90^\circ$ and $MM' \perp AB$. 
Dropping perpendicular

**Theorem.** From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

**Uniqueness.**
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

**Uniqueness.** Assume there is another perpendicular $MD$. 
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

**Uniqueness.** Assume there is another perpendicular $MD$. 

![Diagram showing a point M outside a line AB, with perpendiculars MD and M'D, and another line CD parallel to AB.]
**Theorem.** From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

**Uniqueness.** Assume there is another perpendicular $MD$. Take its image under the symmetry about $AB$. 

![Diagram](image-url)
**Theorem.** From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

**Uniqueness.** Assume there is another perpendicular $MD$. Take its image under the symmetry about $AB$. 

![Diagram with points A, B, C, D, M, and M']
Dropping perpendicular

**Theorem.** *From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.*

**Uniqueness.** Assume there is another perpendicular $MD$. Take its image under the symmetry about $AB$. Angles $\angle MDA$ and $\angle ADM'$ are right, therefore $\angle MDM'$ is straight.
**Theorem.** From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

**Uniqueness.** Assume there is another perpendicular $MD$. Take its image under the symmetry about $AB$. Angles $\angle MDA$ and $\angle ADM'$ are right, therefore $\angle MDM'$ is straight. Hence $MDM' = MM'$ and $D = C$.
**Theorem.** From any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.

**Uniqueness.** Assume there is another perpendicular $MD$. Take its image under the symmetry about $AB$. Angles $\angle MDA$ and $\angle ADM'$ are right, therefore $\angle MDM'$ is straight. Hence $MDM' = MM'$ and $D = C$. 
Theorem. If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.
Theorem. If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.

Proof. Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles such that $AC = A'C'$, $AB = A'B'$, $\angle A = \angle A'$. 
SAS-test

Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that $A$ would coincide with $A'$
Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that $A$ would coincide with $A'$
SAS-test

Superimpose \( \triangle ABC \) onto \( \triangle A'B'C'' \) in such a way that \( A \) would coincide with \( A' \), the side \( AC \) would go along \( A'C'' \)
SAS-test

Superimpose $\triangle ABC$ onto $\triangle A'B'C''$ in such a way that $A$ would coincide with $A'$, the side $AC$ would go along $A'C''$.
Superimpose $\triangle ABC$ onto $\triangle A'B'C''$ in such a way that $A$ would coincide with $A'$, the side $AC$ would go along $A'C''$, and the side $AB$ would lie on the same side of $A'C''$ as $A'B'$.
Superimpose $\triangle ABC$ onto $\triangle A'B'C''$ in such a way that $A$ would coincide with $A'$, the side $AC$ would go along $A'C''$, and the side $AB$ would lie on the same side of $A'C''$ as $A'B'$. 
Pons asinorum

**Theorem.** In isosceles triangles the angles at the base equal one another.
Pons asinorum

**Theorem.** In isosceles triangles the angles at the base equal one another.
Pons asinorum

**Theorem.** In isosceles triangles the angles at the base equal one another.
**Theorem.** In isosceles triangles the angles at the base equal one another.
Pons asinorum

Theorem. In isosceles triangles the angles at the base equal one another.
Theorem. In isosceles triangles the angles at the base equal one another.
Pons asinorum

Theorem. In isosceles triangles the angles at the base equal one another.
Pons asinorum

Theorem. In isosceles triangles the angles at the base equal one another.
**Theorem.** In isosceles triangles the angles at the base equal one another.
Lines in triangle

Triangle $ABC$ with altitude $BD$. 

$A$, $B$, $C$, $D$ are the vertices of the triangle. $BD$ is the altitude.
Lines in triangle

median $BE$
Lines in triangle

![Triangle with bisector](image)

bisector $BF$
Lines in triangle

Altitude $BD$, bisector $BF$, median $BE$
Lines in triangle

Theorem. If the triangle is isosceles (i.e., $AB$ is congruent to $BC$), then $D = F = E$ and all three lines coincide.
Lines in triangle

Theorem. If the triangle is isosceles (i.e., $AB$ is congruent to $BC$), then $D = F = E$ and all three lines coincide.

Lemma. If $AB$ is congruent to $BC$, then the triangle $ABC$ is symmetric about its bisector $BF$.

altitude = bisector = median
Lines in triangle

altitude = bisector = median

**Theorem.** If the triangle is isosceles (i.e., $AB$ is congruent to $BC$), then $D = F = E$ and all three lines coincide.

**Theorem.** If $AB$ is congruent to $BC$, then $\angle A = \angle C$. 
Lines in triangle

Theorem. If the triangle is isosceles (i.e., $AB$ is congruent to $BC$), then $D = F = E$ and all three lines coincide.

Theorem. If $AB$ is congruent to $BC$, then $\angle A = \angle C$.  

altitude = bisector = median
SSS-test

Theorem. SSS-test. If three sides of one triangle are congruent respectively to three sides of another triangle, then the triangles are congruent.
Theorem. **SSS-test.** If three sides of one triangle are congruent respectively to three sides of another triangle, then the triangles are congruent.
SSS-test

Juxtapose $ABC$ and $A'B'C'$ in such a way that $BC$ and $B'C''$ would coincide, and $A$ and $A'$ would lie on the opposite sides of $B'C''$. 
SSS-test

Juxtapose $ABC$ and $A'B'C'$ in such a way that $BC$ and $B'C''$ would coincide, and $A$ and $A'$ would lie on the opposite sides of $B'C''$. 
SSS-test

Joining $A'$ and $A''$ we obtain isosceles triangles $A'B'A''$ and $A'C'A''$ with the common base $A'A''$. 
Joining $A'$ and $A''$ we obtain isosceles triangles $A'B'A''$ and $A'C'A''$ with the common base $A'A''$. 
The angles at the base are congruent.
SSS-test

The angles at the base are congruent.
**SSS-test**

The angles at the base are congruent. Apply SAS-test.
SSS-test

Another case to consider?
SSS-test
Exterior angle

Theorem. An exterior angle of a triangle is greater than each interior angle not supplementary to it.
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint \( E \) on \( BC \).
Theorem. An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$. 
Theorem. An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$.
Draw the median $AE$ and extend it to $F$ so that $EF = AE$. 
**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$. Draw the median $AE$ and extend it to $F$ so that $EF = AE$. 

![Diagram of a triangle with exterior angle theorem](image)
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$.
Draw the median $AE$ and extend it to $F$ so that $EF = AE$.
Draw segment $CF$.

![Diagram of a triangle with exterior angles](attachment:triangle_diagram.png)
Theorem. An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$.
Draw the median $AE$ and extend it to $F$ so that $EF = AE$.
Draw segment $CF$.
**Exterior angle**

**Theorem.** *An exterior angle of a triangle is greater than each interior angle not supplementary to it.*

Put midpoint $E$ on $BC$. Draw the median $AE$ and extend it to $F$ so that $EF = AE$. Draw segment $CF$. Triangles $ABE$ and $EFC$ are congruent by SAS-test.
Exterior angle

Theorem. An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$.
Draw the median $AE$ and extend it to $F$ so that $EF = AE$.
Draw segment $CF$.
Triangles $ABE$ and $EFC$ are congruent by SAS-test.
$\angle B = \angle ECF < \angle C$. 
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

Put midpoint $E$ on $BC$.
Draw the median $AE$ and extend it to $F$ so that $EF = AE$.
Draw segment $CF$.
Triangles $ABE$ and $EFC$ are congruent by SAS-test.
$\angle B = \angle ECF < \angle C$. 
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

**Corollary.** If in a triangle one angle is not acute, then the other two angles are acute.
Exterior angle

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

**Corollary.** If in a triangle one angle is not acute, then the other two angles are acute.

**Proof.** The exterior angle at the vertex with non-acute angle is not obtuse (i.e., $\leq 90^\circ$).
**Exterior angle**

**Theorem.** An exterior angle of a triangle is greater than each interior angle not supplementary to it.

**Corollary.** If in a triangle one angle is not acute, then the other two angles are acute.

**Proof.** The exterior angle at the vertex with non-acute angle is not obtuse (i.e., $\leq 90^\circ$).

Other interior angles are smaller.
Angle opposite to side

Theorem. *In any triangle the angle opposite to a greater side is greater.*
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

**Proof.** Let \( AB < BC \).
Theorem. *In any triangle the angle opposite to a greater side is greater.*

Proof. Let $AB < BC$.

On $BC$, mark the segment $BD$ congruent to $AB$. 

![Diagram of a triangle with a segment marked equal to one of its sides.](image-url)
Angle opposite to side

**Theorem.** In any triangle the angle opposite to a greater side is greater.

**Proof.** Let $AB < BC$.

On $BC$, mark the segment $BD$ congruent to $AB$. 

![Diagram of a triangle with marked segments](image)
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

**Proof.** Let $AB < BC$.

On $BC$, mark the segment $BD$ congruent to $AB$.

Draw the segment $AD$.
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

**Proof.** Let \( AB < BC \).

On \( BC \), mark the segment \( BD \) congruent to \( AB \).

Draw the segment \( AD \).
Angle opposite to side

Theorem. In any triangle the angle opposite to a greater side is greater.

Proof. Let $AB < BC$.
On $BC$, mark the segment $BD$ congruent to $AB$.
Draw the segment $AD$.
$\angle A > \angle BAD = \angle BDA > \angle C$. 

![Diagram of triangle with marked segments and angles]
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

We have proved earlier that the angles opposite to congruent sides are congruent.
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

We have proved earlier that the angles opposite to congruent sides are congruent.

**Converse Theorem.** *In any triangle*

1. *the sides opposite to congruent angles are congruent;*
2. *the side opposite to a greater angle is greater.*
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

We have proved earlier that the angles opposite to congruent sides are congruent.

**Converse Theorem.** *In any triangle*

1. *the sides opposite to congruent angles are congruent;*
2. *the side opposite to a greater angle is greater.*

Proof by contradiction *reductio ad absurdum.*
Angle opposite to side

**Theorem.** *In any triangle the angle opposite to a greater side is greater.*

We have proved earlier that the angles opposite to congruent sides are congruent.

**Converse Theorem.** *In any triangle*

(1) *the sides opposite to congruent angles are congruent;*

(2) *the side opposite to a greater angle is greater.*

**Corollary.**

(1) *In an equilateral triangle all angles are congruent.*

(2) *In an equiangular triangle all sides are congruent.*
Triangle inequality

**Theorem.** In a triangle, each side is smaller than the sum of other two sides.
Triangle inequality

**Theorem.** In a triangle, each side is smaller than the sum of other two sides.
**Triangle inequality**

**Theorem.** *In a triangle, each side is smaller than the sum of other two sides.*

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. 
Triangle inequality

**Theorem.** *In a triangle, each side is smaller than the sum of other two sides.*

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$, mark on it the segment $BD = BC$.
Triangle inequality

**Theorem.** In a triangle, each side is smaller than the sum of other two sides.

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$. 
Triangle inequality

**Theorem.** *In a triangle, each side is smaller than the sum of other two sides.*

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$. 
Triangle inequality

**Theorem.** *In a triangle, each side is smaller than the sum of other two sides.*

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$. Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$. 

![Diagram of triangle inequality](image)
Triangle inequality

**Theorem.** In a triangle, each side is smaller than the sum of other two sides.

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$.

Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$. 

![Diagram of a triangle with a construction showing the triangle inequality]
Triangle inequality

**Theorem.** *In a triangle, each side is smaller than the sum of other two sides.*

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$.

Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$.

Therefore $\angle D < \angle DCA$.
Triangle inequality

**Theorem.** In a triangle, each side is smaller than the sum of other two sides.

**Proof.** Let the greatest side be $AC$. Continuing the side $AB$ past $B$ mark on it the segment $BD = BC$. Draw $DC$.

Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$.

Therefore $\angle D < \angle DCA$. Hence $AC < AD = AB + BD = AB + BC$.
Table of Contents

- Dropping perpendicular
- SAS-test
- SSS-test
- Pons asinorum
- Lines in triangle
- Exterior angle
- Angle opposite to side
- Triangle inequality