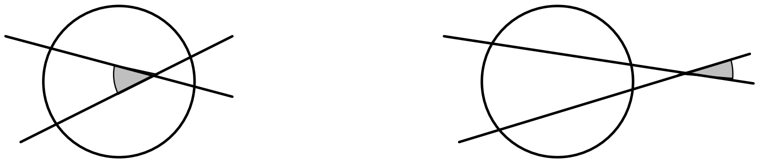
Stony Brook University
Mathematics Department
Oleg Viro

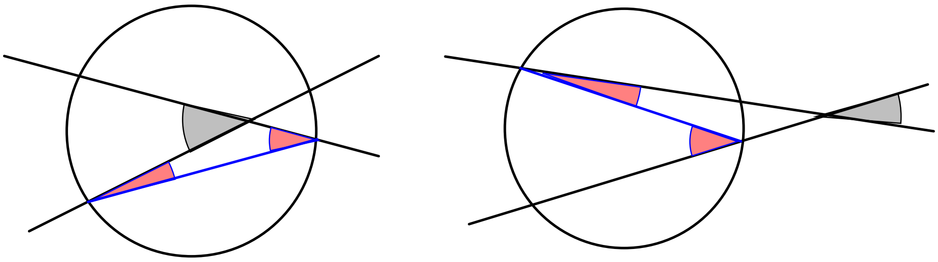
Geometric Structures
MAT 360
May 11, 2010

Final Exam. Solutions

Problem 1. (8 pt) Formulate and prove theorems about relationships between the angles formed by two intersecting lines and the arcs which are cut by the lines on a circle, which is not tangent to the lines and does not pass through their intersection point.



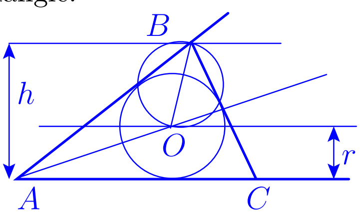
Solution. This is Theorem 126 from the textbook. The proofs become clear from looking at the following pictures and recalling the relation between an exterior angle of a triangle and the two interior angles non-adjacent to the exterior one (the exterior is congruent to the sum of the two interior) and the relation between an inscribed angle and the corresponding central angle (the inscribed is congruent to a half of the central).



There were students who invented in their exams other solutions.

Problem 2. (5 pt) Construct a triangle $\triangle ABC$ given an angle congruent to its interior angle at vertex A, a segment congruent to a radius of inscribed circle, and a segment congruent to the altitude dropped from vertex B.

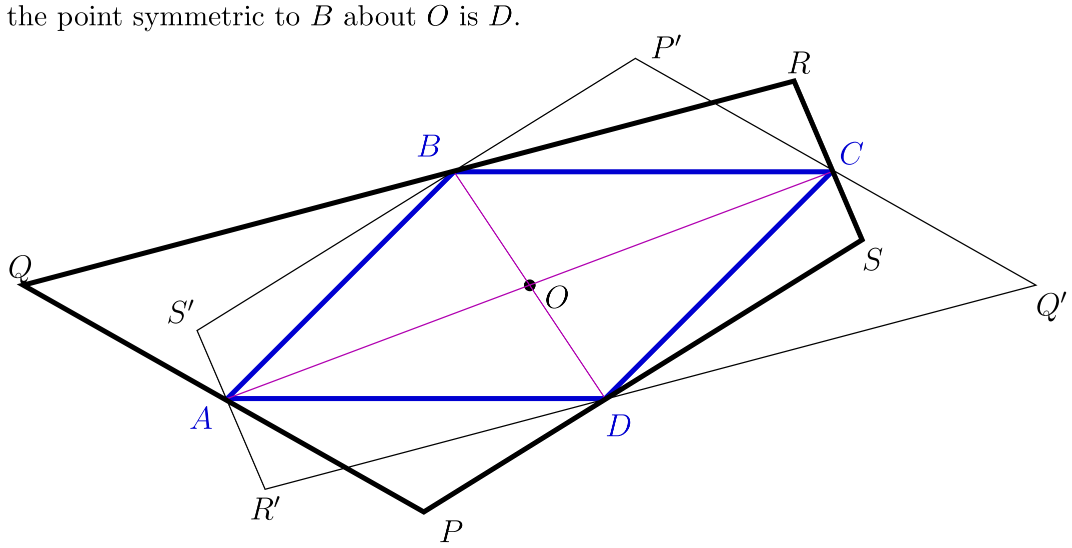
Solution. 1. Draw an angle congruent to the given angle. Mark the vertex of the angle as A.
2. Draw a line parallel to one of the sides of the angle at the distance equal to the given length h of altitude, and intersecting the other side of the angle. Mark the intersection point of the line with the other side as B.
3. Draw a line parallel to one of the sides of the angle at the distance equal to the given radius of inscribed circle r, and intersecting the other side of the angle. 4. Construct the bisector of angle A.
5. Draw a circle of radius r centered at the intersection O of the lines drawn at steps 3 and 4.
6. Draw a line through B tangent to the circle constructed at step 5. (for this draw a circle with diameter BO, draw a line through B and the intersection point of the two circles that is not on the side of $\angle A$).
7. This line is the side BC of the triangle.



A proof that the construction gives the required result is straightforward. The solution is unique. It exists if $2r < h$. If $2r > h$, then the problem has no solution, while the construction gives a triangle in which $\angle A$ is an exterior angle, and r is the radius of an exscribed circle. If $h = 2r$, there is no solution.

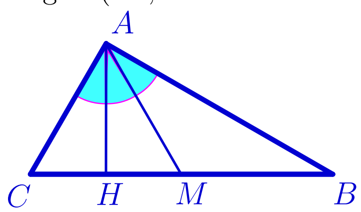
Problem 3. (5 pt) Given a convex quadrilateral $PQRS$ and a point O inside of $PQRS$, construct a parallelogram $ABCD$ such that $A \in PQ$, $B \in QR$, $C \in RS$ and $D \in SP$ and O is the intersection point of diagonals AC and BD.

Solution. Construct a quadrilateral $P'Q'R'S'$ symmetric about O to $PQRS$. If A exists it is intersection point of PQ and $R'S'$. Indeed, $A \in PQ$ should be symmetric to $B \in RS$ about O, because diagonals of a parallelogram meet at their midpoints. The point symmetric to A about O, that is the intersection point of $P'Q'$ with RS, is C. Similarly, the intersection point of QR and $P'S'$ is B, and the point symmetric to B about O is D.



There may be no solution, one solution, or infinitely many solutions, according to the number of intersection points of the segments.

Problem 4. (9 pt) Find the interior angles of a triangle $\triangle ABC$, in which median AM and altitude AH divide angle $\angle A$ into three equal angles (i.e., $\angle CAH = \angle HAM = \angle MAB$).



Solution. In $\triangle ACM$ the segment AH is both altitude and bisector. Therefore, this is an isosceles triangle and AH is also its median, i.e., $|CH| = |HM|$. Since AM is median of $\triangle ABC$, $|CM| = |MB|$ and $|HM| = \frac{1}{2}|MB|$.

In $\triangle AHB$, AM is a bisector. Hence $\frac{AH}{AB} = \frac{HM}{MB} = \frac{1}{2}$. This is a right triangle. A right triangle, in which one of the legs is a half of the hypothenuse, has interior degrees $30°$ and $60°$. (It is a half of a regular triangle.) Hence $\angle B = 30°$, $\angle A = 90°$ and $\angle C = 60°$. □

One could just guess the angles without the arguments above, but a guess does not prove that there is no triangles with other angles satisfying all the conditions of the problem.

Problem 5. (8 pt) Parallelepiped is a polyhedron bounded in the 3-space by three pairs of parallel planes.

(1) Prove that each face of a parallelepiped is a parallelogram.
(2) Formulate properties of a parallelepiped similar to the properties of a parallelogram that were studied in the course and prove one of them.

Solution. (1) Each face of a parallelepiped is bounded by intersection lines of the plane in which the face is contained with two pairs of parallel planes. According to a well-known theorem, parallel planes intersect with a plane in parallel lines. Therefore the face is bounded by two pairs of parallel lines. A parallelogram is defined as a quadrilateral with parallel opposite sides. □

(2) In this course the following properties of parallelograms were studied:

(1) In any parallelogram opposite sides are congruent.
(2) In any parallelogram opposite angles are congruent.
(3) In any parallelogram the diagonals bisects each other.

In a parallelepiped, there are two counter-parts to the notion of side: *face* and *edge*. For both of them one can find counter-parts for property (1):
for faces: In a parallelepiped the opposite faces are parallelograms congruent to each other.
for edges: In a parallelepiped there are 12 edges divided into three groups. Each group consists of four congruent edges parallel to each other.

In a parallelepiped, there are two counter-parts to the notion of angle: *dihedral angles* formed by pairs of planes, and angles at vertices formed by triples of planes. For both of them one can find counter-parts for property (2):
for dihedral angles: The dihedral angles at opposite parallel edges are congruent.
for angles at vertices: The angles at vertices at opposite vertices are congruent.
Both follow from symmetry: a parallelepiped is symmetric about a point, the point is the intersection point of its diagonals (segments connecting the opposite vertices). By the way, this symmetry is a counter-part for the symmetry of a parallelogram about the intersection point of its diagonals.

The symmetry gives a counter-part for bisecting of diagonals in a parallelogram. All diagonals of a parallelepiped intersect in its center of symmetry. Here again by a diagonal I mean a segment connecting the opposite vertices. There are 8 vertices and 4 diagonals.

Besides, a parallelepiped has 2-dimensional diagonals: plane sections passing through two pairs of opposite vertices. These sections are parallelograms. Two such sections intersect each other either along a common diagonal or along a common midline. In both cases, the intersection divides each of the parallelogram sections into two congruent figures (triangles or parallelograms).

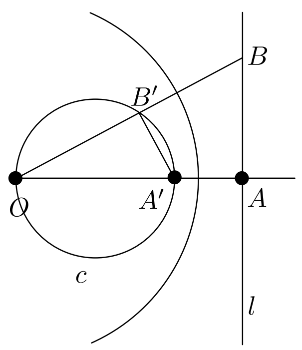
Of course, a solution was not assumed to be so long and include so many theorems.

Problem 6. (8 pt) Prove that the image of a circle under an inversion is either a circle or a line.

Solution. This is a couple of theorems presented in a lecture which can be found on the web page. Here I reproduce this fragment of the lecture.

Obviously, a line passing through the center of an inversion is mapped by the inversion to itself.

Theorem 1. *The image under an inversion of a line l not passing through the center O of the inversion is a circle c passing through O and having at O a tangent line parallel to l.*



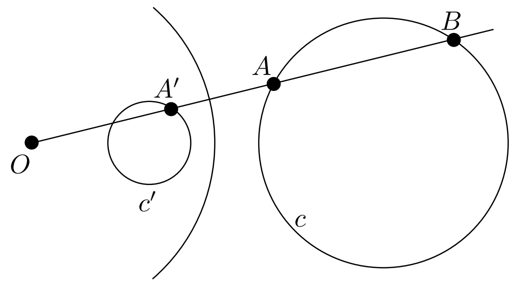
Proof. Drop the perpendicular OA to l from O. Let A be its intersection with l. Let A' be the image of A under the inversion. Take arbitrary point $B \in l$. Denote by B' its image under the inversion. By the definition of inversion $|OA||OA'| = |OB||OB'|$. Therefore $\frac{OA}{OB} = \frac{OB'}{OA'}$. By SAS-test for similar triangles, $\triangle OAB$ is similar to $\triangle OB'A'$. Therefore $\angle A'B'O = \angle OAB$. The latter angle is right, because $OA \perp l$. Hence B' belongs to the circle with diameter OA'.

Vice versa, let us take any point B' of the circle with diameter OA'. Draw a ray OB' and denote the intersection of this ray with l by B. Triangles $\triangle OB'A'$ and $\triangle OAB$ similar by the AA-test. Hence $\frac{OB}{OA} = \frac{OA'}{OB'}$ and $|OB||OB'| = |OA||OA'|$. Therefore, B' is the image of B under the inversion. ☐

Theorem 2. *The image under an inversion of a circle c passing through the center O of the inversion is a line which is parallel to the line tangent to c at O.*

Proof. It follows from Theorem *1*, because an inversion is inverse to itself. ☐

Theorem 3. *The image under an inversion of a circle c that does not pass through the center O of the inversion is a circle c' that is the image of c under a homothety centered at O.*



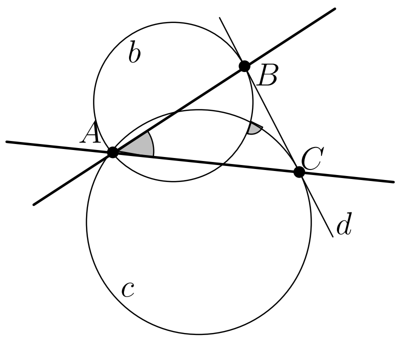
Proof. Let A be a point of circle c, and A' be the image of A under the inversion. Denote by B the second intersection point of the ray OA with c. By definition of inversion, $|OA'| = \frac{R^2}{|OA|}$, where R^2 is the degree of inversion. On the other hand, $|OA| = \frac{d^2}{|OB|}$, where d^2 is the degree of O with respect to the circle c. Recall that d does not depend on the points A and B, this is the length of segment of a tangent line from O to c between O and the point of tangency.

Substituting this formula to the formula for $|OA'|$, we get

$$|OA'| = \frac{R^2}{d^2}|OB|.$$

This means that A' is the image of B under the homothety with center O and ratio $\frac{R^2}{d^2}$. Hence, the image of c under the inversion is the image of c under this homothety. ☐

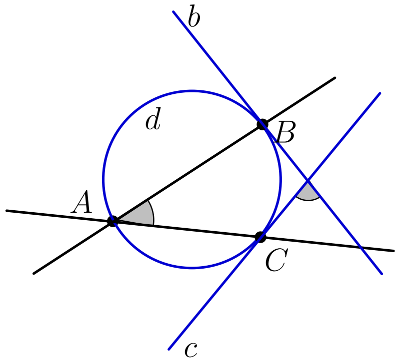
Problem 7. (12 pt) On sides of a fixed angle with vertex A, one chooses points B and C and draws circles b and c passing through A and tangent to BC at points B and C, respectively.



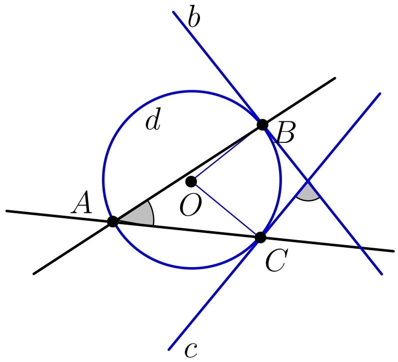
(1) Draw the image of this picture under an inversion centered at A.
(2) Prove that the angle between circles b and c that is marked on the picture above does not depend on the choice of B and C.
(3) Find the relation of the angle between b and c to $\angle A$.

Solution. (1) An inversion centered at A maps lines AB and AC passing through A to themselves. Denote the line BC by b. This line is mapped to the circle passing through A and the images of

B and C. The circles b and c are mapped to lines. They pass through the images of points B and C, respectively, and are tangent to the circle that is the image of line d. In the picture below the configuration of images is shown. The images are marked with the same letters as the original figures.

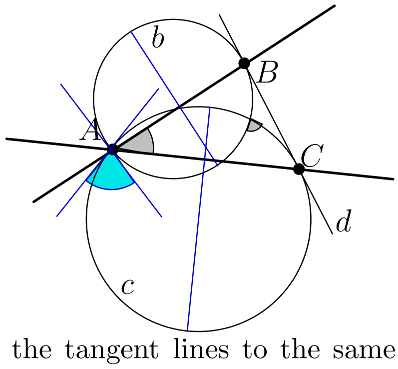


(3) Since the inversion preserves the angles, the angle between lines b and c is equal to the angle between their pre-images, circles b and c. The relation between angles is easier to see in the image. The angle $\angle A$ is inscribed in circle d and equals the half of the central angle $\angle BOC$, where O is the center of circle d.



The sides of the angle between b and c are tangent lines at B and C to circle d. They are perpendicular to the sides of the angle $\angle BOC$. Therefore the angle between b and c is $2\angle BAC$.

Of course, the same result can be obtained in the original picture. Observe that the angle between circles b and c, that is the angle between their tangent lines at the intersection point, does not depend on the intersection point, and can be measured at A as the angle between the lines tangent to circles b and c at A.



These tangent lines are related to the tangent lines to the same circles at points B and C (which coincide with each other: this is line $BC = d$). Namely, the tangent lines to b and c at A are the images of BC under reflections about the diameters of circles b and c perpendicular to the chords AB and AC, respectively. Thus one of these tangent lines can be obtained from the other one by the composition of reflections in these diameters. The diameters are perpendicular to the sides of $\angle BAC$. Therefore the composition of reflections in them is a rotation by the angle $2\angle BAC$. □

(2) Independence of the angle on position of B and C on the sides of $\angle A$ follows from the calculation above. □

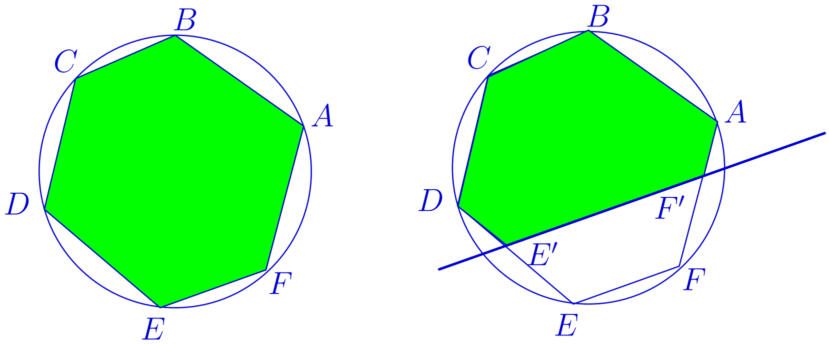
Certainly, it was not expected that all the arguments presented above will be necessary for the full credit. It would suffice to provide one calculation of the angle. The calculation in the image of inversion is more straightforward.

Problem 8. (5 pt) It is known that if a convex hexagon $ABCDEF$ can be inscribed in a circle, then the sum of interior angles at A, C and E is $360°$.

Show that the converse is not true. Even more, prove that it is impossible to recognize whether a convex hexagon can be inscribed in a circle if only interior angles are known.

Solution. For any convex hexagon $ABCDEF$ inscribed in a circle there exists a hexagon with exactly the same interior angles which cannot be inscribed in a circle.

Indeed, choose one of the sides of an inscribed hexagon, say, EF. Draw a line parallel to EF, intersecting the hexagon and sufficiently close to EF so that it would separate EF from the other vertices of the hexagon. Denote the intersection points of this line with DE and FA by E' and F', respectively. Cut out from $ABCDEF$ the quadrilateral $E'EFF'$.



The new hexagon $ABCDE'F'$ has the same interior angles as the original one. It cannot be inscribed in a circle, because otherwise the circle would pass through ABC, but this would determine the old circle, and the new vertices are not on it. □

There are many other ways to solve this problem.

The set of hexagons which can be inscribed in a circle is quite meager in the set of all hexagons. A general hexagon depends on 12 real parameters (coordinates of the vertices), while a hexagon which can be inscribed depends on $6+3=9$ parameters, because the first 3 vertices depend on 6 parameters and determine a circle, while each next vertex should be contained in this circle and therefore depends on 1 parameter each. Therefore one non-degenerate equation on angles cannot describe the set of hexagons that can be inscribed.

We showed above that interior angles are not good parameters, in terms of which restrictions can be stated. On the other hand, the problem belongs to the similarity geometry: a hexagon can or cannot be inscribed in a circle simultaneously with all hexagons similar to it. Therefore, it would be natural to expect that the answer will be formulated in terms of some angles. This is correct, but interior angles should be supplemented with angles between diagonals. Exercise: find the system of equations on the angles between sides and diagonals which would describe the set of hexagons inscribed in a circle.