Linear Algebra

Oleg Viro

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distributivity $(S_1 + S_2)T = S_1T + S_2T$ and $(T_1 + T_2)S = T_1S + T_2S$.

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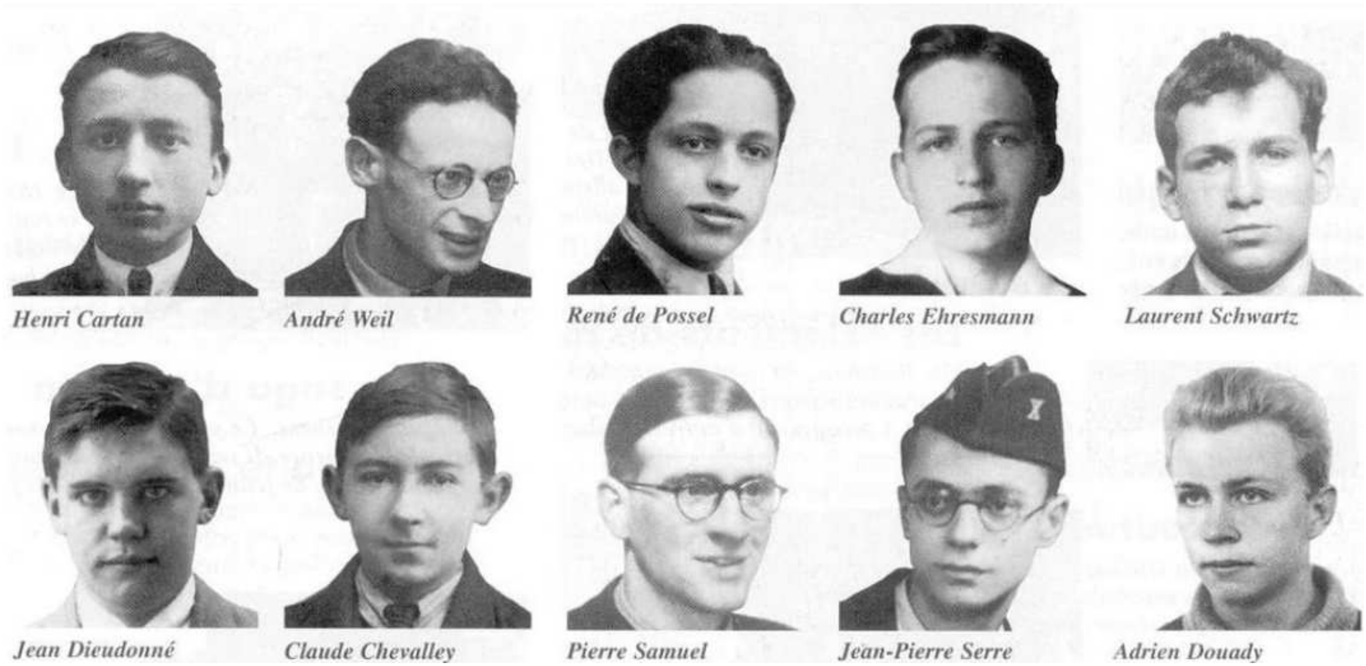
■

liberté, égalité et fraternité



Henri Cartan — *André Weil* — *René de Possel* — *Charles Ehresmann* — *Laurent Schwartz*

Jean Dieudonné — *Claude Chevalley* — *Pierre Samuel* — *Jean-Pierre Serre* — *Adrien Douady*

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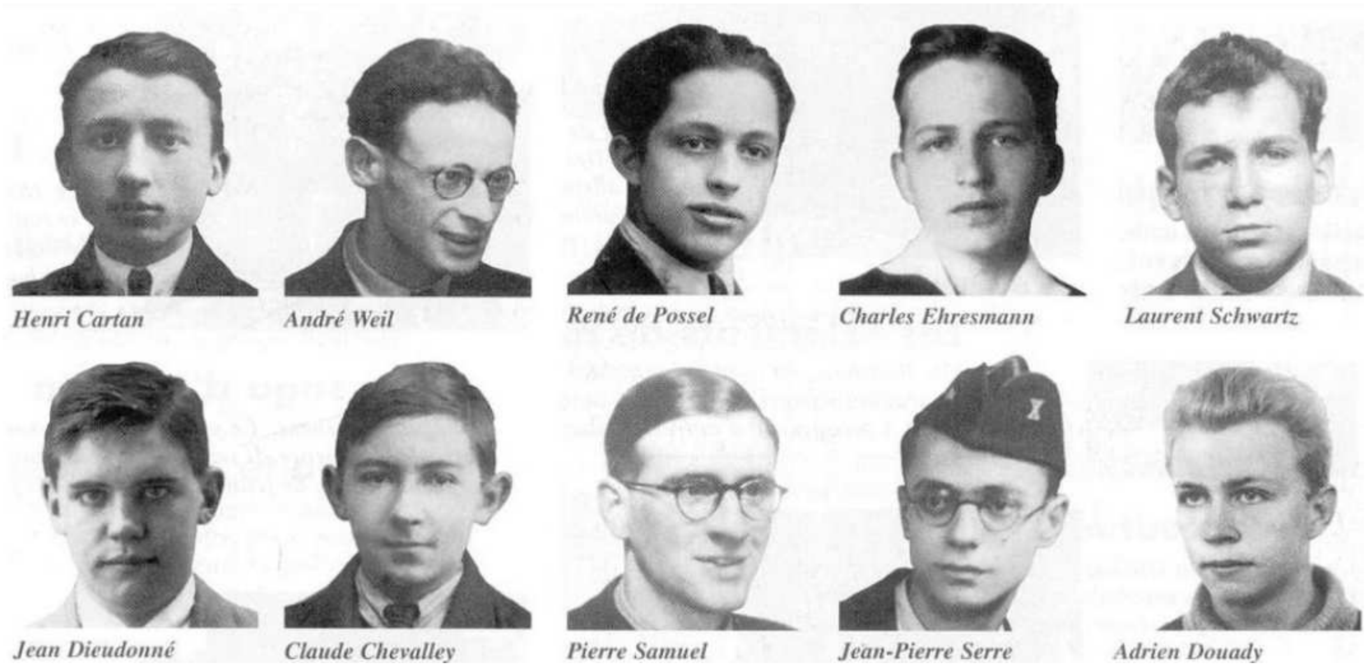

Nicolas Bourbaki

surjectivity, injectivity and bijectivity

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3.20 **Definition** A map $T : V \to W$ is called **surjective** if

surjectivity, injectivity and bijectivity

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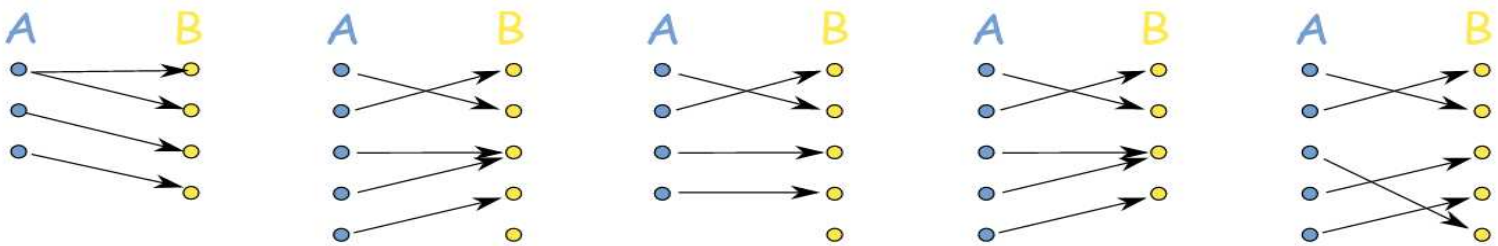

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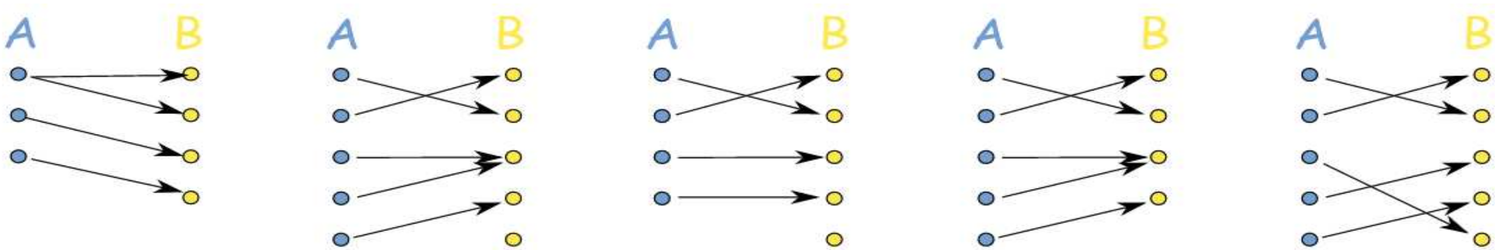

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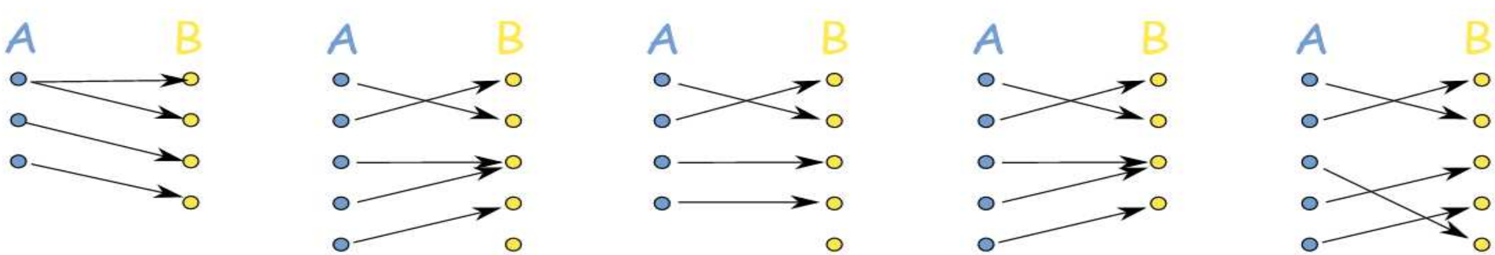

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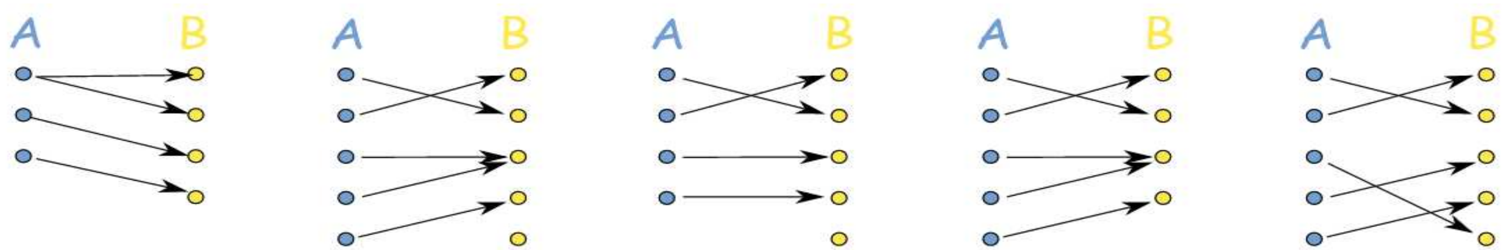

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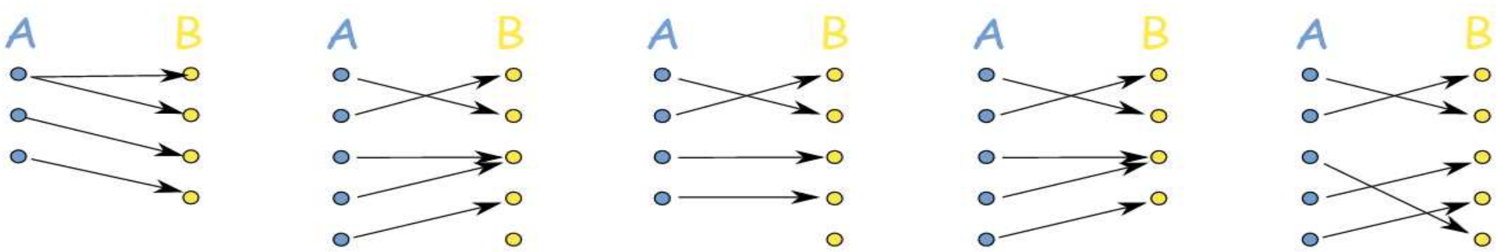

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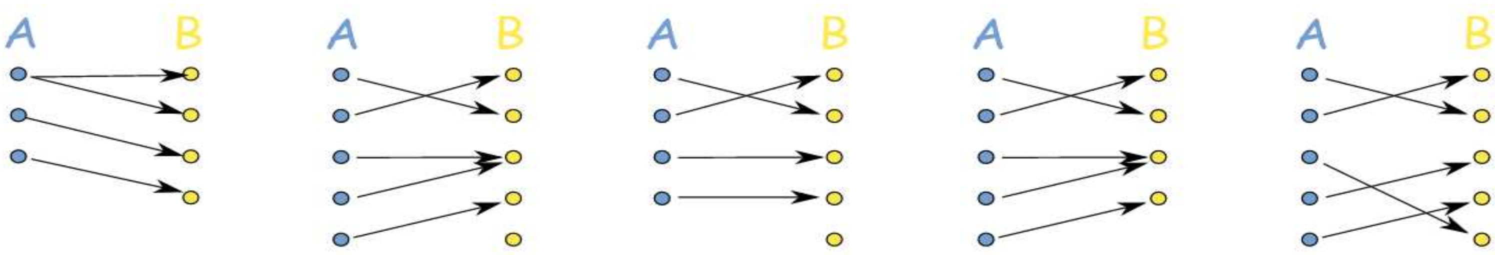

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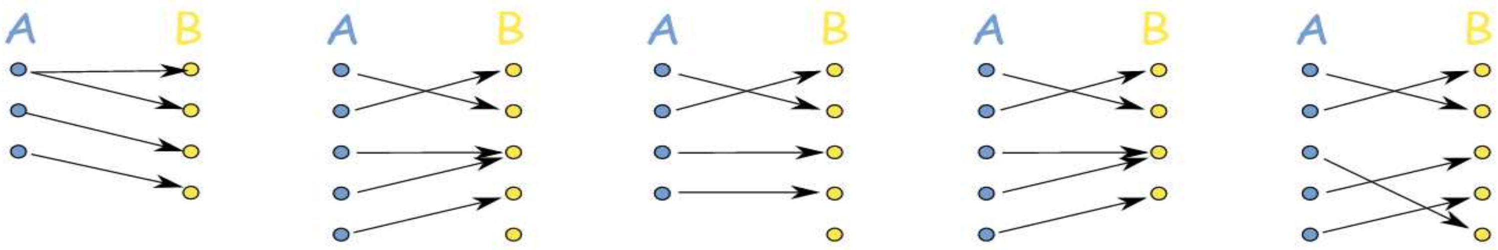

injection, but not surjection

1-to-1

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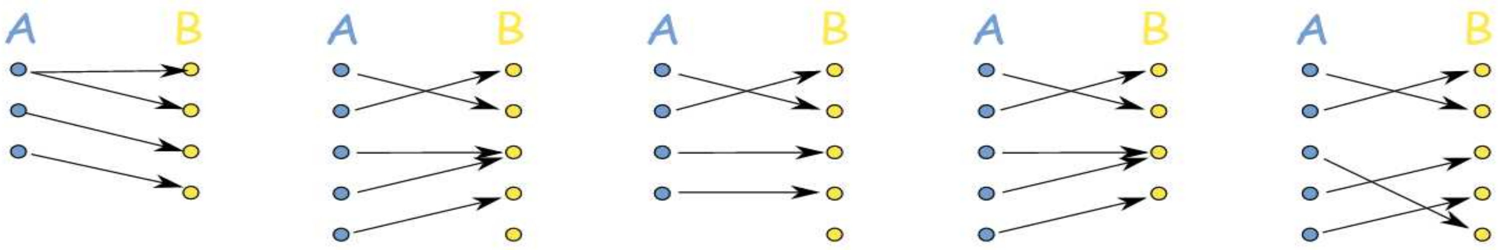

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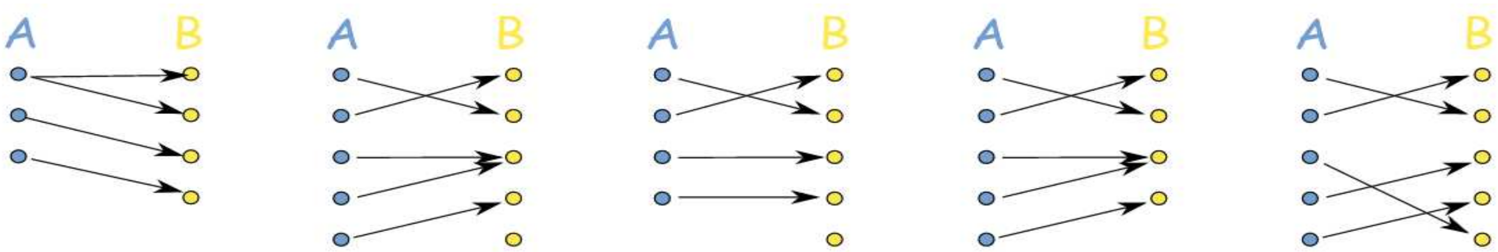

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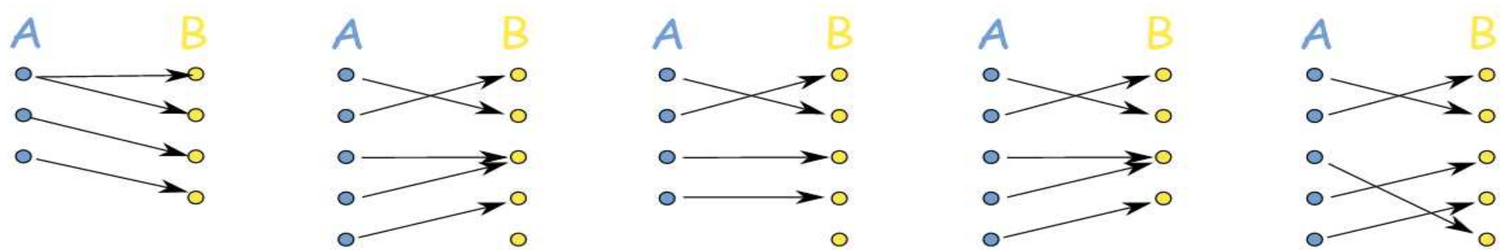

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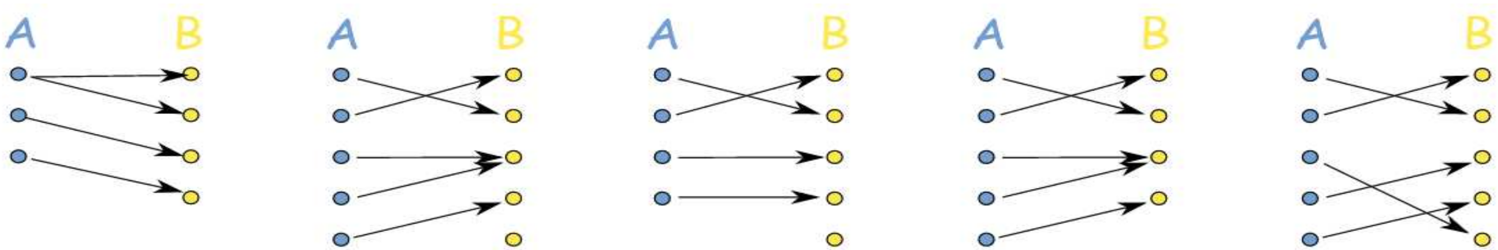

not a map	a map	injection, but not surjection	surjection, but not injection	bijection
		1-to-1	"onto"	invertible

Invertible = bijection

Invertible $=$ bijection

3.56 **Theorem. Invertibility is equivalent to bijectivity.**

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You should know this.

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Isomorphic vector spaces

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Isomorphic finite-dimensional vector spaces have the same dimension.

Linear maps $\mathbb{F}^n \to V$

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Let V be a vector space

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