MAT 150, Introduction to Advanced Mathematics Name ___________________________________
Homework 8, due by 11/3

Score_______________

In the last class, the Pythagorean triples problem was reduced to the problem of finding all rational solutions of the equation $x^2 + y^2 = 1$. The solution was found in the following form:

$$(*) \qquad x = \frac{1-t^2}{1+t^2} \quad y = \frac{2t}{1+t^2}, \quad \text{where } t \in \mathbb{Q}$$

(Since the original problem was about Pythagorean triples of *natural* numbers, we were interested in positive solutions. All the positive rational solutions of the equation $x^2 + y^2 = 1$ are obtained in this way for $t \in \mathbb{Q} \cap (0,1)$. Moreover, all the non-positive rational solutions but one, were obtained either. The missing non-positive solution $x = -1, y = 0$ appears as the limit of the general solution above as $t \to \infty$.)

In this hometask, I suggest a series of problems that allows to generalize the solution (*) of $x^2+y^2 = 1$ to other similar equations, and, in particular, to obtain the solution (*) of $x^2+y^2 = 1$ independently. The whole collection of problems may happen to be too long for a single homework. Make as many problems as you can. The rest will be either discussed in class or included into the future homeworks.

1. Parametric equation for a line. Find formulas similar to (*) (but much simpler!) providing all points of the line with a slope k passing through the point with coordinate (x_0, y_0).

If k, x_0 and y_0 are rational, how to obtain all the points on the line with rational coordinates?

2. Given a polynomial

$$P(x,y) = ax^2 + by^2 + cxy + dx + ey + f$$

of degree two with rational coefficients a, b, c, d, e, f, and a line L with rational slope k,

(1) prove that either the coordinates of each point of L satisfy the equation $P(x,y) = 0$, or there are at most two points of the intersection of L with the curve C defined by the equation $P(x,y) = 0$;
(2) prove that if $L \cap C$ consists of two real points and the coordinates of one of these points are rational, then the coordintes of the other point are also rational.

3. Let $P(x,y)$ be (as above) a polynomial of degree two with rational coefficients, and let x_0, y_0 be rational numbers such that $P(x_0, y_0) = 0$. Find all other rational solutions of the equation $P(x,y) = 0$. (Find formulas similar to (*) expressing the solutions.)

4. Show how the solutions (*) for $x^2 + y^2 = 1$ can be obtained in this way.

5. Generalize the problem 1 to a line in the 3-space, and to a line in n-space. Solve the problems obtained.

6. Generalize the problem 2 to polynomials in n variables. Solve the problem obtained.

7. Let $P(x_1, x_2, \dots x_n)$ be a polynomial of degree two with rational coefficients in n variables $x_1, x_2, \dots x_n$. Assume that the equation $P(x_1, x_2, \dots x_n) = 0$ has a rational particular solution $(a_1, a_2, \dots, a_n)$. How to find all other rational solutions?