4.5. The Cotangent

On the contents of the lecture. In this lecture we perform what was promised at the beginning: we sum up the Euler series and expand $\sin x$ into the product. We will see that sums of series of reciprocal powers are expressed via Bernoulli numbers. And we will see that the function responsible for the summation of the series is the cotangent.

An ingenious idea, which led Euler to finding the sum $\sum_{k=1}^{\infty}\frac{1}{k^2}$, is the following. One can consider $\sin x$ as a polynomial of infinite degree. This polynomial has as roots all points of the type $k\pi$. Any ordinary polynomial can be expanded into a product $\prod(x-x_k)$ where x_k are its roots. By analogy, Euler conjectured that $\sin x$ can be expanded into the product

$$\sin x = \prod_{k=-\infty}^{\infty}(x-k\pi).$$

This product diverges, but can be modified to a convergent one by division of the n-th term by $-n\pi$. The division does not change the roots. The modified product is

$$\prod_{k=-\infty}^{\infty}\left(1-\frac{x}{k\pi}\right) = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right). \tag{4.5.1}$$

Two polynomials with the same roots can differ by a multiplicative constant. To find the constant, consider $x=\frac{\pi}{2}$. In this case we get the inverse to the Wallis product in (4.5.1) multiplied by $x=\frac{\pi}{2}$. Hence the value of (4.5.1) is 1, which coincides with $\sin\frac{\pi}{2}$. Thus it is natural to expect that $\sin x$ coincides with the product (4.5.1).

There is another way to tame $\prod_{k=-\infty}^{\infty}(x-k\pi)$. Taking the logarithm, we get a divergent series $\sum_{k=-\infty}^{\infty}\ln(x-k\pi)$, but achieve convergence by termwise differentiation. Since the derivative of $\ln\sin x$ is $\cot x$, it is natural to expect that $\cot x$ coincides with the following function

$$\operatorname{ctg}(x) = \sum_{k=-\infty}^{\infty}\frac{1}{x-k\pi} = \frac{1}{x}+\sum_{k=1}^{\infty}\frac{2x}{x^2-k^2\pi^2}. \tag{4.5.2}$$

Cotangent expansion. The expansion $\frac{z}{e^z-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}z^k$ allows us to get a power expansion for $\cot z$. Indeed, representing $\cot z$ by Euler's formula one gets

$$i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{e^{2iz}+1}{e^{2iz}-1} = i+\frac{2i}{e^{2iz}-1} = i+\frac{1}{z}\frac{2iz}{e^{2iz}-1} = i+\frac{1}{z}\sum_{k=0}^{\infty}\frac{B_k}{k!}(2iz)^k.$$

The term of the last series corresponding to $k=1$ is $2izB_1=-iz$. Multiplied by $\frac{1}{z}$, it turns into $-i$, which eliminates the first i. The summand corresponding to $k=0$ is 1. Taking into account that $B_{2k+1}=0$ for $k>0$, we get

$$\cot z = \frac{1}{z}+\sum_{k=1}^{\infty}(-1)^k\frac{4^kB_{2k}}{(2k)!}z^{2k-1}.$$

Power expansion of $\operatorname{ctg}(z)$**.** Substituting

$$\frac{1}{z^2-n^2\pi^2} = -\frac{1}{n^2\pi^2}\frac{1}{1-\frac{z^2}{n^2\pi^2}} = -\sum_{k=0}^{\infty}\frac{z^{2k}}{(n\pi)^{2k+2}}$$

into (4.5.2) and changing the order of summation, one gets:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{z^{2k}}{(n\pi)^{2k+2}} = \sum_{k=0}^{\infty}\frac{z^{2k}}{\pi^{2k+2}}\sum_{n=1}^{\infty}\frac{1}{n^{2k+2}}.$$

The change of summation order is legitimate in the disk $|z|<1$, because the series absolutely converges there. This proves the following:

LEMMA 4.5.1. $\operatorname{ctg}(z)-\frac{1}{z}$ *is an analytic function in the disk* $|z|<1$*. The* n*-th coefficient of the Taylor series of* $\operatorname{ctg}(z)-\frac{1}{z}$ *at* 0 *is equal to* 0 *for even* n *and is equal to* $\frac{1}{\pi^{n+1}}\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}$ *for any odd* n*.*

Thus the equality $\cot z = \operatorname{ctg}(z)$ would imply the following remarkable equality:

$$\boxed{(-1)^n\frac{4^nB_{2n}}{2n!} = -\frac{1}{\pi^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}}$$

In particular, for $n=1$ it gives the sum of Euler series as $\frac{\pi^2}{6}$.

Exploring the cotangent.

LEMMA 4.5.2. $|\cot z|\le 2$ *provided* $|\operatorname{Im} z|\ge 1$*.*

PROOF. Set $z=x+iy$. Then $|e^{iz}|=|e^{ix-y}|=e^{-y}$. Therefore if $y\ge 1$, then $|e^{2iz}|=e^{-2y}\le\frac{1}{e^2}<\frac{1}{3}$. Hence $|e^{2iz}+1|\le\frac{1}{e^2}+1<\frac{4}{3}$ and $|e^{2iz}-1|\ge 1-\frac{1}{e^2}>\frac{2}{3}$. Thus the absolute value of

$$\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{e^{2iz}+1}{e^{2iz}-1}$$

is less than 2. For $y\ge 1$ the same arguments work for the representation of $\cot z$ as $i\frac{1+e^{-2iz}}{1-e^{-2iz}}$. □

LEMMA 4.5.3. $|\cot(\pi/2+iy)|\le 4$ *for all* y*.*

PROOF. $\cot(\pi/2+iy)=\frac{\cos(\pi/2+iy)}{\sin(\pi/2+iy)}=\frac{-\sin iy}{\cos iy}=\frac{e^t-e^{-t}}{e^t+e^{-t}}$. The module of the numerator of this fraction does not exceed $e-e^{-1}$ for $t\in[-1,1]$ and the denominator is greater than 1. This proves the inequality for $y\in[-1,1]$. For other y this is the previous lemma. □

Let us denote by $\pi\mathbb{Z}$ the set $\{k\pi \mid k\in\mathbb{Z}\}$ of π*-integers.*

LEMMA 4.5.4. *The set of singular points of* $\cot z$ *is* $\pi\mathbb{Z}$*. All these points are simple poles with residue* 1*.*

PROOF. The singular points of $\cot z$ coincide with the roots of $\sin z$. The roots of $\sin z$ are roots of the equation $e^{iz}=e^{-iz}$ which is equivalent to $e^{2iz}=1$. Since $|e^{2iz}|=|e^{-2\operatorname{Im} z}|$ one gets $\operatorname{Im} z=0$. Hence $\sin z$ has no roots beyond the real line. And all its real roots as we know have the form $\{k\pi\}$. Since $\lim_{z\to 0} z\cot z = \lim_{z\to 0}\frac{z\cos z}{\sin z}=\lim_{z\to 0}\frac{z}{\sin z}=\frac{1}{\sin' 0}=1$, we get that 0 is a simple pole of $\cot z$

with residue 1 and the other poles have the same residue because of periodicity of $\cot z$. $\square$

LEMMA 4.5.5. *Let $f(z)$ be an analytic function on a domain D. Suppose that f has in D finitely many singular points, they are not π-integers and D has no π-integer point on its boundary. Then*

$$\oint_{\partial D} f(\zeta)\cot\zeta d\zeta = 2\pi i\sum_{k=-\infty}^{\infty} f(k\pi)[k\pi\in D] + 2\pi i\sum_{z\in D}\operatorname{res}_z(f(z)\cot z)[z\notin\pi\mathbb{Z}].$$

PROOF. In our situation every singular point of $f(z)\cot z$ in D is either a π-integer or a singular point of $f(z)$. Since $\operatorname{res}_{z=k\pi}\cot z = 1$, it follows that $\operatorname{res}_{z=k\pi} f(z)\cot z = f(k\pi)$. Hence the conclusion of the lemma is a direct consequence of Residue Theory. $\square$

Exploring $\operatorname{ctg}(z)$**.**

LEMMA 4.5.6. $\operatorname{ctg}(z+\pi) = \operatorname{ctg}(z)$ *for any* z.

PROOF.

$$\begin{aligned}
\operatorname{ctg}(z+\pi) &= \lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{z+\pi-k\pi}\\
&= \lim_{n\to\infty}\sum_{k=-n-1}^{n-1}\frac{1}{z+k\pi}\\
&= \lim_{n\to\infty}\frac{1}{z-(n+1)\pi} + \lim_{n\to\infty}\frac{1}{z-n\pi} + \lim_{n\to\infty}\sum_{k=-(n-1)}^{(n-1)}\frac{1}{z+\pi-k\pi}\\
&= 0+0+\operatorname{ctg}(z).
\end{aligned}$$

$\square$

LEMMA 4.5.7. *The series representing* $\operatorname{ctg}(z)$ *converges for any z which is not a π-integer.* $|\operatorname{ctg}(z)|\le 2$ *for all z such that* $|\operatorname{Im} z| > \pi$.

PROOF. For any z one has $|z^2-k^2\pi^2|\ge k^2$ for $k>|z|$. This provides the convergence of the series. Since $\operatorname{ctg}(z)$ has period π, it is sufficient to prove the inequality of the lemma in the case $x\in[0,\pi]$, where $z=x+iy$. In this case $|y|\ge|x|$ and $\operatorname{Re} z^2 = x^2-y^2\le 0$. Then $\operatorname{Re}(z^2-k^2\pi^2)\le -k^2\pi^2$. It follows that $|z^2-k^2\pi^2|\ge k^2\pi^2$. Hence $|\operatorname{ctg}(z)|$ is termwise majorized by $\frac{1}{\pi}+\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2} < 2$. $\square$

LEMMA 4.5.8. $|\operatorname{ctg}(z)|\le 3$ *for any z with* $\operatorname{Re} z = \frac{\pi}{2}$.

PROOF. In this case $\operatorname{Re}(z^2-k^2\pi^2) = \frac{\pi^2}{4} - y^2 - k^2\pi^2 \le -k^2$ for all $k\ge 1$. Hence $|C(z)|\le\frac{2}{\pi}+\sum_{k=1}^{\infty}\frac{1}{k^2}\le 1+2=3$. $\square$

LEMMA 4.5.9. *For any $z\ne k\pi$ and domain D which contains z and whose boundary does not contain π-integers, one has*

$$\oint_{\partial D}\frac{\operatorname{ctg}(\zeta)}{\zeta-z}d\zeta = 2\pi i\operatorname{ctg}(z) + 2\pi i\sum_{k=-\infty}^{\infty}\frac{1}{k\pi-z}[k\pi\in D]. \tag{4.5.3}$$

PROOF. As was proved in Lecture 3.6, the series $\sum_{k=-\infty}^{\infty} \frac{1}{(\zeta-z)(\zeta-k\pi)}$ admits termwise integration. The residues of $\frac{1}{(\zeta-z)(\zeta-k\pi)}$ are $\frac{1}{k\pi-z}$ at $k\pi$ and $\frac{1}{z-k\pi}$ at z. Hence

$$\oint_{\partial D} \frac{1}{(\zeta-z)(\zeta-k\pi)} d\zeta = \begin{cases} 2\pi i \frac{1}{z-k\pi} & \text{for } k\pi \notin D, \\ 0 & \text{if } k\pi \in D. \end{cases}$$

It follows that

$$\begin{aligned} \oint_{\partial D} \frac{\operatorname{ctg}(\zeta)}{\zeta - z} d\zeta &= 2\pi i \sum_{k=-\infty}^{\infty} \frac{1}{z-k\pi}[k\pi \notin D] \\ &= 2\pi i \operatorname{ctg}(z) - \sum_{k=-\infty}^{\infty} \frac{1}{z-k\pi}[k\pi \in D]. \end{aligned}$$

□

LEMMA 4.5.10. $\operatorname{ctg}(z)$ *is an analytic function defined on the whole plane, having all* π*-integers as its singular points, where it has residues* 1.

PROOF. Consider a point $z \notin \pi\mathbb{Z}$. Consider a disk D, not containing π-integers with center at z. Then formula (4.5.3) transforms to the Cauchy Integral Formula. And our assertion is proved by termwise integration of the power expansion of $\frac{1}{\zeta-z}$ just with the same arguments as was applied there. The same formula (4.5.3) allows us to evaluate the residues. □

THEOREM 4.5.11. $\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2\pi^2}$.

PROOF. Consider the difference $R(z) = \cot z - \operatorname{ctg}(z)$. This is an analytic function which has π-integers as singular points and has residues 0 in all of these. Hence $R(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{R(\zeta)}{\zeta-z} d\zeta$ for any $z \notin \pi\mathbb{Z}$. We will prove that $R(z)$ is constant. Let z_0 and ζ be a pair of different points not belonging to $\pi\mathbb{Z}$. Then for any D such that $\partial D \cap \pi\mathbb{Z} = \varnothing$ one has

$$\begin{aligned} R(z) - R(z_0) &= \frac{1}{2\pi i}\oint_{\partial D} R(\zeta)\left(\frac{1}{\zeta-z} - \frac{1}{\zeta-z_0}\right) d\zeta \\ &= \frac{1}{2\pi i}\oint_{\partial D} \frac{R(z)(z-z_0)}{(\zeta-z)(\zeta-z_0)}. \end{aligned} \tag{4.5.4}$$

Let us define D_n for a natural $n > 3$ as the rectangle bounded by the lines $\operatorname{Re} z = \pm(\pi/2 - n\pi)$, $\operatorname{Im} z = \pm n\pi$. Since $|R(z)| \leq 7$ by Lemmas 4.5.2, 4.5.3, 4.5.7, and 4.5.8 the integrand of (4.5.4) eventually is bounded by $\frac{7|z-z_0|}{n^2}$. The contour of integration consists of four monotone curves of diameter $< 2n\pi$. By the Estimation Lemma 3.5.4, the integral can be estimated from above by $\frac{32\pi n 7|z-z_0|}{n^2}$. Hence the limit of our integral as n tends to infinity is 0. This implies $R(z) = R(z_0)$. Hence $R(z)$ is constant and the value of the constant we find by putting $z = \pi/2$. As $\cot \pi/2 = 0$, the value of the constant is

$$\operatorname{ctg}(\pi/2) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{\pi/2 - k\pi} = \frac{2}{\pi}\lim_{n\to\infty}\sum_{k=-n}^{n} \frac{1}{1-2k}.$$

This limit is zero because

$$\sum_{k=-n}^{n}\frac{1}{1-2k}=\sum_{k=-n}^{0}\frac{1}{1-2k}+\sum_{k=1}^{n}\frac{1}{1-2k}=\sum_{k=0}^{n}\frac{1}{2k+1}+\sum_{k=1}^{n}-\frac{1}{2k-1}=\frac{1}{2n+1}.$$

□

Summation of series by $\cot z$**.**

THEOREM 4.5.12. *For any rational function $R(z)$, which is not singular in integers and has degree ≤ -2, one has $\sum_{k=-\infty}^{\infty} R(n) = -\sum_z \operatorname{res} \pi\cot(\pi z)R(z)$.*

PROOF. In this case the integral $\lim_{n\to\infty}\oint_{\partial D_n/pi} R(z)\pi\cot\pi z = 0$. Hence the sum of all residues of $R(z)\pi\cot\pi z$ is zero. The residues at π-integers gives $\sum_{k=-\infty}^{\infty} R(k)$. The rest gives $-\sum_z \operatorname{res}\pi\cot(\pi z)R(z)$. □

Factorization of $\sin x$**.** Theorem 4.5.11 with πz substituted for z gives the series $\pi\cot\pi z = \sum_{k=-\infty}^{\infty}\frac{1}{z-k}$. The half of the series on the right-hand side consisting of terms with nonnegative indices represents a function, which formally telescopes $-\frac{1}{z}$. The negative half telescopes $\frac{1}{z}$. Let us bisect the series into nonnegative and negative halves and add $\sum_{k=-\infty}^{\infty}\frac{1}{k}[k\neq 0]$ to provide convergence:

$$\sum_{k=-\infty}^{-1}\left(\frac{1}{z-k}+\frac{1}{k}\right)+\sum_{k=0}^{\infty}\left(\frac{1}{z-k}+\frac{1}{k+1}\right)$$
$$=\sum_{k=1}^{\infty}\left(-\frac{1}{k}+\frac{1}{z+k}\right)+\sum_{k=1}^{\infty}\left(\frac{1}{z+1-k}+\frac{1}{k}\right).$$

The first of the series on the right-hand side represents $-\digamma(z)-\gamma$, the second is $\digamma(-z+1)+\gamma$. We get the following *complement formula* for the digamma function:

$$-\digamma(z)+\digamma(1-z)=\pi\cot\pi z.$$

Since $\Theta''(z+1)=\digamma'(z)=\varGamma(z)$ (Lemma 4.4.11) it follows that $\Theta'(1+z)=\digamma(z)+c$ and $\Theta'(-z)=-(\digamma(1-z)+c)$. Therefore $\Theta'(1+z)+\Theta'(-z)=\pi\cot\pi z$. Integration of the latter equality gives $-\Theta(1+z)-\Theta(-z)=\ln\sin\pi z+c$. Changing z by $-z$ we get $\Theta(1-z)+\Theta(z)=-\ln\sin\pi z+c$. Exponentiating gives $\Gamma(1-z)\Gamma(-z)=\frac{1}{\sin\pi z}c$. One defines the constant by putting $z=\frac{1}{2}$. On the left-hand side one gets $\Gamma(\frac{1}{2})^2=\pi$, on the right-hand side, c. Finally we get the *complement formula for the Gamma-function*:

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin\pi z}. \tag{4.5.5}$$

Now consider the product $\prod_{k=1}^{\infty}(1-\frac{x^2}{k^2})$. Its canonical form is

$$\prod_{n=1}^{\infty}\left\{\left(1-\frac{x}{n}\right)e^{\frac{x}{n}}\right\}^{-1}\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}^{-1}. \tag{4.5.6}$$

The first product of (4.5.6) is equal to $-\frac{e^{\gamma x}}{x\Gamma(-x)}$, and the second one is $\frac{e^{-\gamma x}}{x\Gamma(-x)}$. Therefore the whole product is $-\frac{1}{x^2\Gamma(x)\Gamma(-x)}$. Since $\Gamma(1-x)=-x\Gamma(-x)$ we get the following result

$$\frac{1}{\Gamma(x)\Gamma(1-x)}=x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right).$$

Comparing this to (4.5.5) and substituting πx for x we get the Euler formula:

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right).$$

Problems.

1. Expand $\tan z$ into a power series.
2. Evaluate $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.
3. Evaluate $\sum_{k=1}^{\infty} \frac{1}{1+k^4}$.