4.4. Gamma Function

On the contents of the lecture. Euler's Gamma-function is the function responsible for infinite products. An infinite product whose terms are values of a rational function at integers is expressed in terms of the Gamma-function. In particular it will help us prove Euler's factorization of sin.

Telescoping problem. Given a function $f(x)$, find a function $F(x)$ such that $\delta F = f$. This is the *telescoping problem* for functions. In particular, for $f = 0$ any periodic function of period 1 is a solution. In the general case, to any solution of the problem we can add a 1-periodic function and get another solution. The general solution has the form $F(x) + k(t)$ where $F(x)$ is a particular solution and $k(t)$ is a 1-periodic function, called the periodic constant.

The Euler-Maclaurin formula gives a formal solution of the problem, but the Euler-Maclaurin series rarely converges. Another formal solution is

$$F(x) = -\sum_{k=0}^{\infty} f(x+k). \tag{4.4.1}$$

Trigamma. Now let us try to telescope the Euler series. The series (4.4.1) converges for $f(x) = \frac{1}{x^m}$ provided $m \geq 2$ and $x \neq -n$ for natural $n > 1$. In particular, the function

$$\Gamma(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} \tag{4.4.2}$$

is analytic; it is called the *trigamma* function and it telescopes $-\frac{1}{(1+x)^2}$. Its value $\Gamma(0)$ is just the sum of the Euler series.

This function is distinguished among others functions telescoping $-\frac{1}{(1+x)^2}$ by its finite variation.

THEOREM 4.4.1. *There is a unique function $\Gamma(x)$ such that $\delta\Gamma(x) = -\frac{1}{(1+x)^2}$, $\operatorname{var}_\Gamma[0,\infty] < \infty$ and $\Gamma(0) = \sum_{k=1}^{\infty} \frac{1}{k^2}$.*

PROOF. Since Γ is monotone, one has $\operatorname{var}_\Gamma[0,\infty] = \sum_{k=0}^{\infty} |\delta\Gamma| = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Suppose $f(x)$ is another function of finite variation telescoping $\frac{1}{(1+x)^2}$. Then $f(x) - \Gamma(x)$ is a periodic function of finite variation. It is obvious that such a function is constant, and this constant is 0 if $f(1) = \Gamma(1)$. □

Digamma. The series $-\sum_{k=0}^{\infty} \frac{1}{x+k}$, which formally telescopes $\frac{1}{x}$, is divergent. However the series $-\sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k}[k \neq 0]\right)$ is convergent and it telescopes $\frac{1}{x}$, because adding a constant does not affect the differences. Indeed,

$$-\sum_{k=0}^{\infty} \left(\tfrac{1}{x+1+k} - \tfrac{1}{k}[k \neq 0]\right) + \sum_{k=0}^{\infty} \left(\tfrac{1}{x+k} - \tfrac{1}{k}[k \neq 0]\right) = -\sum_{k=0}^{\infty} \delta \tfrac{1}{x+k} = \tfrac{1}{x}.$$

The function

$$\digamma(x) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k}\right) \tag{4.4.3}$$

is called the*digamma* function. Here γ is the Euler constant. The digamma function is an analytic function, whose derivative is the trigamma function, and whose difference is $\frac{1}{1+x}$.

Monotonicity distinguishes $\digamma$ among others function telescoping $\frac{1}{1+x}$.

THEOREM 4.4.2. *There is a unique monotone function $\digamma(x)$ such that $\delta\digamma(x) = \frac{1}{1+x}$ and $\digamma(0) = -\gamma$.*

PROOF. Suppose $f(x)$ is a monotone function telescoping $\frac{1}{1+x}$. Denote by v the variation of $f-\digamma$ on $[0,1]$. Then the variation of $f-\digamma$ over $[1,n]$ is nv. On the other hand, $\operatorname{var}_f[1,n] = \sum_{k=1}^{n} \frac{1}{k} < \ln n + \gamma$. Hence the variation of $f(x) - \digamma(x)$ on $[1,n]$ is less than $2(\gamma + \ln n)$. Hence v for any n satisfies the inequality $nv \leq 2(\gamma + \ln n)$. Since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, we get $v = 0$. Hence $f - \digamma$ is constant, and it is zero if $f(1) = \digamma(1)$. $\square$

LEMMA 4.4.3. $\digamma' = \Gamma$.

PROOF. To prove that $\digamma'(x) = \Gamma(x)$, consider $F(x) = \int_1^x \Gamma(t)\,dt$. This function is monotone, because $F'(x) = \Gamma(x) \geq 0$. Further $(\delta F)' = \delta F' = \delta\Gamma(x) = -\frac{1}{(1+x)^2}$. It follows that $\delta F = \frac{1}{1+x} + c$, where c is a constant. By Theorem 4.4.2 it follows that $F(x+1) - cx - \gamma = \digamma(x)$. Hence $\digamma(x)' = F'(x+1) + c = \Gamma(x)$. This proves that $\digamma'$ is differentiable and has finite variation. As $\delta\digamma(x) = \frac{1}{1+x}$ it follows that $\delta\digamma'(x) = -\frac{1}{(1+x)^2}$. We get that $\digamma'(x) = \Gamma(x)$ by Theorem 4.4.1. $\square$

Telescoping the logarithm. To telescope the logarithm, we start with the formal solution $-\sum_{k=0}^{\infty} \ln(x+k)$. To decrease the divergence, add $\sum_{k=1}^{\infty} \ln k$ termwise. We get $-\ln x - \sum_{k=1}^{\infty}(\ln(x+k) - \ln k) = -\ln x - \sum_{k=1}^{\infty} \ln(1+\frac{x}{k})$. We know that $\ln(1+x)$ is close to x, but the series still diverges. Now convergence can be reached by the subtraction of $\frac{x}{k}$ from the k-th term of the series. This substraction changes the difference. Let us evaluate the difference of $F(x) = -\ln x - \sum_{k=1}^{\infty}(\ln(1+\frac{x}{k}) - \frac{x}{k})$. The difference of the n-th term of the series is

$$\begin{aligned}
&\left(\ln\left(1 + \tfrac{x+1}{k}\right) - \tfrac{x+1}{k}\right) - \left(\ln\left(1 + \tfrac{x}{k}\right) - \tfrac{x}{k}\right) \\
&\quad = \left(\ln(x+k+1) - \ln k - \tfrac{x+1}{k}\right) - \left(\ln(x+k) - \ln k - \tfrac{x}{k}\right) \\
&\quad = \delta \ln(x+k) - \tfrac{1}{k}.
\end{aligned}$$

Hence

$$\begin{aligned}
\delta F(x) &= -\delta \ln x - \textstyle\sum_{k=1}^{\infty} \left(\delta \ln(x+k) - \frac{1}{k}\right) \\
&= \lim_{n\to\infty} \left(-\delta \ln x - \textstyle\sum_{k=1}^{n-1} \left(\delta \ln(x+k) - \frac{1}{k}\right)\right) \\
&= \lim_{n\to\infty} \left(\ln x - \ln(n+x) + \textstyle\sum_{k=1}^{n-1} \frac{1}{k}\right) \\
&= \ln x + \lim_{n\to\infty}(\ln(n) - \ln(n+x)) + \lim_{n\to\infty} \left(\textstyle\sum_{k=1}^{n-1} \frac{1}{k} - \ln n\right) \\
&= \ln x + \gamma.
\end{aligned}$$

As a result, we get the following formula for a function, which telescopes the logarithm:

$$\Theta(x) = -\gamma x - \ln x - \sum_{k=1}^{\infty} \left(\ln\left(1 + \frac{x}{k}\right) - \frac{x}{k}\right). \tag{4.4.4}$$

THEOREM 4.4.4. *The series* (4.4.4) *converges absolutely for all* x *except negative integers. It presents a function* $\Theta(x)$ *such that* $\Theta(1) = 0$ *and* $\delta\Theta(x) = \ln x$.

PROOF. The inequality $\frac{x}{1+x} \le \ln(1+x) \le x$ implies

$$|\ln(1+x) - x| \le \left|\frac{x}{1+x} - x\right| = \left|\frac{x^2}{1+x}\right|. \tag{4.4.5}$$

Denote by ε the distance from x to the closest negative integer. Then due to (4.4.5), the series $\sum_{k=1}^{\infty} \ln\left(\left(1+\frac{y}{k}\right) - \frac{y}{k}\right)$ is termwise majorized by the convergent series $\sum_{k=1}^{\infty} \frac{x^2}{\varepsilon k^2}$. This proves the absolute convergence of (4.4.4).

Since $\lim_{n\to\infty} \sum_{k=1}^{n-1}(\ln(1+\frac{1}{k}) - \frac{1}{k}) = \lim_{n\to\infty}(\ln n - \sum_{k=1}^{n-1}\frac{1}{k}) = -\gamma$, one gets $\Theta(1) = 0$. □

Convexity. There are a lot of functions that telescope the logarithm. The property which distinguishes Θ among others is convexity.

Throughout the lecture θ and $\overline{\theta}$ are nonnegative and *complementary* to each other, that is $\theta + \overline{\theta} = 1$. The function f is called *convex* if, for any x, y, it satisfies the inequality:

$$f(\theta x + \overline{\theta} y) \le \theta f(x) + \overline{\theta} f(y) \quad \forall\theta \in [0,1]. \tag{4.4.6}$$

Immediately from the definition it follows that

LEMMA 4.4.5. *Any linear function* $ax + b$ *is convex.*

LEMMA 4.4.6. *Any sum (even infinite) of convex functions is a convex function. The product of a convex function by a positive constant is a convex function.*

LEMMA 4.4.7. *If* $f(p) = f(q) = 0$ *and* f *is convex, then* $f(x) \ge 0$ *for all* $x \notin [p, q]$.

PROOF. If $x > q$ then $q = x\theta + p\overline{\theta}$ for $\theta = \frac{q-p}{x-p}$. Hence $f(q) \le f(x)\theta + f(p)\overline{\theta} = f(x)$, and it follows that $f(x) \ge f(q) = 0$. For $x < p$ one has $p = x\theta + q\overline{\theta}$ for $\theta = \frac{q-p}{q-x}$. Hence $0 = f(p) \le f(x)\theta + f(q)\overline{\theta} = f(x)$. □

LEMMA 4.4.8. *If* f'' *is nonnegative then* f *is convex.*

PROOF. Consider the function $F(t) = f(l(t))$, where $l(t) = x\overline{\theta} + y\theta$. Newton's formula for $F(t)$ with nodes 0, 1 gives $F(t) = F(0) + \delta F(0)t + \frac{1}{2}F''(\xi)t^{\underline{2}}$. Since $F''(\xi) = (y-x)^2 f''(\xi) > 0$, and $t^{\underline{2}} = t(t-1) < 0$ we get the inequality $F(t) \le F(0)\overline{\theta} + tF(1)$. Since $F(\theta) = f(x\overline{\theta} + y\theta)$ this is just the inequality of convexity. □

LEMMA 4.4.9. *If* f *is convex, then* $0 \le f(a) + \theta\delta f(a) - f(a+\theta) \le \delta^2 f(a-1)$ *for any* a *and any* $\theta \in [0,1]$

PROOF. Since $a + \theta = \overline{\theta}a + \theta(a+1)$ we get $f(a+\theta) \le f(a)\overline{\theta} + f(a+1)\theta = f(a) + \theta\delta f(a)$. On the other hand, the convex function $f(a+x) - f(a) - x\delta f(a-1)$ has roots -1 and 0. By Lemma 4.4.7 it is nonnegative for $x > 0$. Hence $f(a+\theta) \ge f(a) + \theta\delta f(a-1)$. It follows that $f(a) + \theta\delta f(a) - f(a+\theta) \ge f(a) + \theta\delta f(a) - f(a) - \theta\delta f(a-1) = \theta\delta^2 f(a-1)$. □

THEOREM 4.4.10. $\Theta(x)$ *is the unique convex function that telescopes* $\ln x$ *and satisfies* $\Theta(1) = 1$.

PROOF. Convexity of Θ follows from the convexity of the summands of its series. The summands are convex because their second derivatives are nonnegative.

Suppose there is another convex function $f(x)$ which telescopes the logarithm too. Then $\phi(x) = f(x) - \Theta(x)$ is a periodic function, $\delta\phi = 0$. Let us prove that $\phi(x)$ is convex. Consider a pair c, d, such that $|c-d| \leq 1$. Since $f(c\theta + d\overline{\theta}) - \theta f(c) - \overline{\theta} f(d) \leq 0$, as f is convex, one has

$$\begin{aligned}\phi(c\theta + d\overline{\theta}) - \theta\phi(c) - \overline{\theta}\phi(d) &= (f(c\theta + d\overline{\theta}) - \theta f(c) - \overline{\theta} f(d)) \\ &\quad - (\Theta(c\theta + d\overline{\theta}) - \theta\Theta(c) - \overline{\theta}\Theta(d)) \\ &\leq \theta\Theta(c) + \overline{\theta}\Theta(d) - \Theta(c\theta + d\overline{\theta}).\end{aligned}$$

First, prove that ϕ satisfies the following ε-relaxed inequality of convexity:

$$\phi(c\theta + d\overline{\theta}) \leq \theta\phi(c) + \overline{\theta}\phi(d) + \varepsilon. \tag{4.4.7}$$

Increasing c and d by 1, we do not change the inequality as $\delta\phi = 0$. Due to this fact, we can increase c and d to satisfy $\frac{1}{c-1} < \frac{\varepsilon}{3}$. Set $L(x) = \Theta(c) + (x - c)\ln c$. By Lemma 4.4.9 for $x \in [c, c+1]$ one has $|\Theta x - L(x)| \leq \delta^2\Theta(c-1) = \ln c - \ln(c-1) = \ln(1 + \frac{1}{c-1}) \leq \frac{1}{c-1} < \frac{\varepsilon}{3}$. Since $|\Theta(x) - L(x)| < \frac{\varepsilon}{3}$ for $x = c, d, \frac{c+d}{2}$, it follows that $\theta\Theta(c) + \overline{\theta}\Theta(d) - \Theta(c\theta + d\overline{\theta})$ differs from $\theta L(c) + \overline{\theta} L(d) - L(c\theta + d\overline{\theta}) = 0$ by less than by ε. The inequality (4.4.7) is proved. Passing to the limit as ε tends to 0, one eliminates ε.

Hence $\phi(x)$ is convex on any interval of length 1 and has period 1. Then $\phi(x)$ is constant. Indeed, consider a pair a, b with condition $b - 1 < a < b$. Then $a = (b-1)\theta + b\overline{\theta}$ for $\theta = b - a$. Hence $f(a) \leq f(b)\theta + f(b-1)\overline{\theta} = f(b)$. $\square$

LEMMA 4.4.11. $\Theta''(1 + x) = \mathit{\Gamma}(x)$.

PROOF. The function $F(x) = \int_1^x \mathit{\Gamma}(t)\,dt$ is convex because its second derivative is $\mathit{\Gamma}$. The difference of $F' = \mathit{\Gamma}$ is $\frac{1}{1+x}$. Hence $\delta F(x) = \ln(x+1) + c$, where c is some constant. It follows that $F(x-1) - cx + c = \Theta(x)$. Hence Θ is twice differentiable and its second derivative is $\mathit{\Gamma}$. $\square$

Gamma function. Now we define Euler's *gamma function* $\Gamma(x)$ as $\exp(\Theta(x))$, where $\Theta(x)$ is the function telescoping the logarithm. Exponentiating (4.4.4) gives a representation of the Gamma function in so-called *canonical Weierstrass form*:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)^{-1} e^{\frac{x}{k}}. \tag{4.4.8}$$

Since $\delta \ln \Gamma(x) = \ln x$, one gets the following *characteristic equation* of the Gamma function

$$\Gamma(x + 1) = x\Gamma(x). \tag{4.4.9}$$

Since $\Theta(1) = 0$, according to (4.4.4), one proves by induction that $\Gamma(n) = (n-1)!$ using (4.4.9).

A nonnegative function f is called *logarithmically convex* if $\ln f(x)$ is convex.

THEOREM 4.4.12 (characterization). $\Gamma(x)$ *is the unique logarithmically convex function defined for* $x > 0$, *which satisfies equation* (4.4.9) *for all* $x > 0$ *and takes the value* 1 *at* 1.

PROOF. Logarithmical convexity of $\Gamma(x)$ follows from the convexity of $\Theta(x)$. Further $\Gamma(1) = \exp\Theta(1) = 1$. If f is a logarithmically convex function satisfying the gamma-equation, then $\ln f$ satisfies all the conditions of Theorem 4.4.4. Hence, $\ln f(x) = \Theta(x)$ and $f(x) = \Gamma(x)$. $\square$

THEOREM 4.4.13 (Euler). *For any $x \geq 0$ one has $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.*

Let us check that the integral satisfies all the conditions of Theorem 4.4.12. For $x = 1$ the integral gives $\int_0^\infty e^{-t}\,dt = -e^{-t}\,|_0^\infty = 1$. The integration by parts $\int_0^\infty t^x e^{-t}\,dt = -\int_0^\infty t^x\,de^{-t} = -t^x e^{-t}\,|_0^\infty + \int_0^\infty e^{-t}xt^{x-1}\,dx$ proves that it satisfies the gamma-equation (4.4.9). It remains to prove logarithmic convexity of the integral.

LEMMA 4.4.14 (mean criterium). *If f is a monotone function which satisfies the following* mean inequality $2f(\frac{x+y}{2}) \leq f(x) + f(y)$ *for all x, y then f is convex.*

PROOF. We have to prove the inequality $f(x\theta + y\overline{\theta}) \leq \theta f(x) + \overline{\theta} f(y) = L(\overline{\theta})$ for all θ, x and y. Set $F(t) = f(x + (y-x)t)$; than F also satisfies the mean inequality. And to prove our lemma it is sufficient to prove that $F(t) \leq L(t)$ for all $t \in [0, 1]$.

First we prove this inequality only for all *binary rational* numbers t, that is for numbers of the type $\frac{m}{2^n}$, $m \leq 2^n$. The proof is by induction on n, the degree of the denominator. If $n = 0$, the statement is true. Suppose the inequality $F(t) \leq L(t)$ is already proved for fractions with denominators of degree $\leq n$. Consider $r = \frac{m}{2^{n+1}}$, with odd $m = 2k+1$. Set $r^- = \frac{k}{2^n}$, $r^+ = \frac{k+1}{2^n}$. By the induction hypothesis $F(r^\pm) \leq L(r^\pm)$. Since $r = \frac{r^+ + r^-}{2}$, by the mean inequality one has $F(r) \leq \frac{f(r^+)+f(r^-)}{2} \leq \frac{L(r^+)+L(r^-)}{2} = L(\frac{r^+ + r^-}{2}) = L(r)$.

Thus our inequality is proved for all binary rational t. Suppose $F(t) > L(t)$ for some t. Consider two binary rational numbers p, q such that $t \in [p, q]$ and $|q-p| < \frac{F(t)-L(t)}{|f(y)-f(x)|}$. In this case $|L(p) - L(t)| \leq |p-t||f(y)-f(x)| < |F(t)-L(t)|$. Therefore $F(p) \leq L(p) < F(t)$. The same arguments give $F(q) < F(t)$. This is a contradiction, because t is between p and q and its image under a monotone mapping has to be between images of p and q. $\square$

LEMMA 4.4.15 (Cauchy-Bunyakovski-Schwarz).

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx. \tag{4.4.10}$$

PROOF. Since $\int_a^b (f(x) + tg(x))^2\,dx \geq 0$ for all t, the discriminant of the following quadratic equation is non-negative:

$$t^2 \int_a^b g^2(x)\,dx + 2t\int_a^b f(x)g(x)\,dx + \int_a^b f^2(x)\,dx = 0. \tag{4.4.11}$$

This discriminant is $4\left(\int_a^b f(x)g(x)\,dx\right)^2 - 4\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx$. $\square$

Now we are ready to prove the logarithmic convexity of the Euler integral. The integral is obviously an increasing function, hence by the mean criterion it is sufficient to prove the following inequality:

$$\left(\int_0^\infty t^{\frac{x+y}{2}-1}e^{-t}\,dt\right)^2 \leq \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty t^{y-1}e^{-t}\,dt. \tag{4.4.12}$$

This inequality turns into the Cauchy-Bunyakovski-Schwarz inequality (4.4.10) for $f(x) = t^{\frac{x-1}{2}} e^{-t/2}$ and $g(t) = t^{\frac{y-1}{2}} e^{-t/2}$.

Evaluation of products. From the canonical Weierstrass form it follows that

$$\prod_{n=1}^{\infty} \{(1 - x/n) \exp(x/n)\} = \frac{-e^{\gamma x}}{x\Gamma(-x)}, \tag{4.4.13}$$

$$\prod_{n=1}^{\infty} \{(1 + x/n) \exp(-x/n)\} = \frac{e^{-\gamma x}}{x\Gamma(x)}.$$

One can evaluate a lot of products by splitting them into parts which have this canonical form (4.4.13). For example, consider the product $\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)$. Division by n^2 transforms it into $\prod_{k=1}^{\infty} (1 - \frac{1}{2n})^{-1}(1 + \frac{1}{2n})^{-1}$. Introducing multipliers $e^{\frac{1}{2n}}$ and $e^{-\frac{1}{2n}}$, one gets a canonical form

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{1}{2n}\right) e^{\frac{1}{2n}} \right\}^{-1} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{1}{2n}\right) e^{-\frac{1}{2n}} \right\}^{-1}. \tag{4.4.14}$$

Now we can apply (4.4.13) for $x = \frac{1}{2}$. The first product of (4.4.14) is equal to $-\frac{1}{2}\Gamma(-1/2)e^{-\gamma/2}$, and the second one is $\frac{1}{2}\Gamma(1/2)e^{\gamma/2}$. Since according to the characteristic equation for Γ-function, $\Gamma(1/2) = -\frac{1}{2}\Gamma(1/2)$, one gets $\Gamma(1/2)^2/2$ as the value of Wallis product. Since the Wallis product is $\frac{\pi}{2}$, we get $\Gamma(1/2) = \sqrt{\pi}$.

Problems.

1. Evaluate the product $\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)\left(1 + \frac{2x}{n}\right)\left(1 - \frac{3x}{n}\right)$.
2. Evaluate the product $\prod_{k=1}^{\infty} \frac{k(5+k)}{(3+k)(2+k)}$.
3. Prove: The sum of logarithmically convex functions is logarithmically convex.
4. Prove $\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x x^{-n}}{x}$.
5. Prove $\prod_{k=1}^{\infty} \frac{k}{x+k} \left(\frac{k+1}{k}\right)^x = \Gamma(x+1)$.
6. Prove Legendre's doubling formula $\Gamma(2x)\Gamma(0.5) = 2^{2x-1}\Gamma(x+0.5)\Gamma(x)$.