3.4. Abel's Theorem

On the contents of the lecture. The expansion of the logarithm into power series will be extended to the complex case. We learn the very important Abel's transformation of sum. This transformation is a discrete analogue of integrations by parts. Abel's theorem on the limit of power series will be applied to the evaluation of trigonometric series related to the logarithm. The concept of Abel's sum of a divergent series will be introduced.

Principal branch of the Logarithm. Since $\exp(x+iy) = e^x(\cos y + i\sin y)$, one gets the following formula for the logarithm: $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$, where $\operatorname{Arg} z = \arg z + 2\pi k$. We see that the logarithm is a multi-valued function, that is why one usually chooses a *branch* of the logarithm to work. For our purposes it is sufficient to consider the *principal* branch of the logarithm:

$$\ln z = \ln|z| + i\arg z, \quad -\pi < \arg z \leq \pi.$$

The principal branch of the logarithm is a differentiable function of a complex variable with derivative $\frac{1}{z}$, inverse to $\exp z$. This branch is not continuous at negative numbers. However its restriction on the upper half-plane is continuous and even differentiable at negative numbers.

LEMMA 3.4.1. *For any nonnegative z one has $\int_1^z \frac{1}{\zeta}\,d\zeta = \ln z$.*

PROOF. If $\operatorname{Im} z \neq 0$, the segment $[0,z]$ is contained in the circle $|\zeta - z_0| < |z_0|$ for $z_0 = \frac{|z|^2}{\operatorname{Im} z}$. In this circle $\frac{1}{\zeta}$ expands into a power series, which one can integrate termwise. Since for z^k the result of integration depends only on the ends of path of integration, the same is true for power series. Hence, we can change the path of integration without changing the result. Consider the following path: $p(t) = \cos t + i\sin t$, $t \in [0, \arg z]$. We know the integral $\int_p \frac{1}{\zeta}\,d\zeta = i\arg z$. This path terminates at $\frac{z}{|z|}$. Continue this path by the linear path to z. The integral satisfies $\int_{z/|z|}^z \frac{1}{\zeta}\,d\zeta = \int_1^{|z|} \frac{1}{z/|z|t}\,dt z/|z| = \int_1^{|z|}\frac{1}{t}\,dt = \ln|z|$. Therefore $\int_1^z \frac{1}{\zeta}\,d\zeta = \int_p \frac{1}{\zeta}\,d\zeta + \int_{z/|z|}^z \frac{1}{\zeta}\,d\zeta = i\arg z + \ln|z|$. □

Logarithmic series. In particular for $|1-z| < 1$ termwise integration of the series $\frac{1}{\zeta} = \sum_{k=0}^\infty (1-\zeta)^k$ gives the complex Mercator series:

$$\ln(1+z) = \sum_{k=1}^\infty (-1)^{k+1}\frac{z^k}{k}. \tag{3.4.1}$$

Substitute in this series $-z$ for z and subtract the obtained series from (3.4.1) to get the complex Gregory series:

$$\frac{1}{2}\ln\frac{1+z}{1-z} = \sum_{k=0}^\infty (-1)^k \frac{z^{2k+1}}{2k+1}.$$

In particular for $z = ix$, one has $\left|\frac{1+ix}{1-ix}\right| = 1$ and $\arg\frac{1+ix}{1-ix} = 2\operatorname{arctg} x$. Therefore one gets

$$\operatorname{arctg} x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Since $\arg(1+e^{i\phi}) = \operatorname{arctg}\frac{\sin\phi}{1+\cos\phi} = \operatorname{arctg}\tan(\phi/2) = \frac{\phi}{2}$, the substitution of $\exp(i\phi)$ for z in the Mercator series $\ln(1+e^{i\phi}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^{ik\phi}}{k}$ gives for the imaginary parts:

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\sin k\phi}{k} = \frac{\phi}{2}. \tag{3.4.2}$$

However the last substitution is not correct, because $|e^{i\phi}| = 1$ and (3.4.1) is proved only for $|z| < 1$. To justify it we will prove a general theorem, due to Abel.

Summation by parts. Consider two sequences $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$. The difference of their product $\delta a_k b_k = a_{k+1}b_{k+1} - a_k b_k$ can be presented as

$$\delta(a_k b_k) = a_{k+1}\delta b_k + b_k\delta a_k.$$

Summation of these equalities gives

$$a_n b_n - a_1 b_1 = \sum_{k=1}^{n-1} a_{k+1}\delta b_k + \sum_{k=1}^{n-1} b_k\delta a_k.$$

A permutation of the latter equality gives the so-called *Abel's transformation* of sums

$$\sum_{k=1}^{n-1} b_k\Delta a_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}\Delta b_k.$$

Abel's theorem. One writes $x \to a - 0$ instead of $x \to a$ and $x < a$, and $x \to a + 0$ means $x > a$ and $x \to a$.

THEOREM 3.4.2 (Abel).

$$\textit{If } \sum_{k=0}^{\infty} a_k \textit{ converges, then } \lim_{x\to 1-0}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

PROOF. $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < 1$, because of the boundedness of $\{a_k\}$.

Suppose $\varepsilon > 0$. Set $A(n,m) = \sum_{k=n}^{m} a_k$, $A(n,m)(x) = \sum_{k=n}^{m} a_k x^k$. Choose N so large that

$$|A(0,n) - A(0,\infty)| < \frac{\varepsilon}{9}, \quad \forall n > N. \tag{3.4.3}$$

Applying the Abel transformation for any $m > n$ one gets

$$\begin{aligned}
A(n,m) - A(n,m)(x) &= \sum_{k=n}^{m} a_k(1-x^k) \\
&= (1-x)\sum_{k=n}^{m}\delta A(n-1,k-1)\sum_{j=0}^{k-1} x^j \\
&= (1-x)\Big[A(n-1,m)\sum_{j=0}^{m} x^j - A(n-1,n)\sum_{j=0}^{n} x^j - \sum_{k=n}^{m} A(n-1,k)x^k\Big].
\end{aligned}$$

By (3.4.3) for $n > N$, one gets $|A(n-1,m)| = |(A(0,m) - A) + (A - A(0,n))| \leq \varepsilon/9 + \varepsilon/9 = 2\varepsilon/9$. Hence, we can estimate from above by $\frac{2\varepsilon/3}{1-x}$ the absolute value of the expression in the brackets of the previous equation for $A(n,m) - A(n,m)(x)$. As a result we get

$$|A(n,m) - A(n,m)(x)| \leq \frac{2\varepsilon}{3}, \quad \forall m \geq n > N, \forall x. \tag{3.4.4}$$

Since $\lim_{x \to 1-0} A(0,N)(x) = A(0,N)$ one chooses δ so small that for $x > 1-\delta$ the following inequality holds:

$$|A(0,N) - A(0,N)(x)| < \frac{\varepsilon}{3}.$$

Summing up this inequality with (3.4.4) for $n = N+1$ one gets:

$$|A(0,m) - A(0,m)(x)| < \varepsilon, \quad \forall m > N, |1-x| < \delta.$$

Passing to limits as m tends to infinity the latter inequality gives

$$|A(0,\infty) - A(0,\infty)(x)| \leq \varepsilon, \quad \text{for } |1-x| < \delta.$$

□

Leibniz series. As the first application of the Abel Theorem we evaluate the Leibniz series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. This series converges by the Leibniz Theorem 2.4.3. By the Abel Theorem its sum is

$$\lim_{x \to 1-0} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2k+1} = \lim_{x \to 1-0} \operatorname{arctg} x = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

We get the following remarkable equality:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \dots.$$

Abel sum of a series. One defines the *Abel sum* of a series $\sum_{k=0}^{\infty} a_k$ as the limit $\lim_{x \to 1-0} \sum_{k=0}^{\infty} a_k x^k$. The series which have an Abel sum are called *Abel summable*. The Abel Theorem shows that all convergent series have Abel sums coinciding with their usual sums. However there are a lot of series, which have an Abel sum, but do not converge.

Abel's inequality. Consider a series $\sum_{k=1}^{\infty} a_k b_k$, where the partial sums $A_n = \sum_{k=1}^{n-1} a_k$ are bounded by some constant A and the sequence $\{b_k\}$ is monotone. Then $\sum_{k=1}^{n-1} a_k b_k = \sum_{k=1}^{n-1} b_k \delta A_k = A_n b_n - A_1 b_1 + \sum_{k=1}^{n-1} A_{k+1} \delta b_k$. Since $\sum_{k=1}^{n-1} |\delta b_k| = |b_n - b_1|$, one gets the following inequality:

$$\left| \sum_{k=1}^{n-1} a_k b_k \right| \leq 3A \max\{|b_k|\}.$$

Convergence test.

THEOREM 3.4.3. *Let the sequence of partial sums $\sum_{k=1}^{n-1} a_k$ be bounded, and let $\{b_k\}$ be non-increasing and infinitesimally small. Then $\sum_{k=1}^{\infty} a_k b_k$ converges to its Abel sum, if the latter exists.*

PROOF. The difference between a partial sum $\sum_{k=1}^{n-1} a_k b_k$ and the Abel sum is equal to

$$\lim_{x \to 1-0} \sum_{k=1}^{n-1} a_k b_k (1 - x^k) + \lim_{x \to 1-0} \sum_{k=n}^{\infty} a_k b_k x^k.$$

The first limit is zero, the second limit can be estimated by Abel's inequality from above by $3Ab_n$. It tends to 0 as n tends to infinity. $\square$

Application. Now we are ready to prove the equality (3.4.2). The series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$ has an Abel sum. Indeed,

$$\begin{aligned}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{q^k\sin kx}{k} &= \operatorname{Im}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(qe^{ix})^k}{k}\\ &= \operatorname{Im}\lim_{q\to 1-0}\ln(1+qe^{ix})\\ &= \operatorname{Im}\ln(1+e^{ix}).\end{aligned}$$

The sums $\sum_{k=1}^{n-1}\sin kx = \operatorname{Im}\sum_{k=1}^{n-1}e^{ikx} = \operatorname{Im}\frac{1-e^{inx}}{1-e^{ix}}$ are bounded. And $\frac{1}{k}$ is decreasing and infinitesimally small. Hence we can apply Theorem 3.4.3.

Problems.

1. Evaluate $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\dots$.

2. Evaluate $\sum_{k=1}^{\infty}\frac{\sin 2k}{k}$.

3. $\sum_{k=1}^{\infty}\frac{\cos k\phi}{k} = -\ln|2\sin\frac{\phi}{2}|$, $(0<|\phi|\le\pi)$.

4. $\sum_{k=1}^{\infty}\frac{\sin k\phi}{k} = \frac{\pi-\phi}{2}$, $(0<\phi<2\pi)$.

5. $\sum_{k=0}^{\infty}\frac{\cos(2k+1)\phi}{2k+1} = \frac{1}{2}\ln|2\cot\frac{\phi}{2}|$, $(0<|\phi|<\pi)$

6. $\sum_{k=0}^{\infty}\frac{\sin(2k+1)\phi}{2k+1} = \frac{\pi}{4}$, $(0<\phi<\pi)$

7. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos k\phi}{k} = \ln\left(2\cos\frac{\phi}{2}\right)$, $(-\pi<\phi<\pi)$

8. Find the Abel sum of $1-1+1-1+\dots$.

9. Find the Abel sum of $1-1+0+1-1+0+\dots$.

10. Prove: A periodic series, such that the sum of the period is zero, has an Abel sum.

11. Telescope $\sum_{k=1}^{\infty}\dfrac{k^2}{2^k}$.

12. Evaluate $\sum_{k=0}^{n-1}k\cos kx$.

13. Estimate from above $\sum_{k=n}^{\infty}\frac{\sin kx}{k^2}$.

***14.** Prove: If $\sum_{k=0}^{\infty}a_k$, $\sum_{k=0}^{\infty}b_k$ and their convolution $\sum_{k=0}^{\infty}c_k$ converge, then $\sum_{k=0}^{\infty}c_k = \sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k$.