## 3.3. Euler Formula

**On the contents of the lecture.** The reader becomes acquainted with the most famous Euler formula. Its special case $e^{i\pi} = -1$ symbolizes the unity of mathematics: here $e$ represents Analysis, $i$ represents Algebra, and $\pi$ represents Geometry. As a direct consequence of the Euler formula we get power series for sin and cos, which we need to sum up the Euler series.

**Complex Newton-Leibniz.** For a function of a complex variable $f(z)$ the derivative is defined by the same formula $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We will denote it also by $\frac{df(z)}{dz}$, to distinguish from derivatives of paths: complex valued functions of real variable. For a path $p(t)$ its derivative will be denoted either $p'(t)$ or $\frac{dp(t)}{dt}$. The Newton-Leibniz formula for real functions can be expressed by the equality $\frac{df(t)}{dt}dt = df(t)$. Now we extend this formula to complex functions.

The linearization of a complex function $f(z)$ at $z_0$ has the same form $f(z_0) + f'(z_0)(z - z_0) + o(z)(z - z_0)$, where $o(z)$ is an infinitesimally small function of complex variable. The same arguments as for real numbers prove the basic rules of differentiation: the derivative of sums, products and compositions.

THEOREM 3.3.1. $\frac{dz^n}{dz} = nz^{n-1}$.

PROOF. $\frac{dz}{dz} = 1$ one gets immediately from the definition of the derivative. Suppose the equality $\frac{dz^n}{dz} = nz^{n-1}$ is proved for $n$. Then $\frac{dz^{n+1}}{dz} = \frac{dzz^n}{dz} = z\frac{dz^n}{dz} + z^n\frac{dz}{dz} = znz^{n-1} + z^n = (n+1)z^n dz$. And the theorem is proved by induction. □

A *smooth path* is a differentiable mapping $p\colon [a,b] \to \mathbb{C}$ with a continuous bounded derivative. A function $f(z)$ of a complex variable is called *virtually monotone* if for any smooth path $p(t)$ the functions $\operatorname{Re} f(p(t))$ and $\operatorname{Im} f(p(t))$ are virtually monotone.

LEMMA 3.3.2. *If $f'(z)$ is bounded, then $f(z)$ is virtually monotone.*

PROOF. Consider a smooth path $p$. Then $\frac{df(p(t))}{dt} = f'(p(t))p'(t)$ is bounded by some $K$. Due to Lemma 3.1.15 one has $|f(p(t)) - f(p(t_0))| \le K|t - t_0|$. Hence any partial variation of $f(p(t))$ does not exceed $K(b - a)$. Therefore $\operatorname{var}_{f(p(t))}[a,b] \le K$. □

THEOREM 3.3.3. *If a complex function $f(z)$ has a bounded virtually monotone continuous complex derivative over the image of a smooth path $p\colon [a,b] \to \mathbb{C}$, then $\int_p f'(z)\,dz = f(p(b)) - f(p(a))$.*

PROOF. $\frac{df(p(t))}{dt} = f'(p(t))p'(t) = \frac{d\operatorname{Re} f(p(t))}{dt} + i\frac{d\operatorname{Im} f(p(t))}{dt}$. All functions here are continuous and virtually monotone by hypothesis. Passing to differential forms one gets

$$\begin{aligned}
\frac{df(p(t))}{dt}\,dt &= \frac{d\operatorname{Re} f(p(t))}{dt}\,dt + i\,\frac{d\operatorname{Im} f(p(t))}{dt}\,dt \\
&= d(\operatorname{Re} f(p(t))) + i\,d(\operatorname{Im} f(p(t))) \\
&= d(\operatorname{Re} f(p(t)) + i\operatorname{Im} f(p(t))) \\
&= d(f(p(t))).
\end{aligned}$$

Hence $\int_p f'(z)\,dz = \int_p df(z)$. □

COROLLARY 3.3.4. *If $f'(z) = 0$ then $f(z)$ is constant.*

PROOF. Consider $p(t) = z_0 + (z - z_0)t$, then $f(z) - f(z_0) = \int_p f'(\zeta)\, d\zeta = 0$. $\square$

**Differentiation of series.** Let us say that a complex series $\sum_{k=1}^{\infty} a_k$ majorizes (eventually) another such series $\sum_{k=1}^{\infty} b_k$ if $|b_k| \leq |a_k|$ for all $k$ (resp. for $k$ beyond some $n$).

The series $\sum_{k=1}^{\infty} kc_k(z - z_0)^{k-1}$ is called a *formal derivative* of $\sum_{k=0}^{\infty} c_k(z - z_0)^k$.

LEMMA 3.3.5. *Any power series $\sum_{k=0}^{\infty} c_k(z - z_0)^k$ eventually majorizes its formal derivative $\sum_{k=0}^{\infty} kc_k(z_1 - z_0)^{k-1}$ if $|z_1 - z_0| < |z - z_0|$.*

PROOF. The ratio of the $n$-th term of the derivative to the $n$-th term of the series tends to 0 as $n$ tends to infinity. Indeed, this ratio is $\frac{k(z_1-z_0)^k}{(z-z_0)^k} = kq^k$, where $|q| < 1$ since $|z_1 - z_0| < |z - z_0|$. The fact that $\lim_{n\to\infty} nq^n = 0$ follows from the convergence of $\sum_{k=1}^{\infty} kq^k$ which we already have proved before. This series is eventually majorized by any geometric series $\sum_{k=0}^{\infty} AQ^k$ with $Q > q$. $\square$

A path $p(t)$ is called *monotone* if both $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$ are monotone.

LEMMA 3.3.6. *Let $p\colon [a, b] \to \mathbb{C}$ be a smooth monotone path, and let $f(z)$ be virtually monotone. If $|f(p(t))| \leq c$ for $t \in [a, b]$ then $\left|\int_p f(z)\, dz\right| \leq 4c|p(b) - p(a)|$.*

PROOF. Integration of the inequalities $-c \leq \operatorname{Re} f(p(t)) \leq c$ against $d \operatorname{Re} z$ along the path gives $|\int_p \operatorname{Re} f(z)\, d\operatorname{Re} z| \leq c|\operatorname{Re} p(b) - \operatorname{Re} p(a)| \leq c|p(b) - p(a)|$. The same arguments prove $|\int_p \operatorname{Im} f(z)\, d\operatorname{Im} z| \leq c|\operatorname{Im} p(b) - \operatorname{Im} p(a)| \leq c|p(b) - p(a)|$. The sum of these inequalities gives $|\operatorname{Re} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. The same arguments yields $|\operatorname{Im} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. And the addition of the two last inequalities allows us to accomplish the proof of the Lemma because $|\int_p f(z)\, dz| \leq |\operatorname{Re} \int_p f(z)\, dz| + |\int_p f(z)\, dz|$. $\square$

LEMMA 3.3.7. $|z^n - \zeta^n| \leq n|z - \zeta| \max\{|z^{n-1}|, |\zeta^{n-1}|\}$.

PROOF. $(z^n - \zeta^n) = (z - \zeta)\sum_{k=0}^{n-1} z^k \zeta^{n-k-1}$ and $|z^k \zeta^{n-k-1}| \leq \max\{|z^{n-1}|, |\zeta^{n-1}|\}$. $\square$

A *linear path* from $z_0$ to $z_1$ is defined as a linear mapping $p\colon [a, b] \to \mathbb{C}$, such that $p(a) = z_0$ and $p(b) = z_1$, that is $p(t) = z_0(t - a) + (z_1 - z_0)(t - a)/(b - a)$.

We denote by $\int_a^b f(z)\, dz$ the integral along the linear path from $a$ to $b$.

LEMMA 3.3.8. *For any complex $z$, $\zeta$ and natural $n > 0$ one has*

$$|z^n - z_0^n - nz_0^{n-1}(z - z_0)| \leq 2n(n-1)|z - z_0|^2 \max\{|z|^{n-2}, |z_0|^{n-2}\}. \tag{3.3.1}$$

PROOF. By the Newton-Leibniz formula, $z^n - z_0^n = \int_{z_0}^{z} n\zeta^{n-1}\, d\zeta$. Further,

$$\begin{aligned}\int_{z_0}^{z} n\zeta^{n-1}\, d\zeta &= \int_{z_0}^{z} nz_0^{n-1}\, d\zeta + \int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta \\ &= nz_0^{n-1} + \int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta.\end{aligned}$$

Consequently, the left-hand side of (3.3.1) is equal to $\left|\int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta\right|$. Due to Lemma 3.3.7 the absolute value of the integrand along the linear path does not exceed $(n-1)|z-z_0|\max\{|z^{n-2}|,|z_0^{n-2}|\}$. Now the estimation of the integral by Lemma 3.3.6 gives just the inequality (3.3.1). $\square$

THEOREM 3.3.9. *If $\sum_{k=0}^{\infty} c_k(z_1-z_0)^k$ converges absolutely, then $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ and $\sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ absolutely converge provided by $|z-z_0|<|z_1-z_0|$, and the function $\sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ is the complex derivative of $\sum_{k=0}^{\infty} c_k(z-z_0)^k$.*

PROOF. The series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ and its formal derivative are eventually majorized by $\sum_{k=0}^{\infty} c_k(z_1-z_0)^k$ if $|z-z_0|\le|z_1-z_0|$ by the Lemma 3.3.5. Hence they absolutely converge in the circle $|z-z_0|\le|z_1-z_0|$. Consider

$$R(z)=\sum_{k=0}^{\infty} c_k(z-z_0)^k-\sum_{k=0}^{\infty} c_k(\zeta-z_0)^k-(z-\zeta)\sum_{k=1}^{\infty} kc_k(\zeta-z_0)^{k-1}.$$

To prove that the formal derivative is the derivative of $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ at $\zeta$ it is sufficient to prove that $R(z)=o(z)(z-\zeta)$, where $o(z)$ is infinitesimally small at $\zeta$. One has $R(z)=\sum_{k=1}^{\infty} c_k\left((z-z_0)^k-(\zeta-z_0)^k-k(\zeta-z_0)^{k-1}\right)$. By Lemma 3.3.8 one gets the following estimate: $|R(z)|\le\sum_{k=1}^{\infty} 2|c_k|k(k-1)|z-\zeta|^2|z_2-z_0|^{n-2}$, where $|z_2-z_0|=\max\{|z-z_0|,|\zeta-z_0|\}$. Hence all we need now is to prove that $\sum_{k=1}^{\infty} 2k(k-1)|c_k||z_2-z_0|^{k-2}|z-\zeta|$ is infinitesimally small at $\zeta$. And this in its turn follows from the convergence of $\sum_{k=1}^{\infty} 2k(k-1)|c_k||z_2-z_0|^{k-2}$. The latter may be deduced from Lemma 3.3.5. Indeed, consider $z_3$, such that $|z_2-z_0|<|z_3-z_0|<|z_1-z_0|$. The convergence of $\sum_{k=1}^{\infty} k|c_k||z_3-z_0|^{k-1}$ follows from the convergence of $\sum_{k=0}^{\infty}|c_k||z_1-z_0|^k$ by Lemma 3.3.5. And the convergence of $\sum_{k=2}^{\infty} k(k-1)|c_k||z_2-z_0|^{k-2}$ follows from the convergence of $\sum_{k=1}^{\infty} k|c_k||z_3-z_0|^{k-1}$ by the same lemma. $\square$

COROLLARY 3.3.10. *Let $f(z)=\sum_{k=0}^{\infty} c_kz^k$ converge absolutely for $|z|<r$, and let $a,b$ have absolute values less then $r$. Then $\int_a^b f(z)\,dz=\sum_{k=0}^{\infty}\frac{c_k}{k+1}(b^{k+1}-a^{k+1})$.*

PROOF. Consider $F(z)=\sum_{k=0}^{\infty}\frac{c_kz^{k+1}}{k+1}$. This series is termwise majorized by the series of $f(z)$, hence it converges absolutely for $|z|<r$. By Theorem 3.3.9 $f(z)$ is its derivative for $|z|<r$. In our case $f(z)$ is differentiable and its derivative is bounded by $\sum_{k=0}^{\infty} k|c_k|r_0^k$, where $r_0=\max\{|a|,|b|\}$. Hence $f(z)$ is continuous and virtually monotone and our result now follows from Theorem 3.3.3. $\square$

**Exponenta in $\mathbb{C}$.** The exponenta for any complex number $z$ is defined as $\exp z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$. The definition works because the series $\sum_{k=0}^{\infty}\frac{z^k}{k!}$ absolutely converges for any $z\in\mathbb{C}$.

THEOREM 3.3.11. *The exponenta is a differentiable function of a complex variable with derivative $\exp' z=\exp z$, such that for all complex $z$, $\zeta$ the following* addition formula *holds:* $\exp(z+\zeta)=\exp z\exp\zeta$.

PROOF. The derivative of the exponenta can be evaluated termwise by Theorem 3.3.9. And this evaluation gives $\exp' z=\exp z$. To prove the addition formula consider the following function $r(z)=\frac{\exp(z+\zeta)}{\exp z}$. Differentiation of the equality $r(z)\exp z=\exp(z+\zeta)$ gives $r'(z)\exp z+r(z)\exp z=\exp(z+\zeta)$. Division by $\exp z$ gives $r'(z)+r(z)=r(z)$. Hence $r(z)$ is constant. This constant is determined by substitution $z=0$ as $r(z)=\exp\zeta$. This proves the addition formula. $\square$

LEMMA 3.3.12. *Let $p\colon [a,b] \to \mathbb{C}$ be a smooth path contained in the complement of a neighborhood of* 0*. Then* $\exp \int_p \frac{1}{\zeta}\, d\zeta = \frac{p(b)}{p(a)}$.

PROOF. First consider the case when $p$ is contained in a circle $|z - z_0| < |z_0|$ with center $z_0 \neq 0$. In this circle, $\frac{1}{z}$ expands in a power series:

$$\frac{1}{\zeta} = \frac{1}{z_0 - (z_0 - \zeta)} = \frac{1}{z_0}\frac{1}{1 - \frac{z_0 - \zeta}{z_0}} = \sum_{k=0}^{\infty} \frac{(z_0 - \zeta)^k}{z_0^{k+1}}.$$

Integration of this series is possible to do termwise due to Corollary 3.3.10. Hence the result of the integration does not depend on the path. And Theorem 3.3.9 provides differentiability of the termwise integral and the possibility of its termwise differentiation. Such differentiation simply gives the original series, which represents $\frac{1}{z}$ in this circle.

Consider the function $l(z) = \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta$. Then $l'(z) = \frac{1}{z}$. The derivative of the composition $\exp l(z)$ is $\frac{\exp l(z)}{z}$. Hence the composition satisfies the differential equation $y'z = y$. We search for a solution of this equation in the form $y = zw$. Then $y' = w + w'z$ and our equation turns into $wz + w'z^2 = wz$. Therefore $w' = 0$ and $w$ is constant. To find this constant substitute $z = z_0$ and get $1 = \exp 0 = \exp l(z_0) = wz_0$. Hence $w = \frac{1}{z_0}$ and $\exp l(z) = \frac{z}{z_0}$.

To prove the general case consider a partition $\{x_k\}_{k=0}^{n}$ of $[a,b]$. Denote by $p_k$ the restriction of $p$ over $[x_k, x_{k+1}]$. Choose the partition so small that $|p(x) - p(x_k)| < |p(x_k)|$ for all $x \in [x_k, x_{k+1}]$. Then any $p_k$ satisfies the requirement of the above considered case. Hence $\exp \int_{p_k} \frac{1}{\zeta}\, d\zeta = \frac{p(x_{k+1})}{p(x_k)}$. Further $\exp \int_p \frac{1}{\zeta}\, d\zeta = \exp \sum_{k=0}^{n-1} \int_{p_k} \frac{1}{\zeta}\, d\zeta = \prod_{k=0}^{n-1} \frac{p(x_{k+1})}{p(x_k)} = p(b)/p(a)$. □

THEOREM 3.3.13 (Euler Formula). *For any real $\phi$ one has*

$$\boxed{\exp i\phi = \cos\phi + i\sin\phi}$$

PROOF. In Lecture 2.5 we have evaluated $\int_p \frac{1}{z}\, dz = i\phi$ for $p(t) = \cos t + i \sin t$, $t \in [0, \phi]$. Hence Lemma 3.3.12 applied to $p(t)$ immediately gives the Euler formula. □

**Trigonometric functions in $\mathbb{C}$.** The Euler formula gives power series expansions for $\sin x$ and $\cos x$:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

These expansions are used to define trigonometric functions for complex variable. On the other hand the Euler formula allows us to express trigonometric functions via the exponenta:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \qquad \cos z = \frac{\exp(iz) + \exp(-iz)}{2}.$$

The other trigonometric functions tan, cot, sec, cosec are defined for complex variables by the usual formulas via sin and cos.

**Problems.**

**1.** Evaluate $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

**2.** Prove the formula of Joh. Bernoulli $\int_0^1 x^x \, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$.

**3.** Find $\ln(-1)$.

**4.** Solve the equation $\exp z = i$.

**5.** Evaluate $i^i$.

**6.** Prove $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

**7.** Prove the identity $\sin^2 z + \cos^2 z = 1$.

**8.** Solve the equation $\sin z = 5/3$.

**9.** Solve the equation $\cos z = 2$.

**10.** Evaluate $\sum_{k=0}^{\infty} \frac{\cos k}{k!}$.

**11.** Evaluate $\oint_{|z|=1} \dfrac{dz}{z^2}$.

**12.** Evaluate $\sum_{k=1}^{\infty} q^k \dfrac{\sin kx}{k}$.

**13.** Expand into a power series $e^x \cos x$.