3.1. Newton-Leibniz Formula

On the contents of the lecture. In this lecture appears the celebrated Newton-Leibniz formula — the main tool in the evaluation of integrals. It is accompanied with such fundamental concepts as the derivative, the limit of a function and continuity.

Motivation. Consider the following problem: for a given function F find a function f such that $dF(x) = f(x)\,dx$, over $[a,b]$, that is, $\int_c^d f(t)\,dt = F(d) - F(c)$ for any subinterval $[c,d]$ of $[a,b]$.

Suppose that such an f exists. Since the value of f at a single point does not affects the integral, we cannot say anything about the value of f at any given point. But if f is continuous at a point x_0, its value is uniquely defined by F.

To be precise, the difference quotient $\frac{F(x)-F(x_0)}{x-x_0}$ tends to $f(x_0)$ as x tends to x_0. Indeed, $F(x) = F(x_0) + \int_{x_0}^x f(t)\,dt$. Furthermore, $\int_{x_0}^x f(t)\,dt = f(x_0)(x-x_0) + \int_{x_0}^x (f(t) - f(x_0))\,dt$. Also, $|\int_{x_0}^x (f(t) - f(x_0)\,dt| \leq \operatorname{var}_f[x_0,x]|x-x_0|$. Consequently

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \leq \operatorname{var}_f[x,x_0]. \tag{3.1.1}$$

However, $\operatorname{var}_f[x,x_0]$ can be made arbitrarily small by choosing x sufficiently close to x_0, since $\operatorname{var}_f x_0 = 0$.

Infinitesimally small functions. A set is called a *neighborhood* of a point x if it contains all points *sufficiently close* to x, that is, all points y such that $|y-x|$ is less then a positive number ε.

We will say that a function f is *locally bounded* (above) by a constant C at a point x, if $f(x) \leq C$ for all y sufficiently close to x.

A function $o(x)$ is called *infinitesimally small* at x_0, if $|o(x)|$ is locally bounded at x_0 by any $\varepsilon > 0$.

LEMMA 3.1.1. *If the functions o and ω are infinitesimally small at x_0 then $o \pm \omega$ are infinitesimally small at x_0.*

PROOF. Let $\varepsilon > 0$. Let O_1 be a neighborhood of x_0 where $|o(x)| < \varepsilon/2$, and let O_2 be a neighborhood of x_0 where $|\omega(x)| < \varepsilon/2$. Then $O_1 \cap O_2$ is a neighborhood where both inequalities hold. Hence for all $x \in O_1 \cap O_2$ one has $|o(x) \pm \omega(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

LEMMA 3.1.2. *If $o(x)$ is infinitesimally small at x_0 and $f(x)$ is locally bounded at x_0, then $f(x)o(x)$ is infinitesimally small at x_0.*

PROOF. The neighborhood where $|f(x)o(x)|$ is bounded by a given $\varepsilon > 0$ can be constructed as the intersection of a neighborhood U, where $|f(x)|$ is bounded by a constant C, and a neighborhood V, where $|o(x)|$ is bounded by ε/C. □

DEFINITION. *One says that a function $f(x)$* tends to A *as x tends to x_0 and writes* $\lim_{x\to x_0} f(x) = A$, *if $f(x) = A + o(x)$ on the complement of x_0, where $o(x)$ is infinitesimally small at x_0.*

COROLLARY 3.1.3. *If both the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then the limit* $\lim_{x\to x_0}(f(x)+g(x))$ *also exists and* $\lim_{x\to x_0}(f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$.

PROOF. This follows immediately from Lemma 3.1.1. $\square$

LEMMA 3.1.4. *If the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then also* $\lim_{x\to x_0} f(x)g(x)$ *exists and* $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x)\lim_{x\to x_0} g(x)$.

PROOF. If $f(x) = A+o(x)$ and $g(x) = B+\omega(x)$, then $f(x)g(x) = AB+A\omega(x)+Bo(x) + \omega(x)o(x)$, where $A\omega(x)$, $Bo(x)$ and $\omega(x)o(x)$ all are infinitesimally small at x_0 by Lemma 3.1.2, and their sum is infinitesimally small by Lemma 3.1.1. $\square$

DEFINITION. *A function f is called* continuous *at x_0, if* $\lim_{x\to x_0} f(x) = f(x_0)$.

A function is said to be continuous (without mentioning a point), if it is continuous at all points under consideration.

The following lemma gives a lot of examples of continuous functions.

LEMMA 3.1.5. *If f is a monotone function on $[a,b]$ such that $f[a,b] = [f(a), f(b)]$ then f is continuous.*

PROOF. Suppose f is nondecreasing. Suppose a positive ε is given. For a given point x denote by $x^\varepsilon = f^{-1}(f(x)+\varepsilon)$ and $x_\varepsilon = f^{-1}(f(x)-\varepsilon)$. Then $[x_\varepsilon, x^\varepsilon]$ contains a neighborhood of x, and for any $y \in [x_\varepsilon, x^\varepsilon]$ one has $f(x)+\varepsilon = f(x_\varepsilon) \leq f(y) \leq f(x^\varepsilon) = f(x)+\varepsilon$. Hence the inequality $|f(y)-f(x)| < \varepsilon$ holds locally at x for any ε. $\square$

The following theorem immediately follows from Corollary 3.1.3 and Lemma 3.1.4.

THEOREM 3.1.6. *If the functions f and g are continuous at x_0, then $f+g$ and fg are continuous at x_0.*

The following property of continuous functions is very important.

THEOREM 3.1.7. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g(f(x))$ is continuous at x_0.*

PROOF. Given $\varepsilon > 0$, we have to find a neighborhood U of x_0, such that $|g(f(x)) - g(f(x_0))| < \varepsilon$ for $x \in U$. As $\lim_{y\to f(x_0)} g(y) = g(f(x_0))$, there exists a neighborhood V of $f(x_0)$ such that $|g(y)-g(y_0)| < \varepsilon$ for $y \in V$. Thus it is sufficient to find a U such that $f(U) \subset V$. And we can do this. Indeed, by the definition of neighborhood there is $\delta > 0$ such that V contains $V_\delta = \{y \mid |y-f(x_0)| < \delta\}$. Since $\lim_{x\to x_0} f(x) = f(x_0)$, there is a neighborhood U of x_0 such that $|f(x)-f(x_0)| < \delta$ for all $x \in U$. Then $f(U) \subset V_\delta \subset V$. $\square$

DEFINITION. *A function f is called* differentiable *at a point x_0 if the* difference quotient $\frac{f(x)-(f_0)}{x-x_0}$ *has a limit as x tends to x_0. This limit is called the* derivative *of the function F at the point x_0, and denoted* $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

Immediately from the definition one evaluates the derivative of linear function

$$(ax+b)' = a \tag{3.1.2}$$

The following lemma is a direct consequence of Lemma 3.1.3.

LEMMA 3.1.8. *If f and g are differentiable at x_0, then $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0)+g'(x_0)$.*

Linearization. Let f be differentiable at x_0. Denote by $o(x)$ the difference $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0), \tag{3.1.3}$$

where $o(x)$ is infinitesimally small at x_0. We will call such a representation a *linearization* of $f(x)$.

LEMMA 3.1.9. *If f is differentiable at x_0, then it is continuous at x_0.*

PROOF. All summands but $f(x_0)$ on the right-hand side of (3.1.3) are infinitesimally small at x_0; hence $\lim_{x\to x_0} f(x) = f(x_0)$. □

LEMMA 3.1.10 (on uniqueness of linearization). *If $f(x) = a + b(x - x_0) + o(x)(x - x_0)$, where $\lim_{x\to x_0} o(x) = 0$, then f is differentiable at x_0 and $a = f(x_0)$, $b = f'(x_0)$.*

PROOF. The difference $f(x) - f(x_0)$ is infinitesimally small at x_0 because f is continuous at x_0 and the difference $f(x) - a = b(x - x_0) + o(x)(x - x_0)$ is infinitesimally small by the definition of linearization. Hence $f(x_0) - a$ is infinitesimally small. But it is constant, hence $f(x_0) - a = 0$. Thus we established $a = f(x_0)$.

The difference $\frac{f(x)-a}{x-x_0} - b = o(x)$ is infinitesimally small as well as $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. But $\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-a}{x-x_0}$. Therefore $b - f'(x_0)$ is infinitesimally small. That is $b = f'(x_0)$. □

LEMMA 3.1.11. *If f and g are differentiable at x_0, then fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$.*

PROOF. Consider leneariations $f(x_0)+f'(x_0)(x-x_0)+o(x)(x-x_0)$ and $g(x_0)+g'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Their product is $f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) + (f(x)\omega(x) + f(x_0)o(x))(x - x_0)$. This is the linearization of $f(x)g(x)$ at x_0, because $f\omega$ and go are infinitesimally small at x_0. □

THEOREM 3.1.12. *If f is differentiable at x_0, and g is differentiable at $f(x_0)$ then $g(f(x))$ is differentiable at x_0 and $(g(f(x_0)))' = g'(f(x_0))f'(x_0)$.*

PROOF. Denote $f(x_0)$ by y_0 and substitute into the linearization $g(y) = g(y_0)+g'(y_0)(y - y_0) + o(y)(y - y_0)$ another linearization $y = f(x_0) + f'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Since $y - y_0 = f'(x_0)(x - x_0) + \omega(x)(x - x_0)$, we get $g(y) = g(y_0) + g'(y_0)f'(x_0)(x - x_0) + g'(y_0)(x - x_0)\omega(x) + o(f(x))(x - x_0)$. Due to Lemma 3.1.10, it is sufficient to prove that $g'(y_0)\omega(x) + o(f(x))$ is infinitesimally small at x_0. The first summand is obviously infinitesimally small. To prove that the second one also is infinitesimally small, we remark that $o(f(x_0) = 0$ and $o(y)$ is continuous at $f(x_0)$ and that $f(x)$ is continuous at x_0 due to Lemma 3.1.9. Hence by Theorem 3.1.6 the composition is continuous at x_0 and infinitesimally small. □

THEOREM 3.1.13. *Let f be a virtually monotone function on $[a, b]$. Then $F(x) = \int_a^x f(t)\,dt$ is virtually monotone and continuous on $[a, b]$. It is differentiable at any point x_0 where f is continuous, and $F'(x_0) = f(x_0)$.*

PROOF. If f has a constant sign, then F is monotone. So, if $f = f_1 + f_2$ is a monotonization of f, then $\int_a^x f_1(x)\,dx + \int_a^x f_1(x)\,dx$ is a monotonization of $F(x)$. This proves that $F(x)$ is virtually monotone.

To prove continuity of $F(x)$ at x_0, fix a constant C which bounds f in some neighborhood U of x_0. Then for $x \in U$ one proves that $|F(x) - F(x_0)|$ is infinitesimally small via the inequalities $|F(x) - F(x_0)| = |\int_{x_0}^{x} f(x)\,dx| \leq |\int_{x_0}^{x} C\,dx| = C|x - x_0|$.

Now suppose f is continuous at x_0. Then $o(x) = f(x_0) - f(x)$ is infinitesimally small at x_0. Therefore $\lim_{x\to x_0} \frac{1}{x-x_0}\int_{x_0}^{x} o(x)\,dx = 0$. Indeed for any $\varepsilon > 0$ the inequality $|o(x)| \leq \varepsilon$ holds over $[x_\varepsilon, x_0]$ for some x_ε. Hence $|\int_{x_0}^{x} o(x)\,dx| \leq |\int_{x_0}^{x} \varepsilon\,dx| = \varepsilon|x - x_0|$ for any $x \in [x_0, x_\varepsilon]$.

Then $F(x) = F(x_0) + f(x_0)(x-x_0) + (\frac{1}{x-x_0}\int_{x_0}^{x} o(t)\,dt)(x-x_0)$ is a linearization of $F(x)$ at x_0. $\square$

COROLLARY 3.1.14. *The functions* ln, sin, cos *are differentiable and* $\ln'(x) = \frac{1}{x}$, $\sin' = \cos$, $\cos' = -\sin$.

PROOF. Since $d\sin x = \cos x\,dx$, $d\cos x = -\sin x\,dx$, due to Theorem 3.1.13 both $\sin x$ and $\cos x$ are continuous, and, as they are continuous, the result follows from Theorem 3.1.13. And $\ln' x = \frac{1}{x}$, by Theorem 3.1.13, follows from the continuity of $\frac{1}{x}$. The continuity follows from Lemma 3.1.5. $\square$

Since $\sin'(0) = \cos 0 = 1$ and $\sin 0 = 0$, the linearization of $\sin x$ at 0 is $x + xo(x)$. This implies the following very important equality

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \tag{3.1.4}$$

LEMMA 3.1.15. *If* $f'(x) > 0$ *for all* $x \in [a, b]$*, then* $f(b) > f(a)|$.

PROOF. Suppose $f(a) \geq f(b)$. We construct a sequence of intervals $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ such that their lengths tend to 0 and $f(a_k) \geq f(b_k)$. All steps of construction are the same. The general step is: let m be the middle point of $[a_k, b_k]$. If $f(m) \leq f(a_k)$ we set $[a_{k+1}, b_{k+1}] = [a_k, m]$, otherwise $f(m) > f(a_k) \geq f(b_k)$ and we set $[a_{k+1}, b_{k+1}] = [m, b_k]$.

Now consider a point x belonging to all $[a_k, b_k]$. Let $f(y) = f(x) + (f'(x) + o(x))(y - x)$ be the linearization of f at x. Let U be neighborhood where $|o(x)| < f'(x)$. Then $\operatorname{sgn}(f(y) - f(x)) = \operatorname{sgn}(y - x)$ for all $y \in U$. However for some n we get $[a_n, b_n] \subset U$. If $a_n \leq x < b_n$ we get $f(a_n) \leq f(x) < f(b_n)$ else $a_n < x$ and $f(a_n) < f(x) \leq f(b_n)$. In the both cases we get $f(a_n) < f(b_n)$. This is a contradiction with our construction of the sequence of intervals. $\square$

THEOREM 3.1.16. *If* $f'(x) = 0$ *for all* $x \in [a, b]$*, then* $f(x)$ *is constant.*

PROOF. Set $k = \frac{f(b)-f(a)}{b-a}$. If $k < 0$ then $g(x) = f(x) - kx/2$ has derivative $g'(x) = f'(x) - k/2 > 0$ for all x. Hence by Lemma 3.1.15 $g(b) > g(a)$ and further $f(b) - f(a) > k(b - a)/2$. This contradicts the definition of k. If $k > 0$ then one gets the same contradiction considering $g(x) = -f(x) + kx/2$. $\square$

THEOREM 3.1.17 (Newton-Leibniz). *If* $f'(x)$ *is a continuous virtually monotone function on an interval* $[a, b]$*, then* $\int_a^b f'(x)\,dx = f(b) - f(a)$.

PROOF. Due to Theorem 3.1.13, the derivative of the difference $\int_a^x f'(t)\,dt - f(x)$ is zero. Hence the difference is constant by Theorem 3.1.16. Substituting

$x = a$ we find the constant which is $f(a)$. Consequently, $\int_a^x f'(t)\,dt - f(x) = f(a)$ for all x. In particular, for $x = b$ we get the Newton-Leibniz formula. $\square$

Problems.

1. Evaluate $(1/x)'$, $\sqrt{x}'$, $(\sqrt{\sin x^2})'$.
2. Evaluate $\exp' x$.
3. Evaluate $\operatorname{arctg}' x$, $\tan' x$.
4. Evaluate $|x|'$, $\operatorname{Re} z'$.
5. Prove: $f'(x) \equiv 1$ if and only if $f(x) = x + \text{const}$.
6. Evaluate $\left(\int_x^{x^2} \frac{\sin t}{t}\,dt\right)'$ as a function of x.
7. Evaluate $\sqrt{1-x^2}'$.
8. Evaluate $(\int_0^1 \frac{\sin kt}{t}\,dt)'$ as a function of k.
9. Prove: If f is continuous at a and $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} f(x_n) = f(a)$.
10. Evaluate $\left(\int_0^y [x]\,dx\right)'_y$.
11. Evaluate $\arcsin' x$.
12. Evaluate $\int \frac{dx}{2+3x^2}$.
13. Prove: If $f'(x) < 0$ for all $x < m$ and $f'(x) > 0$ for all $x > m$ then $f'(m) = 0$.
14. Prove: If $f'(x)$ is bounded on $[a, b]$ then f is virtually monotone on $[a, b]$.