## 2.1. Natural Logarithm

**On the contents of the lecture.**

> In the beginning of Calculus was the Word, and the Word was with Arithmetic, and the Word was *Logarithm*[1]

**Logarithmic tables.** Multiplication is much more difficult than addition. A logarithm reduces multiplication to addition. The invention of logarithms was one of the great achievements of our civilization.

In early times, when logarithms were unknown instead of them one used trigonometric functions. The following identity

$$2\cos x \cos y = \cos(x+y) + \cos(x-y)$$

can be applied to calculate products via tables of cosines. To multiply numbers $x$ and $y$, one represents them as cosines $x = \cos a$, $y = \cos b$ using the cosine table. Then evaluate $(a+b)$ and $(a-b)$ and find their cosines in the table. Finally, the results are summed and divided by 2. That is all. A single multiplication requires four searches in the table of cosines, two additions, one subtraction and one division by 2.

A logarithmic function $l(x)$ is a function such that $l(xy) = l(x) + l(y)$ for any $x$ and $y$. If one has a logarithmic table, to evaluate the product $xy$ one has to find in the logarithmic table $l(x)$ and $l(y)$ then sum them and find the antilogarithm of the sum. This is much easier.

The idea of logarithms arose in 1544, when M. Stiefel compared geometric and arithmetic progressions. The addition of exponents corresponds to the multiplication of powers. Hence consider a number close to 1, say, 1.000001. Calculate the sequence of its powers and place them in the left column. Place in the right column the corresponding values of exponents, which are just the line numbers. The logarithmic table is ready.

Now to multiply two numbers $x$ and $y$, find them (or their approximations) in the left column of the logarithmic table, and read their logarithms from the right column. Sum the logarithms and find the value of the sum in the right column. Next to this sum in the left column the product $xy$ stands. The first tables of such logarithms were composed by John Napier in 1614.

**Area of a curvilinear trapezium.** Recall that a sequence is said to be monotone, if it is either increasing or decreasing. The minimal interval which contains all elements of a given sequence of points will be called *supporting interval* of the sequence. And a sequence is called *exhausting* for an interval $I$ if $I$ is the supporting interval of the sequence.

Let $f$ be a non-negative function defined on $[a,b]$. The set $\{(x,y) \mid x \in [a,b] \text{ and } 0 \leq y \leq f(x)\}$ is called a *curvilinear trapezium* under the graph of $f$ over the interval $[a,b]$.

To estimate the area of a curvilinear trapezium under the graph of $f$ over $[a,b]$, choose an exhausting sequence $\{x_i\}_{i=0}^{n}$ for $[a,b]$ and consider the following sums:

$$(2.1.1) \qquad \sum_{k=0}^{n-1} f(x_k)|\delta x_k|, \qquad \sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k| \quad (\text{where } \delta x_k = x_{k+1} - x_k).$$

[1] $\lambda o\gamma o\varsigma$ is Greek for "word", $\alpha\varrho\iota\theta\mu o\varsigma$ means "number".

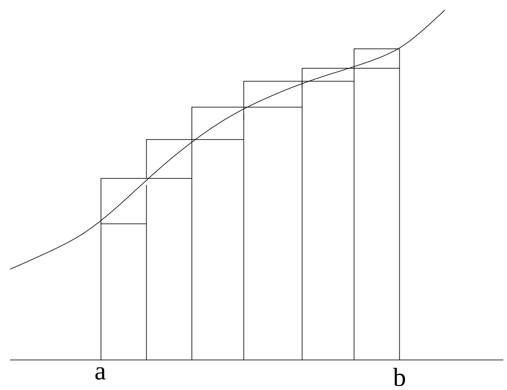


FIGURE 2.1.1. A curvilinear trapezium

We will call the first of them the *receding sum*, and the second the *advancing sum*, of the sequence $\{x_k\}$ for the function $f$. If the function $f$ is monotone the area of the curvilinear trapezium is contained between these two sums. To see this, consider the following step-figures: $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_k)]$ and $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_{k+1})]$. If $f$ and $\{x_k\}$ both increase or both decrease the first step-figure is contained in the curvilinear trapezium and the second step-figure contains the trapezium with possible exception of a vertical segment $[a \times [0, f(a)]$ or $[b \times [0, f(b)]$. If one of $f$ and $\{x_k\}$ increases and the other decreases, then the step-figures switch the roles. The rededing sum equals the area of the first step-figure, and the advancing sum equals the area of the second one. Thus we have proved the following lemma.

LEMMA 2.1.1. *Let $f$ be a monotone function and let $S$ be the area of the curvilinear trapezium under the graph of $f$ over $[a, b]$. Then for any sequence $\{x_k\}_{k=0}^{n}$ exhausting $[a, b]$ the area $S$ is contained between $\sum_{k=0}^{n-1} f(x_k)|\delta x_k|$ and $\sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k|$.*

**Fermat's quadratures of parabolas.** In 1636 Pierre Fermat proposed an ingenious trick to determine the area below the curve $y = x^a$.

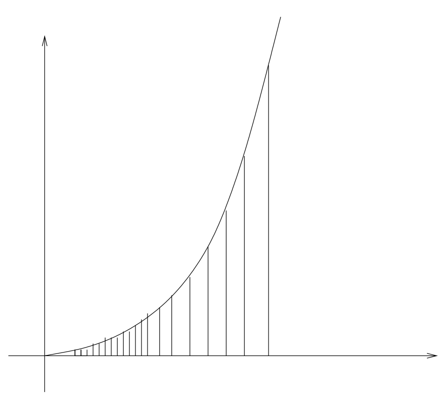

FIGURE 2.1.2. Fermat's quadratures of parabolas

If $a > -1$ then consider any interval of the form $[0, B]$. Choose a positive $q < 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[0, B]$ and the values of the function for this sequence also form a geometric progression $B^a, q^aB^a, q^{2a}B^a, q^{3a}B^a, \dots$. Then both the receding and advancing sums turn into geometric progressions:

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^kB - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(1-q)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^kB - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(1-q)q^a}{1-q^{a+1}}.\end{aligned}$$

For a natural $a$, one has $\frac{1-q}{1-q^{a+1}} = \frac{1}{1+q+q^2+\dots+q^a}$. As $q$ tends to 1 both sums converge to $\frac{B^{a+1}}{a+1}$. This is the area of the curvilinear trapezium. Let us remark that for $a < 0$ this trapezium is unbounded, nevertheless it has finite area if $a > -1$.

If $a < -1$, then consider an interval in the form $[B, \infty]$. Choose a positive $q > 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[B, \infty]$ and the values of the function for this sequence also form a geometric progression $B^a, q^aB^a, q^{2a}B^a, q^{3a}B^a, \dots$. The receding and advancing sums are

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^{k+1}B - q^kB) &= B^{a+1}(q-1)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(q-1)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^{k+1}B - q^kB) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(q-1)q^a}{1-q^{a+1}}.\end{aligned}$$

If $a$ is an integer set $p = q^{-1}$. Then $\frac{q-1}{1-q^{a+1}} = q\frac{1-p}{1-p^{|a|-1}} = q\frac{1}{1+p+p^2+\dots+p^{n-2}}$. As $q$ tends to 1 both sums converge to $\frac{B^{a+1}}{|a|-1}$. This is the area of the curvilinear trapezium.

For $a > -1$ the area of the curvilinear trapezium under the graph of $x^a$ over $[A, B]$ is equal to the difference between the areas of trapezia over $[0, B]$ and $[0, A]$. Hence this area is $\frac{B^{a+1}-A^{a+1}}{a+1}$.

For $a < -1$ one can evaluate the area of the curvilinear trapezium under the graph of $x^a$ over $[A, B]$ as the difference between the areas of trapezia over $[A, \infty]$ and $[B, \infty]$. The result is expressed by the same formula $\frac{B^{a+1}-A^{a+1}}{a+1}$.

THEOREM 2.1.2 (Fermat). *The area below the curve $y = x^a$ over the interval $[A, B]$ is equal to $\frac{B^{a+1}-A^{a+1}}{a+1}$ for $a \neq 1$.*

We have proved this theorem for integer $a$, but Fermat proved it for all real $a \neq -1$.

**The Natural Logarithm.** In the case $a = -1$ the geometric progression for areas of step-figures turns into an arithmetic progression. This means that the area below a hyperbola is a logarithm! This discovery was made by Gregory in 1647.

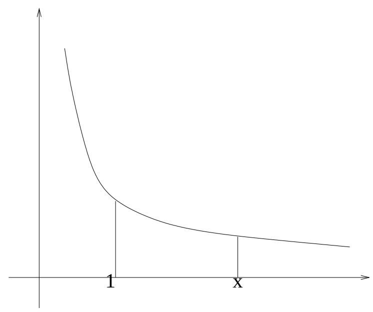


FIGURE 2.1.3. The hyperbolic trapezium over $[1, x]$

The figure bounded from above by the graph of hyperbola $y = 1/x$, from below by segment $[a, b]$ of the axis of abscissas, and on each side by vertical lines passing through the end points of the interval, is called a *hyperbolic trapezium over* $[a, b]$.

The area of hyperbolic trapezium over $[1, x]$ with $x > 1$ is called the *natural logarithm* of $x$, and it is denoted by $\ln x$. For a positive number $x < 1$ its logarithm is defined as the negative number whose absolute value coincides with the area of hyperbolic trapezium over $[x, 1]$. At last, $\ln 1$ is defined as 0.

THEOREM 2.1.3 (on logarithm). *The natural logarithm is an increasing function defined for all positive numbers. For each pair of positive numbers $x$, $y$*

$$\ln xy = \ln x + \ln y.$$

PROOF. Consider the case $x, y > 1$. The difference $\ln xy - \ln y$ is the area of the hyperbolic trapezium over $[y, xy]$. And we have to prove that it is equal to $\ln x$, the area of trapezium over $[1, x]$. Choose a large number $n$. Let $q = x^{1/n}$. Then $q^n = x$. The finite geometric progression $\{q^k\}_{k=0}^n$ exhausts $[1, x]$. Then the receding and advancing sums are

$$(2.1.2) \qquad \sum_{k=0}^{n-1} q^{-k}(q^{k+1} - q^k) = n(q-1) \qquad \sum_{k=0}^{n-1} q^{-k-1}(q^{k+1} - q^k) = \frac{n(q-1)}{q}.$$

Now consider the sequence $\{xq^k\}_{k=0}^n$ exhausting $[x, xy]$. Its receding sum

$$\sum_{k=0}^{n-1} x^{-1}q^{-k}(xq^{k+1} - xq^k) = n(q-1)$$

just coincides with the receding sum (2.1.2) for $\ln x$. The same is true for the advancing sum. As a result we obtain for any natural $n$ the following inequalities:

$$n(q-1) \geq \ln x \geq \frac{n(q-1)}{q} \qquad n(q-1) \geq \ln xy - \ln y \geq \frac{n(q-1)}{q}$$

This implies that $|\ln xy - \ln x - \ln y|$ does not exceed the difference between the the receding and advancing sums. The statement of Theorem 2.1.3 in the case $x, y > 1$ will be proved when we will prove that this difference can be made arbitrarily small by a choice of $n$. This will be deduced from the following general lemma.

LEMMA 2.1.4. *Let $f$ be a monotone function over the interval $[a,b]$ and let $\{x_k\}_{k=0}^n$ be a sequence that exhausts $[a,b]$. Then*

$$\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| \le |f(b)-f(a)| \max_{k<n} |\delta x_k|$$

PROOF OF LEMMA. The proof of the lemma is a straightforward calculation. To shorten the notation, set $\delta f(x_k) = f(x_{k+1}) - f(x_k)$.

$$\begin{aligned}\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| &= \left|\sum_{k=0}^{n-1} \delta f(x_k)\delta x_k\right| \\ &\le \sum_{k=0}^{n-1} |\delta f(x_k)| \max|\delta x_k| \\ &= \max|\delta x_k| \sum_{k=0}^{n-1} |\delta f(x_k)| \\ &= \max|\delta x_k| \left|\sum_{k=0}^{n-1} \delta f(x_k)\right| \\ &= \max|\delta x_k||f(b)-f(a)|.\end{aligned}$$

The equality $\left|\sum_{k=0}^{n-1} \delta f(x_k)\right| = \sum_{k=0}^{n-1} |\delta f(x_k)|$ holds, as $\delta f(x_k)$ have the same signs due to the monotonicity of $f$. □

The value $\max|\delta x_k|$ is called *maximal step* of the sequence $\{x_k\}$. For the sequence $\{q^k\}$ of $[1,x]$ its maximal step is equal to $q^n - q^{n-1} = q^n(1-q^{-1}) = x(1-q)/q$. It tends to 0 as $q$ tends to 1. In our case $|f(b)-f(a)| = 1 - \frac{1}{x} < 1$. By Lemma 2.1.4 the difference between the receding and advancing sums could be made arbitrarily small. This completes the proof in the case $x, y > 1$.

Consider the case $xy = 1$, $x > 1$. We need to prove the following

(inversion rule) $\qquad \ln 1/x = -\ln x.$

As above, put $q^n = x > 1$. The sequence $\{q^{-k}\}_{k=0}^n$ exhausts $[1/x, 1]$. The corresponding receding sum $\sum_{k=0}^{n-1} q^{k+1}(q^{-k} - q^{-k-1}) = \sum_{k=0}^{n-1}(q-1) = n(q-1)$ coincides with its counterpart for $\ln x$. The same is true for the advancing one. The same arguments as above prove $|\ln 1/x| = \ln x$. The sign of $\ln 1/x$ is defined as minus because $1/x < 1$. This proves the inversion rule.

Now consider the case $x < 1$, $y < 1$. Then $1/x > 1$ and $1/y > 1$ and by the first case $\ln 1/xy = (\ln 1/x + \ln 1/y)$. Replacing all terms of this equation according to the inversion rule, one gets $-\ln xy = -\ln x - \ln y$ and finally $\ln xy = \ln x + \ln y$.

The next case is $x > 1$, $y < 1$, $xy < 1$. Since both $1/x$ and $xy$ are less then 1, then by the previous case $\ln xy + \ln 1/x = \ln \frac{xy}{x} = \ln y$. Replacing $\ln 1/x$ by $-\ln x$ one gets $\ln xy - \ln x = \ln y$ and finally $\ln xy = \ln x + \ln y$.

The last case, $x > 1$, $y < 1$, $xy > 1$ is proved by $\ln xy + \ln 1/y = \ln x$ and replacing $\ln 1/y$ by $-\ln y$. □

**Base of a logarithm.** Natural or hyperbolic logarithms are not the only logarithmic functions. Other popular logarithms are decimal ones. In computer science one prefers binary logarithms. Different logarithmic functions are distinguished by their *bases*. The base of a logarithmic function $l(x)$ is defined as the number $b$ for which $l(b) = 1$. Logarithms with the base $b$ are denoted by $\log_b x$. What is the base of the natural logarithm? This is the second most important constant in mathematics (after $\pi$). It is an irrational number denoted by $e$ which is equal to $2.71828182845905\ldots$. It was Euler who introduced this number and this notation.

Well, $e$ is the number such that the area of hyperbolic trapezium over $[1, e]$ is 1. Consider the geometric progression $q^n$ for $q = 1 + \frac{1}{n}$. All summands in the corresponding hyperbolic receding sum for this progression are equal to $\frac{q^{k+1}-q^k}{q^k} = q - 1 = \frac{1}{n}$. Hence the receding sum for the interval $[1, q^n]$ is equal to 1 and it is greater than $\ln q^n$. Consequently $e > q^n$. The summands of the advancing sum in this case are equal to $\frac{q^{k+1}-q^k}{q^{k+1}} = 1 - \frac{1}{q} = \frac{1}{n+1}$. Hence the advancing sum for the interval $[1, q^{n+1}]$ is equal to 1. It is less than the corresponding logarithm. Consequently, $e < q^{n+1}$. Thus we have proved the following estimates for $e$:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

We see that $\left(1 + \frac{1}{n}\right)^n$ rapidly tends to $e$ as $n$ tends to infinity.

**Problems.**

**1.** Prove that $\ln x/y = \ln x - \ln y$.
**2.** Prove that $\ln 2 < 1$.
**3.** Prove that $\ln 3 > 1$.
**4.** Prove that $x > y$ implies $\ln x > \ln y$.
**5.** Is $\ln x$ bounded?
**6.** Prove that $\frac{1}{n+1} < \ln(1 + 1/n) < \frac{1}{n}$.
**7.** Prove that $\frac{x}{1+x} < \ln(1 + x) < x$.
**8.** Prove the Theorem 2.1.2 (Fermat) for $a = 1/2, 1/3, 2/3$.
**9.** Prove the unboundedness of $\frac{n}{\ln n}$.
**10.** Compare $\left(1 + \frac{1}{n}\right)^n$ and $(1 + \frac{1}{n+1})^{n+1}$.
**11.** Prove the monotonicity of $\frac{n}{\ln n}$.
**12.** Prove that $\sum_{k=2}^{n-1} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$.
**13.** Prove that $\ln(1 + x) > x - \frac{x^2}{2}$.
**14.** Estimate integral part of $\ln 1000000$.
**15.** Prove that $\ln \frac{x+y}{2} \geq \frac{\ln x + \ln y}{2}$.
**16.** Prove the convergence of $\sum_{k=1}^{\infty}(\frac{1}{k} - \ln(1 + \frac{1}{k}))$.
**17.** Prove that $(n + \frac{1}{2})^{-1} \leq \ln(1 + \frac{1}{n}) < \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})$.
***18.** Prove that $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \ln 2$.