1.4. Infinite Products

On the contents of the lecture. In this lecture we become acquainted with infinite products. The famous Euler Identity will be proved. We will find out that $\zeta(2)$ is another name for the Euler series. And we will see how Euler's decomposition of the sine function into a product works to sum up the Euler Series.

DEFINITION. *The product of an infinite sequence of numbers* $\{a_k\}$*, such that* $a_k \geq 1$ *for all* k*, is defined as the least number majorizing all partial products* $\prod_{k=1}^n a_k = a_1 a_2 \dots a_n$.

A sequence of natural numbers is called *essentially finite* if all but finitely many of its elements are equal to zero. Denote by $\mathbb{N}^\infty$ the set of all essentially finite sequences of natural numbers.

THEOREM 1.4.1. *For any given sequence of positive series* $\sum_{k=0}^\infty a_k^j$, $j = 1, 2, \dots$ *such that* $a_0^j = 1$ *for all* j *one has*

$$\prod_{j=1}^\infty \sum_{k=0}^\infty a_k^j = \sum_{\{k_j\}\in\mathbb{N}^\infty} \prod_{j=1}^\infty a_{k_j}^j. \tag{1.4.1}$$

The summands on the right-hand side of (1.4.1) usually contain factors which are less than one. But each of the summands contains only finitely many factors different from 1. So the summands are in fact finite products.

PROOF. For a sequence $\{k_j\} \in \mathbb{N}^\infty$ define its *length* as maximal j for which $k_j \neq 0$ and its *maximum* as the value of its maximal term. The length of the zero sequence is defined as 0.

Consider a finite subset $S \subset \mathbb{N}^\infty$. Consider the partial sum

$$\sum_{\{k_j\}\in S} \prod_{k=1}^\infty a_{k_j}^j.$$

To estimate it, denote by L the maximal length of elements of S and denote by M the greatest of maxima of $\{k_j\} \in S$. In this case

$$\sum_{\{k_j\}\in S} \prod_{j=1}^\infty a_{k_j}^j = \sum_{\{k_j\}\in S} \prod_{j=1}^L a_{k_j}^j \leq \sum_{\{k_j\}\in \mathbb{N}_M^L} \prod_{j=1}^L a_{k_j}^j = \prod_{j=1}^L \sum_{k=0}^M a_k^j \leq \prod_{j=1}^\infty \sum_{k=0}^\infty a_k^j,$$

where $\mathbb{N}_M^L$ denotes the set of all finite sequences $\{k_1, k_2, \dots, k_L\}$ of natural numbers such that $k_i \leq M$. By All-for-One this implies one of the required inequalities, namely, $\geq$.

To prove the opposite inequality, we prove that for any natural L one has

$$\prod_{j=1}^L \sum_{k=0}^\infty a_k^j = \sum_{\{k_j\}\in\mathbb{N}^L} \prod_{j=1}^L a_{k_j}^j, \tag{1.4.2}$$

where $\mathbb{N}^L$ denotes the set of all finite sequences $\{k_1, \dots k_L\}$ of natural numbers. The proof is by induction on L.

LEMMA 1.4.2. *For any families $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$ of nonnegative numbers, one has*

$$\sum_{i\in I} a_i \sum_{j\in J} b_j = \sum_{(i,j)\in I\times J} a_i b_j.$$

PROOF OF LEMMA 1.4.2. Since $I \times J = \bigsqcup_{i\in I}\{i\}\times J$ by the Sum Partition Theorem one gets:

$$\begin{aligned}\sum_{(i,j)\in I\times J} a_i b_j &= \sum_{i\in I}\sum_{(i,j)\in\{i\}\times J} a_i b_j\\ &= \sum_{i\in I}\sum_{j\in J} a_i b_j\\ &= \sum_{i\in I} a_i \sum_{j\in J} b_j\\ &= \sum_{j\in J} b_j \sum_{i\in I} a_i.\end{aligned}$$

□

Case $L = 2$ follows from Lemma 1.4.2, because $\mathbb{N}^2 = \mathbb{N}\times\mathbb{N}$. The induction step is done as follows

$$\begin{aligned}\prod_{j=1}^{L+1}\sum_{k=0}^{\infty} a_k^j &= \sum_{k=0}^{\infty} a_k^{L+1}\prod_{j=1}^{L}\sum_{k=0}^{\infty} a_k^j\\ &= \sum_{k\in\mathbb{N}} a_k^{L+1}\sum_{\{k_j\}\in\mathbb{N}^L}\prod_{j=1}^{L} a_{k_j}^j\\ &= \sum_{\{k_j\}\in\mathbb{N}^{L+1}}\prod_{j=1}^{L+1} a_{k_j}^j.\end{aligned}$$

The left-hand side of (1.4.2) is a partial product for the left-hand side of (1.4.1) and the right-hand side of (1.4.2) is a subsum of the right-hand side of (1.4.1). Consequently, all partial products of the right-hand side in (1.4.1) do not exceed its left-hand side. This proves the inequality $\leq$. □

Euler's Identity. Our next goal is to prove the *Euler Identity.*

$$\boxed{\sum_{k=1}^{\infty}\frac{1}{k^\alpha} = \prod_{p=1}^{\infty}\left(1-\frac{1}{p^\alpha}\right)^{-[p \text{ is prime}]}}$$

Here α is any rational (or even irrational) positive number.

The product on the right-hand side is called the *Euler Product.* The series on the left-hand side is called the *Dirichlet series.* Each factor of the Euler Product expands into the geometric series $\sum_{k=0}^{\infty}\frac{1}{p^{k\alpha}}$. By Theorem 1.4.1, the product of these geometric series is equal to the sum of products of the type $p_1^{-k_1\alpha}p_2^{-k_2\alpha}\dots p_n^{-k_n\alpha}$ $= N^{-\alpha}$. Here $\{p_i\}$ are different prime numbers, $\{k_i\}$ are positive natural numbers and $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$. But each product $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$ is a natural number, different products represent different numbers and any natural number has a unique representation of this sort. This is exactly what is called Principal Theorem of

Arithmetic. That is, the above decomposition of the Euler product expands in the Dirichlet series.

Convergence of the Dirichlet series.

THEOREM 1.4.3. *The Dirichlet series* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *converges if and only if* $s > 1$.

PROOF. Consider a $\{2^k\}$ packing of the series. Then the n-th term of the packed series one estimates from above as

$$\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^s} \leq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{(2^n)^s} = 2^n \frac{1}{2^{ns}} = 2^{n-ns} = (2^{1-s})^n.$$

If $s > 1$ then $2^{1-s} < 1$ and the packed series is termwise majorized by a convergent geometric progression. Hence it converges. In the case of the Harmonic series $(s = 1)$ the n-th term of its packing one estimates from below as

$$\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \geq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{2^{n+1}} = 2^n \frac{1}{2^{n+1}} = \frac{1}{2}.$$

That is why the harmonic series diverges. A Dirichlet series for $s < 1$ termwise majorizes the Harmonic series and so diverges. □

The Riemann ζ-function. The function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is called the *Riemann ζ-function*. It is of great importance in number theory.

The simplest application of Euler's Identity represents Euler's proof of the infinity of the set of primes. The divergence of the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ implies the Euler Product has to contain infinitely many factors to diverge.

Euler proved an essentially more exact result: the series of reciprocal primes diverges $\sum \frac{1}{p} = \infty$.

Summing via multiplication. Multiplication of series gives rise to a new approach to evaluating their sums. Consider the geometric series $\sum_{k=0}^{\infty} x^k$. Then

$$\left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{j,k \in \mathbb{N}^2} x^j x^k = \sum_{m=0}^{\infty} \sum_{j+k=m} x^j x^k = \sum_{m=0}^{\infty} (m+1) x^m.$$

As $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ one gets $\sum_{k=0}^{\infty} (k+1) x^k = \frac{1}{(1-x)^2}$.

Sine-product. Now we are ready to understand how two formulas

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right), \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{1.4.3}$$

which appeared in the Legends, yield an evaluation of the Euler Series. Since at the moment we do not know how to multiply infinite sequences of numbers which are less than one, we invert the product in the first formula. We get

$$\frac{x}{\sin x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right)^{-1} = \prod_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{x^{2j}}{k^{2j} \pi^{2j}}. \tag{1.4.4}$$

To avoid negative numbers, we interpret the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

in the second formula of (1.4.3) as the difference

$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$

Substituting this expression for $\sin x$ in $\frac{x}{\sin x}$ and cancelling out x, we get

$$\frac{x}{\sin x} = \frac{1}{1 - \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}} = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}\right)^j.$$

All terms on the right-hand side starting with $j = 2$ are divisible by x^4. Consequently the only summand with x^2 on the right-hand side is $\frac{x^2}{6}$. On the other hand in (1.4.4) after an expansion into a sum by Theorem 1.4.1, the terms with x^2 give the series $\sum_{k=1}^{\infty} \frac{x^2}{k^2\pi^2}$. Comparing these results, one gets $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Problems.

1. Prove $\prod_{n=1}^{\infty} 1.1 = \infty$.

2. Prove the identity $\prod_{n=1}^{\infty} a_n^2 = (\prod_{n=1}^{\infty} a_n)^2$ $(a_n \geq 1)$.

3. Does $\prod_{n=1}^{\infty}(1 + \frac{1}{n})$ converge?

4. Evaluate $\prod_{n=2}^{\infty} \frac{n^2}{n^2-1}$.

5. Prove the divergence of $\prod_{1}^{\infty}(1 + \frac{1}{k})^{[k \text{ is prime}]}$.

6. Evaluate $\prod_{n=3}^{\infty} \frac{n(n+1)}{(n-2)(n+3)}$.

7. Evaluate $\prod_{n=3}^{\infty} \frac{n^2-1}{n^2-4}$.

8. Evaluate $\prod_{n=1}^{\infty}(1 + \frac{1}{n(n+2)})$.

9. Evaluate $\prod_{n=1}^{\infty} \frac{(2n+1)(2n+7)}{(2n+3)(2n+5)}$.

10. Evaluate $\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1}$.

11. Prove the inequality $\prod_{k=2}^{\infty}(1 + \frac{1}{k^2}) \geq \sum_{k=2}^{\infty} \frac{1}{k^2}$.

12. Prove the convergence of the Wallis product $\prod \frac{4k^2}{4k^2-1}$.

13. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k^4}$ by applying (1.4.3).

14. Prove $\prod_{n=2}^{\infty} \frac{n^2+1}{n^2} < \infty$.

15. Multiply a geometric series by itself and get a power series expansion for $(1 - x)^{-2}$.

16. Define $\tau(n)$ as the number of divisors of n. Prove $\zeta^2(x) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^x}$.

17. Define $\phi(n)$ as the number of numbers which are less than n are relatively prime to n. Prove $\frac{\zeta(x-1)}{\zeta(x)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^x}$.

18. Define $\mu(n)$ (Möbius function) as follows: $\mu(1) = 1$, $\mu(n) = 0$, if n is divisible by the square of a prime number, $\mu(n) = (-1)^k$, if n is the product of k different prime numbers. Prove $\frac{1}{\zeta(x)} = \sum_{k=1}^{\infty} \frac{\mu(n)}{n^x}$.

***19.** Prove $\sum_{k=1}^{\infty} \frac{[k \text{ is prime}]}{k} = \infty$.

***20.** Prove the identity $\prod_{n=0}^{\infty}(1 + x^{2^n}) = \frac{1}{1-x}$.