Summing the numbers column-wise (i.e., by the Termwise Addition Formula), we get

$$s + (s-1) + (s-1-0) = (1+0-1) + (0-1+1) + (-1+1+0) + (1+0-1) + (0-1+1) + (-1+1+0) + \dots.$$

The left-hand side is $3s - 2$. The right-hand side is the zero series. That is why $s = \frac{2}{3}$.

The series $1 - 1 + 1 - 1 + \dots$ arises as Zeno's series in the case of a blind Achilles directed by a cruel Zeno, who is interested, as always, only in proving his claim, and a foolish, but merciful turtle. The blind Achilles is not fast, his velocity equals the velocity of the turtle. At the first moment Zeno tells the blind Achilles where the turtle is. Achilles starts the rally. But the merciful turtle wishing to help him goes towards him instead of running away. Achilles meets the turtle half-way. But he misses it, being busy to perform the first step of the algorithm. When he accomplishes this step, Zeno orders: "Turn about!" and surprises Achilles by saying that the turtle is on Achilles' initial position. The turtle discovers that Achilles turns about and does the same. The situation repeats ad infinitum. Now we see that assigning the sum $\frac{1}{2}$ to the series $1 - 1 + 1 - 1 + \dots$ makes sense. It predicts accurately the time of the first meeting of Achilles and turtle.

Positivity. The paradoxes discussed above are discouraging. Our intuition based on handling finite sums fails when we turn to infinite ones. Observe that all paradoxes above involve negative numbers. And to eliminate the evil in its root, let us consider only nonnegative numbers.

We return to the ancient Greeks. They simply did not know what a negative number is. But in contrast to the Greeks, we will retain zero. A series with nonnegative terms will be called a *positive* series. We will show that for positive series all familiar laws, including associativity and commutativity, hold true and zero terms do not affect the sum.

Definition of Infinite Sum. Let us consider what Euler's equality could mean:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The natural answer is: the *partial sums* $\sum_{k=1}^{n} \frac{1}{k^2}$, which contain more and more reciprocal squares, approach closer and closer the value $\frac{\pi^2}{6}$. Consequently, all partial sums have to be less than $\frac{\pi^2}{6}$, its *ultimate sum*. Indeed, if some partial sum exceeds or coincides with $\frac{\pi^2}{6}$ then all subsequent sums will move away from $\frac{\pi^2}{6}$. Furthermore, any number c which is less than $\frac{\pi^2}{6}$ has to be surpassed by partial sums eventually, when they approach $\frac{\pi^2}{6}$ closer than by $\frac{\pi^2}{6} - c$. Hence the ultimate sum majorizes all partial ones, and any lesser number does not. This means that the ultimate sum is the smallest number which majorizes all partial sums.

Geometric motivation. Imagine a sequence $[a_{i-1}, a_i]$ of intervals of the real line. Denote by l_i the length of i-th interval. Let $a_0 = 0$ be the left end point of the first interval. Let $[0, A]$ be the smallest interval containing the whole sequence. Its length is naturally interpreted as the sum $\sum_{i=1}^{\infty} l_i$

This motivates the following definition.

DEFINITION. *If the partial sums of the positive series* $\sum_{k=1}^{\infty} a_k$ *increase without bound, its sum is defined to be* ∞ *and the series is called* divergent. *In the opposite case the series called* convergent, *and its sum is defined as the smallest number* A *such that* $A \geq \sum_{k=1}^{n} a_k$ *for all* n.

This Definition is equivalent to the following couple of principles. The first principle limits the ultimate sum from below:

PRINCIPLE (One-for-All). *The ultimate sum of a positive series majorizes all partial sums.*

And the second principle limits the ultimate sum from above:

PRINCIPLE (All-for-One). *If all partial sums of a positive series do not exceed a number, then the ultimate sum also does not exceed it.*

THEOREM 1.2.1 (Termwise Addition Formula).

$$\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k).$$

PROOF. The inequality $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k)$ is equivalent to $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k$. By All-for-One, the last is equivalent to the system of inequalities

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k \quad N = 1, 2, \ldots.$$

This system is equivalent to the following system

$$\sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad N = 1, 2, \ldots.$$

Each inequality of the last system, in its turn, is equivalent to the system of inequalities

$$\sum_{k=1}^{M} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad M = 1, 2, \ldots.$$

But these inequalities are true for all N and M, as the following computations show.

$$\sum_{k=1}^{M} b_k + \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{M+N} b_k + \sum_{k=1}^{M+N} a_k = \sum_{k=1}^{M+N} (a_k + b_k) \leq \sum_{k=1}^{\infty}(a_k + b_k).$$

In the opposite direction, we see that any partial sum on the right-hand side $\sum_{k=1}^{n}(a_k + b_k)$ splits into $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. And by virtue of the One-for-All principle, this does not exceed $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$. Now, the All-for-One principle provides the inequality in the opposite direction. □

THEOREM 1.2.2 (Shift Formula).

$$\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k.$$

PROOF. The Shift Formula immediately follows from the Termwise Addition formula. To be precise, immediately from the definition, one gets the following: $a_0 + 0 + 0 + 0 + 0 + \cdots = a_0$ and that $0 + a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$. Termwise Addition of these series gives

$$a_0 + \sum_{k=1}^{\infty} a_k = (a_0 + 0) + (0 + a_1) + (0 + a_2) + (0 + a_3) + \cdots = \sum_{k=0}^{\infty} a_k.$$

□

THEOREM 1.2.3 (Termwise Multiplication Formula).

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

PROOF. For any partial sum from the right-hand side one has

$$\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k \leq \lambda \sum_{k=1}^{\infty} a_k$$

by the Distributivity Law for finite sums and One-for-All. This implies the inequality $\lambda \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} \lambda a_k$ by All-for-One. The opposite inequality is equivalent to $\sum_{k=1}^{\infty} a_k \geq \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$. As any partial sum $\sum_{k=1}^{n} a_k$ is equal to $\frac{1}{\lambda} \sum_{k=1}^{n} \lambda a_k$, which does not exceed $\frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$, one gets the opposite inequality. □

Geometric series. We have to return to the geometric series, because the autorecursion equation produced by shift and multiplication formulas says nothing about convergence. So we have to prove convergence for $\sum_{k=0}^{\infty} q^k$ with positive $q < 1$. It is sufficient to prove the following inequality for all n

$$1 + q + q^2 + q^3 + \cdots + q^n < \tfrac{1}{1-q}.$$

Multiplying both sides by $1 - q$ one gets on the left-hand side

$$\begin{aligned}
&(1 - q) + (q - q^2) + (q^2 - q^3) + \cdots + (q^{n-1} - q^n) + (q^n - q^{n+1}) \\
&= 1 - q + q - q^2 + q^2 - q^3 + q^3 - \cdots - q^n + q^n - q^{n+1} \\
&= 1 - q^{n+1}
\end{aligned}$$

and 1 on the right-hand side. The inequality $1 - q^{n+1} < 1$ is obvious. Hence we have proved the convergence. Now the autorecursion equation $x = 1 + qx$ for $\sum_{k=0}^{\infty} q^k$ is constructed in usual way by the shift formula and termwise multiplication. It leaves only two possibilities for $\sum_{k=0}^{\infty} q^k$, either $\frac{1}{q-1}$ or ∞. For $q < 1$ we have proved convergence, and for $q \geq 1$ infinity is the true answer.

Let us pay special attention to the case $q = 0$. We adopt a common convention:

$$0^0 = 1.$$

This means that the series $\sum_{k=0}^{\infty} 0^k$ satisfies the common formula for a convergent geometric series $\sum_{k=0}^{\infty} 0^k = \frac{1}{1-0} = 1$. Finally we state the theorem, which is essentially due to Eudoxus, who proved the convergence of the geometric series with ratio $q < 1$.

THEOREM 1.2.4 (Eudoxus). *For every nonnegative q one has*

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \text{for } q < 1, \text{ and} \quad \sum_{k=0}^{\infty} q^k = \infty \quad \text{for } q \geq 1.$$

Comparison of series. Quite often exact summation of series is too difficult, and for practical purposes it is enough to know the sum approximatively. In this case one usually compares the series with another one whose sum is known. Such a comparison is based on the following *Termwise Comparison Principle*, which immediately follows from the definition of a sum.

PRINCIPLE (Termwise Comparison). *If $a_k \leq b_k$ for k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

The only series we have so far to compare with are the geometric ones. The following lemma is very useful for this purposes.

LEMMA 1.2.5 (Ratio Test). *If $a_{k+1} \leq qa_k$ for k holds for some $q < 1$ then*

$$\sum_{k=0}^{\infty} a_k \leq \frac{a_0}{1-q}.$$

PROOF. By induction one proves the inequality $a_k \leq a_0 q^k$. Now by Termwise Comparison one estimates $\sum_{k=0}^{\infty} a_k$ from above by the geometric series $\sum_{k=0}^{\infty} a_0 q^k = \frac{a_0}{1-q}$ □

If the series under consideration satisfies an autorecursion equation, to prove its convergence usually means to evaluate it exactly. For proving convergence, the Termwise Comparison Principle can be strengthened. Let us say that the series $\sum_{k=1}^{\infty} a_k$ is *eventually* majorized by the series $\sum_{k=1}^{\infty} b_k$, if the inequality $b_k \geq a_k$ holds for each k starting from $k = n$ for some n. The following lemma is very useful to prove convergence.

PRINCIPLE (Eventual Comparison). *A series $\sum_{k=1}^{\infty} a_k$, which is eventually majorized by a convergent series $\sum_{k=1}^{\infty} b_k$, is convergent.*

PROOF. Consider a tail $\sum_{k=n}^{\infty} b_k$, which termwise majorizes $\sum_{k=n}^{\infty} a_k$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} a_k \\ &\leq \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} b_k \\ &\leq \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{\infty} b_k \\ &< \infty.\end{aligned}$$

□

Consider the series $\sum_{k=1}^{\infty} k2^{-k}$. The ratio of two successive terms $\frac{a_{k+1}}{a_k}$ of the series is $\frac{k+1}{2k}$. This ratio is less or equal to $\frac{2}{3}$ starting with $k = 3$. Hence this series is eventually majorizes by the geometric series $\sum_{k=0}^{\infty} a_3 \frac{2^k}{3^k}$, $(a_3 = \frac{2}{3})$. This proves its convergence. And now by autorecursion equation one gets its sum.

Harmonic series paradox. Now we have a solid background to evaluate positive series. Nevertheless, we must be careful about infinity! Consider the following calculation:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} &= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{2k} \right) \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}.
\end{aligned}$$

We get that the sum $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k}$ satisfies the equation $s = \frac{s}{2}$. This equation has two roots 0 and ∞. But s satisfies the inequalities $\frac{1}{2} < s < \frac{\pi^2}{6}$. What is wrong?

Problems.

1. Prove $\sum_{k=1}^{\infty} 0 = 0$.
2. Prove $\sum_{k=1}^{\infty} 0^k = 1$.
3. Prove $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}$.
4. Prove $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$ for convergent series.
5. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.
6. Prove $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots = 1 - [(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + \cdots$.
7. Prove the convergence of $\sum_{k=0}^{\infty} \frac{2^k}{k!}$.
8. Prove the convergence of $\sum_{k=1}^{\infty} \frac{1000^k}{k!}$.
9. Prove the convergence of $\sum_{k=1}^{\infty} \frac{k^{1000}}{2^k}$.
10. Prove that $q^n < \frac{1}{n(1-q)}$ for $0 < q < 1$.
11. Prove that for any positive $q < 1$ there is an n that $q^n < \frac{1}{2}$.
12. Prove $\sum_{k=1}^{\infty} \frac{1}{k!} \leq 2$.
13. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.
14. Prove the convergence of the Euler series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.
***15.** Prove that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ for $a_{ij} \geq 0$.