## 1.1. Autorecursion of Infinite Expressions

**On the contents of the lecture.** The lecture presents a romantic style of early analytics. The motto of the lecture could be "infinity, equality and no definitions!". *Infinity* is the main personage we will play with today. We demonstrate how infinite expressions (i.e., infinite sums, products, fractions) arise in solutions of simple equations, how it is possible to calculate them, and how the results of such calculations apply to finite mathematics. In particular, we will deduce the Euler-Binet formula for Fibonacci numbers, the first Euler's formula of the course. We become acquainted with geometric series and the golden section.

**Achilles and the turtle.** The ancient Greek philosopher Zeno claimed that Achilles pursuing a turtle could never pass it by, in spite of the fact that his velocity was much greater than the velocity of the turtle. His arguments adopted to our purposes are the following.

First Zeno proposed a pursuing algorithm for Achilles:

**Initialization.** Assign to the variable *goal* the original position of the turtle.

**Action.** Reach the *goal*.

**Correction.** If the current turtle's position is *goal*, then stop, else reassign to the variable *goal* the current position of the turtle and go to **Action**.

Secondly, Zeno remarks that this algorithm never stops if the turtle constantly moves in one direction.

And finally, he notes that Achilles has to follow his algorithm if he want pass the turtle by. He may be not aware of this algorithm, but unconsciously he must perform it. Because he cannot run the turtle down without reaching the original position of the turtle and then all positions of the turtle which the variable *goal* takes.

Zeno's algorithm generates a sequence of times $\{t_k\}$, where $t_k$ is the time of execution of the $k$-th action of the algorithm. And the whole time of work of the algorithm is the infinite sum $\sum_{k=1}^{\infty} t_k$; and this sum expresses the time Achilles needs to run the turtle down. (The corrections take zero time, because Achilles really does not think about them.) Let us name this sum the *Zeno series*.

Assume that both Achilles and the turtle run with constant velocities $v$ and $w$, respectively. Denote the initial distance between Achilles and the turtle by $d_0$. Then $t_1 = \frac{d_0}{v}$. The turtle in this time moves by the distance $d_1 = t_1 w = \frac{w}{v} d_0$. By his second action Achilles overcomes this distance in time $t_2 = \frac{d_1}{v} = \frac{w}{v} t_1$, while the turtle moves away by the distance $d_2 = t_2 w = \frac{w}{v} d_1$. So we see that the sequences of times $\{t_k\}$ and distances $\{d_k\}$ satisfy the following *recurrence relations*: $t_k = \frac{w}{v} t_{k-1}$, $d_k = \frac{w}{v} d_{k-1}$.

Hence $\{t_k\}$ as well as $\{d_k\}$ are *geometric progressions* with ratio $\frac{w}{v}$. And the time $t$ which Achilles needs to run the turtle down is

$$t = t_1 + t_2 + t_3 + \cdots = t_1 + \tfrac{w}{v} t_1 + \tfrac{w^2}{v^2} t_1 + \cdots = t_1 \left(1 + \tfrac{w}{v} + \tfrac{w^2}{v^2} + \cdots\right).$$

In spite of Zeno, we know that Achilles does catch up with the turtle. And one easily gets the time $t$ he needs to do it by the following argument: the distance between Achilles and the turtle permanently decreases with the velocity $v - w$. Consequently it becomes 0 in the time $t = \frac{d_0}{v-w} = t_1 \frac{v}{v-w}$. Comparing the results we come to the following conclusion

$$\tfrac{v}{v-w} = 1 + \tfrac{w}{v} + \tfrac{w^2}{v^2} + \tfrac{w^3}{v^3} + \cdots . \tag{1.1.1}$$

**Infinite substitution.** We see that some infinite expressions represent finite values. The fraction in the left-hand side of (1.1.1) expands into the infinite series on the right-hand side. Infinite expressions play a key rôle in mathematics and physics. Solutions of equations quite often are presented as infinite expressions.

For example let us consider the following simple equation

$$t = 1 + qt. \tag{1.1.2}$$

Substituting on the right-hand side $1 + qt$ instead of $t$, one gets a new equation $t = 1 + q(1 + qt) = 1 + q + q^2 t$. Any solution of the original equation satisfies this one. Repeating this trick, one gets $t = 1 + q(1 + q(1 + qt)) = 1 + q + q^2 + q^3 t$. Repeating this infinitely many times, one eliminates $t$ on the right hand side and gets a solution of (1.1.2) in an infinite form

$$t = 1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{\infty} q^k.$$

On the other hand, the equation (1.1.2) solved in the usual way gives $t = \frac{1}{1-q}$. As a result, we obtain the following formula

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \cdots = \sum_{k=0}^{\infty} q^k. \tag{1.1.3}$$

which represents a special case of (1.1.1) for $v = 1$, $w = q$.

**Autorecursion.** An infinite expression of the form $a_1 + a_2 + a_3 + \dots$ is called a *series* and is concisely denoted by $\sum_{k=1}^{\infty} a_k$. Now we consider a summation method for series which is inverse to the above method of infinite substitution. To find the sum of a series we shall construct an equation which is satisfied by its sum. We name this method *autorecursion*. Recursion means "return to something known". Autorecursion is "return to oneself".

The series $a_2 + a_3 + \cdots = \sum_{k=2}^{\infty} a_k$ obtained from $\sum_{k=1}^{\infty} a_k$ by dropping its first term is called the *shift* of $\sum_{k=1}^{\infty} a_k$.

We will call the following equality the *shift formula*:

$$\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} a_k.$$

Another basic formula we need is the following *multiplication formula*:

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

These two formulas are all one needs to find the sum of geometric series $\sum_{k=0}^{\infty} q^k$. To be exact, the multiplication formula gives the equality $\sum_{k=1}^{\infty} q^k = q \sum_{k=0}^{\infty} q^k$. Hence the shift formula turns into equation $x = 1 + qx$, where $x$ is $\sum_{k=0}^{\infty} q^k$. The solution of this equation gives us the formula (1.1.3) for the sum of the geometric series again.

From this formula, one can deduce the formula for the sum of a finite geometric progression. By $\sum_{k=0}^{n} a_k$ is denoted the sum $a_0 + a_1 + a_2 + \cdots + a_n$. One has

$$\sum_{k=0}^{n-1} q^k = \sum_{k=0}^{\infty} q^k - \sum_{k=n}^{\infty} q^k = \frac{1}{1-q} - \frac{q^n}{1-q} = \frac{1-q^n}{1-q}.$$

This is an important formula which was traditionally studied in school.

**The series $\sum_{k=0}^{\infty} kx^k$.** To find the sum of $\sum_{k=1}^{\infty} kx^k$ we have to apply additionally the following *addition formula*,

$$\sum_{k=1}^{\infty}(a_k+b_k)=\sum_{k=1}^{\infty}a_k+\sum_{k=1}^{\infty}b_k$$

which is the last general formula for series we introduce in the first lecture.

Reindexing the shift $\sum_{k=2}^{\infty} kx^k$ we give it the form $\sum_{k=1}^{\infty}(k+1)x^{k+1}$. Further it splits into two parts

$$x\sum_{k=1}^{\infty}(k+1)x^k=x\sum_{k=1}^{\infty}kx^k+x\sum_{k=1}^{\infty}x^k=x\sum_{k=1}^{\infty}kx^k+x\frac{x}{1-x}$$

by the addition formula. The first summand is the original sum multiplied by $x$. The second is a geometric series. We already know its sum. Now the shift formula for the sum $s(x)$ of the original series turns into the equation $s(x) = x+x\frac{x}{1-x}+xs(x)$. Its solution is $s(x)=\frac{x}{(1-x)^2}$.

**Fibonacci Numbers.** Starting with $\phi_0 = 0$, $\phi_1 = 1$ and applying the recurrence relation

$$\phi_{n+1}=\phi_n+\phi_{n-1},$$

one constructs an infinite sequence of numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$, called *Fibonacci numbers*. We are going to get a formula for $\phi_n$.

To do this let us consider the following function $\Phi(x) = \sum_{k=0}^{\infty}\phi_k x^k$, which is called the *generating function* for the sequence $\{\phi_k\}$. Since $\phi_0 = 0$, the sum $\Phi(x)+x\Phi(x)$ transforms in the following way:

$$\sum_{k=1}^{\infty}\phi_k x^k+\sum_{k=1}^{\infty}\phi_{k-1}x^k=\sum_{k=1}^{\infty}\phi_{k+1}x^k=\frac{\Phi(x)-x}{x}.$$

Multiplying both sides of the above equation by $x$ and collecting all terms containing $\Phi(x)$ on the right-hand side, one gets $x = \Phi(x) - x\Phi(x) - x^2\Phi(x) = x$. It leads to

$$\Phi(x)=\frac{x}{1-x-x^2}.$$

The roots of the equation $1-x-x^2=0$ are $\frac{-1\pm\sqrt{5}}{2}$. More famous is the pair of their inverses $\frac{1\pm\sqrt{5}}{2}$. The number $\phi=\frac{-1+\sqrt{5}}{2}$ is the so-called *golden section* or *golden mean*. It plays a significant rôle in mathematics, architecture and biology. Its dual is $\hat\phi=\frac{-1-\sqrt{5}}{2}$. Then $\phi\hat\phi=-1$, and $\phi+\hat\phi=1$. Hence $(1-x\phi)(1-x\hat\phi)=1-x-x^2$, which in turn leads to the following decomposition:

$$\frac{x}{x^2+x-1}=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x}-\frac{1}{1-\hat\phi x}\right).$$

We expand both fractions on the right hand side into geometric series:

$$\frac{1}{1-\phi x}=\sum_{k=0}^{\infty}\phi^k x^k,\qquad\qquad \frac{1}{1-\hat\phi x}=\sum_{k=0}^{\infty}\hat\phi^k x^k.$$

This gives the following representation for the generating function

$$\Phi(x) = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\phi^k - \hat{\phi}^k) x^k.$$

On the other hand the coefficient at $x^k$ in the original presentation of $\Phi(x)$ is $\phi_k$. Hence

$$\phi_k = \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k) = \frac{(\sqrt{5}+1)^k + (-1)^k(\sqrt{5}-1)^k}{2^k\sqrt{5}}. \tag{1.1.4}$$

This is called the *Euler-Binet* formula. It is possible to check it for small $k$ and then prove it by induction using Fibonacci recurrence.

**Continued fractions.** The application of the method of infinite substitution to the solution of quadratic equation leads us to a new type of infinite expressions, the so-called *continued fractions*. Let us consider the golden mean equation $x^2 - x - 1 = 0$. Rewrite it as $x = 1 + \frac{1}{x}$. Substituting $1 + \frac{1}{x}$ instead of $x$ on the right-hand side we get $x = 1 + \frac{1}{1+\frac{1}{x}}$. Repeating the substitution infinitely many times we obtain a solution in the form of the *continued fraction*:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1+\dots}}} \tag{1.1.5}$$

As this fraction seems to represent a positive number and the golden mean is the unique positive root of the golden mean equation, it is natural to conclude that this fraction is equal to $\phi = \frac{1+\sqrt{5}}{2}$. This is true and this representation allows one to calculate the golden mean and $\sqrt{5}$ effectively with great precision.

To be precise, consider the sequence

$$1, \quad 1 + \tfrac{1}{1}, \quad 1 + \cfrac{1}{1 + \frac{1}{1}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}, \quad \dots \tag{1.1.6}$$

of so-called *convergents* of the continued fraction (1.1.5). Let us remark that all odd convergents are less than $\phi$ and all even convergents are greater than $\phi$. To see this, compare the $n$-th convergent with the corresponding term of the following sequence of fractions:

$$1 + \tfrac{1}{x}, \quad 1 + \cfrac{1}{1 + \frac{1}{x}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{x}}}, \quad \dots. \tag{1.1.7}$$

We know that for $x = \phi$ all terms of the above sequence are equal to $\phi$. Hence all we need is to observe how the removal of $\frac{1}{x}$ affects the value of the considered fraction. The value of the first fraction of the sequence decreases, the value of the second fraction increases. If we denote the value of $n$-th fraction by $f_n$, then the value of the next fraction is given by the following recurrence relation:

$$f_{n+1} = 1 + \frac{1}{f_n}. \tag{1.1.8}$$

Hence increasing $f_n$ decreases $f_{n+1}$ and decreasing $f_n$ increases $f_{n+1}$. Consequently in general all odd fractions of the sequence (1.1.7) are less than the corresponding convergent, and all even are greater. The recurrence relation (1.1.8) is valid for the golden mean convergent. By this recurrence relation one can quickly calculate the first ten convergents $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$. The golden mean lies between last two fractions, which have the difference $\frac{1}{34\cdot 55}$. This allows us to determine the first four decimal digits after the decimal point of it and of $\sqrt{5}$.

**Problems.**

**1.** Evaluate $\sum_{k=0}^{\infty} \frac{2^{2k}}{3^{3k}}$.
**2.** Evaluate $1 - 1 + 1 - 1 + \cdots$.
**3.** Evaluate $1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + \cdots$.
**4.** Evaluate $\sum_{k=1}^{\infty} \frac{k}{3^k}$.
**5.** Evaluate $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$.
**6.** Decompose the fraction $\frac{1}{a+x}$ into a power series.
**7.** Find the generating function for the sequence $\{2^k\}$.
**8.** Find sum the $\sum_{k=1}^{\infty} \phi_k 3^{-k}$.
**9.** Prove by induction the Euler-Binet formula.
*__10.__ Evaluate $1 - 2 + 1 + 1 - 2 + 1 + \cdots$.
**11.** Approximate $\sqrt{2}$ by a rational with precision 0.0001.
**12.** Find the value of $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}$.
**13.** Find the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$.
**14.** By infinite substitution, solve the equation $x^2 - 2x - 1 = 0$, and represent $\sqrt{2}$ by a continued fraction.
**15.** Find the value of the infinite product $2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot \cdots$.
**16.** Find a formula for $n$-th term of the recurrent sequence $x_{n+1} = 2x_n + x_{n-1}$, $x_0 = x_1 = 1$.
**17.** Find the sum of the Fibonacci numbers $\sum_{k=1}^{\infty} \phi_k$.
**18.** Find sum $1 + 0 - 1 + 1 + 0 - 1 + \cdots$.
**19.** Decompose into the sum of partial fractions $\frac{1}{x^2-3x+2}$.