CHAPTER 3

Derivatives

3.1. Newton-Leibniz Formula

On the contents of the lecture. In this lecture appears the celebrated Newton-Leibniz formula — the main tool in the evaluation of integrals. It is accompanied with such fundamental concepts as the derivative, the limit of a function and continuity.

Motivation. Consider the following problem: for a given function F find a function f such that $dF(x) = f(x)\,dx$, over $[a,b]$, that is, $\int_c^d f(t)\,dt = F(d) - F(c)$ for any subinterval $[c,d]$ of $[a,b]$.

Suppose that such an f exists. Since the value of f at a single point does not affects the integral, we cannot say anything about the value of f at any given point. But if f is continuous at a point x_0, its value is uniquely defined by F.

To be precise, the difference quotient $\frac{F(x)-F(x_0)}{x-x_0}$ tends to $f(x_0)$ as x tends to x_0. Indeed, $F(x) = F(x_0) + \int_{x_0}^x f(t)\,dt$. Furthermore, $\int_{x_0}^x f(t)\,dt = f(x_0)(x-x_0) + \int_{x_0}^x (f(t) - f(x_0))\,dt$. Also, $|\int_{x_0}^x (f(t) - f(x_0)\,dt| \leq \operatorname{var}_f[x_0, x]|x - x_0|$. Consequently

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \leq \operatorname{var}_f[x, x_0]. \tag{3.1.1}$$

However, $\operatorname{var}_f[x, x_0]$ can be made arbitrarily small by choosing x sufficiently close to x_0, since $\operatorname{var}_f x_0 = 0$.

Infinitesimally small functions. A set is called a *neighborhood* of a point x if it contains all points *sufficiently close* to x, that is, all points y such that $|y - x|$ is less then a positive number ε.

We will say that a function f is *locally bounded* (above) by a constant C at a point x, if $f(x) \leq C$ for all y sufficiently close to x.

A function $o(x)$ is called *infinitesimally small* at x_0, if $|o(x)|$ is locally bounded at x_0 by any $\varepsilon > 0$.

LEMMA 3.1.1. *If the functions o and ω are infinitesimally small at x_0 then $o \pm \omega$ are infinitesimally small at x_0.*

PROOF. Let $\varepsilon > 0$. Let O_1 be a neighborhood of x_0 where $|o(x)| < \varepsilon/2$, and let O_2 be a neighborhood of x_0 where $|\omega(x)| < \varepsilon/2$. Then $O_1 \cap O_2$ is a neighborhood where both inequalities hold. Hence for all $x \in O_1 \cap O_2$ one has $|o(x) \pm \omega(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

LEMMA 3.1.2. *If $o(x)$ is infinitesimally small at x_0 and $f(x)$ is locally bounded at x_0, then $f(x)o(x)$ is infinitesimally small at x_0.*

PROOF. The neighborhood where $|f(x)o(x)|$ is bounded by a given $\varepsilon > 0$ can be constructed as the intersection of a neighborhood U, where $|f(x)|$ is bounded by a constant C, and a neighborhood V, where $|o(x)|$ is bounded by ε/C. $\square$

DEFINITION. *One says that a function $f(x)$* tends to A *as x tends to x_0 and writes* $\lim_{x\to x_0} f(x) = A$, *if $f(x) = A + o(x)$ on the complement of x_0, where $o(x)$ is infinitesimally small at x_0.*

COROLLARY 3.1.3. *If both the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then the limit* $\lim_{x\to x_0}(f(x) + g(x))$ *also exists and* $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$.

PROOF. This follows immediately from Lemma 3.1.1. $\square$

LEMMA 3.1.4. *If the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then also* $\lim_{x\to x_0} f(x)g(x)$ *exists and* $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$.

PROOF. If $f(x) = A+o(x)$ and $g(x) = B+\omega(x)$, then $f(x)g(x) = AB+A\omega(x)+Bo(x) + \omega(x)o(x)$, where $A\omega(x)$, $Bo(x)$ and $\omega(x)o(x)$ all are infinitesimally small at x_0 by Lemma 3.1.2, and their sum is infinitesimally small by Lemma 3.1.1. $\square$

DEFINITION. *A function f is called* continuous *at x_0, if* $\lim_{x\to x_0} f(x) = f(x_0)$.

A function is said to be continuous (without mentioning a point), if it is continuous at all points under consideration.

The following lemma gives a lot of examples of continuous functions.

LEMMA 3.1.5. *If f is a monotone function on $[a,b]$ such that $f[a,b] = [f(a), f(b)]$ then f is continuous.*

PROOF. Suppose f is nondecreasing. Suppose a positive ε is given. For a given point x denote by $x^\varepsilon = f^{-1}(f(x)+\varepsilon)$ and $x_\varepsilon = f^{-1}(f(x)-\varepsilon)$. Then $[x_\varepsilon, x^\varepsilon]$ contains a neighborhood of x, and for any $y \in [x_\varepsilon, x^\varepsilon]$ one has $f(x)+\varepsilon = f(x_\varepsilon) \leq f(y) \leq f(x^\varepsilon) = f(x)+\varepsilon$. Hence the inequality $|f(y)-f(x)| < \varepsilon$ holds locally at x for any ε. $\square$

The following theorem immediately follows from Corollary 3.1.3 and Lemma 3.1.4.

THEOREM 3.1.6. *If the functions f and g are continuous at x_0, then $f+g$ and fg are continuous at x_0.*

The following property of continuous functions is very important.

THEOREM 3.1.7. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g(f(x))$ is continuous at x_0.*

PROOF. Given $\varepsilon > 0$, we have to find a neighborhood U of x_0, such that $|g(f(x)) - g(f(x_0))| < \varepsilon$ for $x \in U$. As $\lim_{y\to f(x_0)} g(y) = g(f(x_0))$, there exists a neighborhood V of $f(x_0)$ such that $|g(y)-g(y_0)| < \varepsilon$ for $y \in V$. Thus it is sufficient to find a U such that $f(U) \subset V$. And we can do this. Indeed, by the definition of neighborhood there is $\delta > 0$ such that V contains $V_\delta = \{y \mid |y - f(x_0)| < \delta\}$. Since $\lim_{x\to x_0} f(x) = f(x_0)$, there is a neighborhood U of x_0 such that $|f(x)-f(x_0)| < \delta$ for all $x \in U$. Then $f(U) \subset V_\delta \subset V$. $\square$

DEFINITION. *A function f is called* differentiable *at a point x_0 if the* difference quotient $\frac{f(x)-(f_0)}{x-x_0}$ *has a limit as x tends to x_0. This limit is called the* derivative *of the function F at the point x_0, and denoted* $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

Immediately from the definition one evaluates the derivative of linear function

$$(ax+b)' = a \tag{3.1.2}$$

The following lemma is a direct consequence of Lemma 3.1.3.

LEMMA 3.1.8. *If f and g are differentiable at x_0, then $f+g$ is differentiable at x_0 and* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Linearization. Let f be differentiable at x_0. Denote by $o(x)$ the difference $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0), \tag{3.1.3}$$

where $o(x)$ is infinitesimally small at x_0. We will call such a representation a *linearization* of $f(x)$.

LEMMA 3.1.9. *If f is differentiable at x_0, then it is continuous at x_0.*

PROOF. All summands but $f(x_0)$ on the right-hand side of (3.1.3) are infinitesimally small at x_0; hence $\lim_{x\to x_0} f(x) = f(x_0)$. □

LEMMA 3.1.10 (on uniqueness of linearization). *If $f(x) = a + b(x - x_0) + o(x)(x - x_0)$, where $\lim_{x\to x_0} o(x) = 0$, then f is differentiable at x_0 and $a = f(x_0)$, $b = f'(x_0)$.*

PROOF. The difference $f(x) - f(x_0)$ is infinitesimally small at x_0 because f is continuous at x_0 and the difference $f(x) - a = b(x - x_0) + o(x)(x - x_0)$ is infinitesimally small by the definition of linearization. Hence $f(x_0) - a$ is infinitesimally small. But it is constant, hence $f(x_0) - a = 0$. Thus we established $a = f(x_0)$.

The difference $\frac{f(x)-a}{x-x_0} - b = o(x)$ is infinitesimally small as well as $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. But $\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-a}{x-x_0}$. Therefore $b - f'(x_0)$ is infinitesimally small. That is $b = f'(x_0)$. □

LEMMA 3.1.11. *If f and g are differentiable at x_0, then fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$.*

PROOF. Considerleariations $f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0)$ and $g(x_0) + g'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Their product is $f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) + (f(x)\omega(x) + f(x_0)o(x))(x - x_0)$. This is the linearization of $f(x)g(x)$ at x_0, because $f\omega$ and go are infinitesimally small at x_0. □

THEOREM 3.1.12. *If f is differentiable at x_0, and g is differentiable at $f(x_0)$ then $g(f(x))$ is differentiable at x_0 and $(g(f(x_0)))' = g'(f(x_0))f'(x_0)$.*

PROOF. Denote $f(x_0)$ by y_0 and substitute into the linearization $g(y) = g(y_0) + g'(y_0)(y - y_0) + o(y)(y - y_0)$ another linearization $y = f(x_0) + f'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Since $y - y_0 = f'(x_0)(x - x_0) + \omega(x)(x - x_0)$, we get $g(y) = g(y_0) + g'(y_0)f'(x_0)(x - x_0) + g'(y_0)(x - x_0)\omega(x) + o(f(x))(x - x_0)$. Due to Lemma 3.1.10, it is sufficient to prove that $g'(y_0)\omega(x) + o(f(x))$ is infinitesimally small at x_0. The first summand is obviously infinitesimally small. To prove that the second one also is infinitesimally small, we remark that $o(f(x_0) = 0$ and $o(y)$ is continuous at $f(x_0)$ and that $f(x)$ is continuous at x_0 due to Lemma 3.1.9. Hence by Theorem 3.1.6 the composition is continuous at x_0 and infinitesimally small. □

THEOREM 3.1.13. *Let f be a virtually monotone function on $[a, b]$. Then $F(x) = \int_a^x f(t)\,dt$ is virtually monotone and continuous on $[a, b]$. It is differentiable at any point x_0 where f is continuous, and $F'(x_0) = f(x_0)$.*

PROOF. If f has a constant sign, then F is monotone. So, if $f = f_1 + f_2$ is a monotonization of f, then $\int_a^x f_1(x)\,dx + \int_a^x f_1(x)\,dx$ is a monotonization of $F(x)$. This proves that $F(x)$ is virtually monotone.

To prove continuity of $F(x)$ at x_0, fix a constant C which bounds f in some neighborhood U of x_0. Then for $x \in U$ one proves that $|F(x) - F(x_0)|$ is infinitesimally small via the inequalities $|F(x) - F(x_0)| = |\int_{x_0}^{x} f(x)\,dx| \leq |\int_{x_0}^{x} C\,dx| = C|x - x_0|$.

Now suppose f is continuous at x_0. Then $o(x) = f(x_0) - f(x)$ is infinitesimally small at x_0. Therefore $\lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} o(x)\,dx = 0$. Indeed for any $\varepsilon > 0$ the inequality $|o(x)| \leq \varepsilon$ holds over $[x_\varepsilon, x_0]$ for some x_ε. Hence $|\int_{x_0}^{x} o(x)\,dx| \leq |\int_{x_0}^{x} \varepsilon\,dx| = \varepsilon|x - x_0|$ for any $x \in [x_0, x_\varepsilon]$.

Then $F(x) = F(x_0) + f(x_0)(x - x_0) + (\frac{1}{x - x_0} \int_{x_0}^{x} o(t)\,dt)(x - x_0)$ is a linearization of $F(x)$ at x_0. □

COROLLARY 3.1.14. *The functions* ln, sin, cos *are differentiable and* $\ln'(x) = \frac{1}{x}$, $\sin' = \cos$, $\cos' = -\sin$.

PROOF. Since $d \sin x = \cos x\,dx$, $d \cos x = -\sin x\,dx$, due to Theorem 3.1.13 both $\sin x$ and $\cos x$ are continuous, and, as they are continuous, the result follows from Theorem 3.1.13. And $\ln' x = \frac{1}{x}$, by Theorem 3.1.13, follows from the continuity of $\frac{1}{x}$. The continuity follows from Lemma 3.1.5. □

Since $\sin'(0) = \cos 0 = 1$ and $\sin 0 = 0$, the linearization of $\sin x$ at 0 is $x + xo(x)$. This implies the following very important equality

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{3.1.4}$$

LEMMA 3.1.15. *If* $f'(x) > 0$ *for all* $x \in [a, b]$, *then* $f(b) > f(a)|$.

PROOF. Suppose $f(a) \geq f(b)$. We construct a sequence of intervals $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ such that their lengths tend to 0 and $f(a_k) \geq f(b_k)$. All steps of construction are the same. The general step is: let m be the middle point of $[a_k, b_k]$. If $f(m) \leq f(a_k)$ we set $[a_{k+1}, b_{k+1}] = [a_k, m]$, otherwise $f(m) > f(a_k) \geq f(b_k)$ and we set $[a_{k+1}, b_{k+1}] = [m, b_k]$.

Now consider a point x belonging to all $[a_k, b_k]$. Let $f(y) = f(x) + (f'(x) + o(x))(y - x)$ be the linearization of f at x. Let U be neighborhood where $|o(x)| < f'(x)$. Then $\operatorname{sgn}(f(y) - f(x)) = \operatorname{sgn}(y - x)$ for all $y \in U$. However for some n we get $[a_n, b_n] \subset U$. If $a_n \leq x < b_n$ we get $f(a_n) \leq f(x) < f(b_n)$ else $a_n < x$ and $f(a_n) < f(x) \leq f(b_n)$. In the both cases we get $f(a_n) < f(b_n)$. This is a contradiction with our construction of the sequence of intervals. □

THEOREM 3.1.16. *If* $f'(x) = 0$ *for all* $x \in [a, b]$, *then* $f(x)$ *is constant.*

PROOF. Set $k = \frac{f(b) - f(a)}{b - a}$. If $k < 0$ then $g(x) = f(x) - kx/2$ has derivative $g'(x) = f'(x) - k/2 > 0$ for all x. Hence by Lemma 3.1.15 $g(b) > g(a)$ and further $f(b) - f(a) > k(b - a)/2$. This contradicts the definition of k. If $k > 0$ then one gets the same contradiction considering $g(x) = -f(x) + kx/2$. □

THEOREM 3.1.17 (Newton-Leibniz). *If* $f'(x)$ *is a continuous virtually monotone function on an interval* $[a, b]$, *then* $\int_a^b f'(x)\,dx = f(b) - f(a)$.

PROOF. Due to Theorem 3.1.13, the derivative of the difference $\int_a^x f'(t)\,dt - f(x)$ is zero. Hence the difference is constant by Theorem 3.1.16. Substituting

$x = a$ we find the constant which is $f(a)$. Consequently, $\int_a^x f'(t)\,dt - f(x) = f(a)$ for all x. In particular, for $x = b$ we get the Newton-Leibniz formula. $\square$

Problems.

1. Evaluate $(1/x)'$, $\sqrt{x}'$, $(\sqrt{\sin x^2})'$.
2. Evaluate $\exp' x$.
3. Evaluate $\operatorname{arctg}' x$, $\tan' x$.
4. Evaluate $|x|'$, $\operatorname{Re} z'$.
5. Prove: $f'(x) \equiv 1$ if and only if $f(x) = x + \text{const}$.
6. Evaluate $\left(\int_x^{x^2} \frac{\sin t}{t}\,dt\right)'$ as a function of x.
7. Evaluate $\sqrt{1-x^2}'$.
8. Evaluate $(\int_0^1 \frac{\sin kt}{t}\,dt)'$ as a function of k.
9. Prove: If f is continuous at a and $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} f(x_n) = f(a)$.
10. Evaluate $\left(\int_0^y [x]\,dx\right)'_y$.
11. Evaluate $\arcsin' x$.
12. Evaluate $\int \frac{dx}{2+3x^2}$.
13. Prove: If $f'(x) < 0$ for all $x < m$ and $f'(x) > 0$ for all $x > m$ then $f'(m) = 0$.
14. Prove: If $f'(x)$ is bounded on $[a, b]$ then f is virtually monotone on $[a, b]$.

3.2. Exponential Functions

On the contents of the lecture. We solve the principal differential equation $y' = y$. Its solution, the exponential function, is expanded into a power series. We become acquainted with hyperbolic functions. And, finally, we prove the irrationality of e.

Debeaune's problem. In 1638 F. Debeaune posed Descartes the following geometrical problem: find a curve $y(x)$ such that for each point P the distances between V and T, the points where the vertical and the tangent lines cut the x-axis, are always equal to a given constant a. Despite the efforts of Descartes and Fermat, this problem remained unsolved for nearly 50 years. In 1684 Leibniz solved the problem via infinitesimal analysis of this curve: let x, y be a given point P (see the picture). Then increase x by a small increment of b, so that y increases almost by yb/a. Indeed, in small the curve is considered as the line. Hence the point P' of the curve with vertical projection V', one considers as lying on the line TP. Hence the triangle $TP'V'$ is similar to TPV. As $TV = a$, $TV' = b+a$ this similarity gives the equality $\frac{a+b}{y+\delta y} = \frac{a}{y}$ which gives $\delta y = yb/a$.

Repeating we obtain a sequence of values

$$y,\ y(1+\tfrac{b}{a}),\ y(1+\tfrac{b}{a})^2,\ y(1+\tfrac{b}{a})^3, \dots.$$

We see that "in small" $y(x)$ transforms an arithmetic progression into a geometric one. This is the inverse to what the logarithm does. And the solution is a function which is the inverse to a logarithmic function. Such functions are called *exponential*.

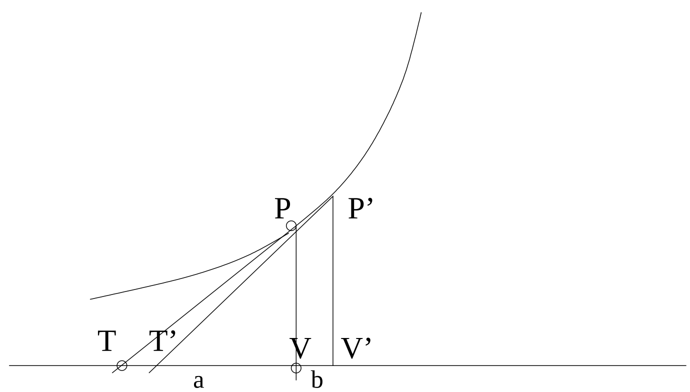

FIGURE 3.2.1. Debeaune's problem

Tangent line and derivative. A tangent line to a smooth convex curve at a point x is defined as a straight line such that the line intersects the curve just at x and the whole curve lies on one side of the line.

We state that the equation of the tangent line to the graph of function f at a point x_0 is just the principal part of linearization of $f(x)$ at x_0. In other words, the equation is $y = f(x_0) + (x - x_0)f'(x_0)$.

First, consider the case of a horizontal tangent line. In this case $f(x_0)$ is either maximal or minimal value of $f(x)$.

LEMMA 3.2.1. *If a function $f(x)$ is differentiable at an extremal point x_0, then $f'(x_0) = 0$.*

PROOF. Consider the linearization $f(x) = f(x_0)+f'(x_0)(x-x_0)+o(x))(x-x_0)$. Denote $x-x_0$ by δx, and $f(x)-f(x_0)$ by $\delta f(x)$. If we suppose that $f'(x_0) \neq 0$, then, for sufficiently small δx, we get $|o(x\pm\delta x)| < |f'(x)|$, hence $\operatorname{sgn}(f'(x_0)+o(x_0+\delta x)) = \operatorname{sgn}(f'(x_0)+o(x_0-\delta x))$, and $\operatorname{sgn}\delta f(x) = \operatorname{sgn}\delta x$. Therefore the sign of $\delta f(x)$ changes whenever the sign of δx changes. The sign of $\delta f(x)$ cannot be positive if $f(x_0)$ is the maximal value of $f(x)$, and it cannot be negative if $f(x_0)$ is the minimal value. This is the contradiction. □

THEOREM 3.2.2. *If a function $f(x)$ is differentiable at x_0 and its graph is convex, then the tangent line to the graph of $f(x)$ at x_0 is $y = f(x_0)+f'(x_0)(x-x_0)$.*

PROOF. Let $y = ax + b$ be the equation of a tangent line to the graph $y = f(x)$ at the point x_0. Since $ax + b$ passes through x_0, one has $ax_0 + b = f(x_0)$, therefore $b = f(x_0) - ax_0$, and it remains to prove that $a = f'(x_0)$. If the tangent line $ax + b$ is not horizontal, consider the function $g(x) = f(x) - ax$. At x_0 it takes either a maximal or a minimal value and $g'(x_0) = 0$ by Lemma 3.2.1. On the other hand, $g'(x_0) = f'(x_0) - a$. □

Differential equation. The Debeaune problem leads to a so-called differential equation on $y(x)$. To be precise, the equation of the tangent line to $y(x)$ at x_0 is $y = y(x_0) + y'(x_0)(x - x_0)$. So the x-coordinate of the point T can be found from the equation $0 = y(x_0) + y'(x_0)(x - x_0)$. The solution is $x = x_0 - \frac{y(x_0)}{y'(x_0)}$. The x-coordinate of V is just x_0. Hence TV is equal to $\frac{y(x_0)}{y'(x_0)}$. And Debeaune's requirement is $\frac{y(x_0)}{y'(x_0)} = a$. Or $ay' = y$. Equations that include derivatives of functions are called *differential equations*. The equation above is the simplest differential equation. Its solution takes one line. Indeed passing to differentials one gets $ay'\,dx = y\,dx$, further $ady = y\,dx$, then $a\frac{dy}{y} = dx$ and $a\,d\ln y = dx$. Hence $a\ln y = x+c$ and finally $y(x) = \exp(c+\frac{x}{a})$, where $\exp x$ denotes the function inverse to the natural logarithm and c is an arbitrary constant.

Exponenta. The function inverse to the natural logarithm is called the *exponential function*. We shall call it the *exponenta* to distinguish it from other exponential functions.

THEOREM 3.2.3. *The exponenta is the unique solution of the differential equation $y' = y$ such that $y'(0) = 1$.*

PROOF. Differentiation of the equality $\ln\exp x = x$ gives $\frac{\exp' x}{\exp x} = 1$. Hence $\exp x$ satisfies the differential equation $y' = y$. For $x = 0$ this equation gives $\exp'(0) = \exp 0$. But $\exp 0 = 1$ as $\ln 1 = 0$.

For the converse, let $y(x)$ be a solution of $y' = y$. The derivative of $\ln y$ is $\frac{y'}{y} = 1$. Hence the derivative of $\ln y(x) - x$ is zero. By Theorem 3.1.16 from the previous lecture, this implies $\ln y(x) - x = c$ for some constant c. If $y'(0) = 1$, then $y(0) = 1$ and $c = \ln 1 - 0 = 0$. Therefore $\ln y(x) = x$ and $y(x) = \exp\ln y(x) = \exp x$. □

Exponential series. Our next goal is to prove that

$$\exp x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{3.2.1}$$

where $0! = 1$. This series is absolutely convergent for any x. Indeed, the ratio of its subsequent terms is $\frac{x}{n}$ and tends to 0, hence it is eventually majorized by any geometric series.

Hyperbolic functions. To prove that the function presented by series (3.2.1) is virtually monotone, consider its odd and even parts. These parts represent the so-called *hyperbolic functions*: hyperbolic sine $\operatorname{sh} x$, and hyperbolic cosine $\operatorname{ch} x$.

$$\operatorname{sh}(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \operatorname{ch}(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

The hyperbolic sine is an increasing function, as all odd powers are increasing over the whole line. The hyperbolic cosine is increasing for positive x and decreasing for negative. Hence both are virtually monotone; and so is their sum.

Consider the integral $\int_0^x \operatorname{sh} t\, dt$. As all terms of the series representing sh are increasing, we can integrate the series termwise. This integration gives $\operatorname{ch} x$. As $\operatorname{sh} x$ is locally bounded, $\operatorname{ch} x$ is continuous by Theorem 3.1.13. Consider the integral $\int_0^x \operatorname{ch} t\, dt$; here we also can integrate the series representing ch termwise, because for positive x all the terms are increasing, and for negative x, decreasing. Integration gives $\operatorname{sh} x$, since the continuity of $\operatorname{ch} x$ was already proved. Further, by Theorem 3.1.13 we get that $\operatorname{sh} x$ is differentiable and $\operatorname{sh}' x = \operatorname{ch} x$. Now returning to the equality $\operatorname{ch} x = \int_0^x \operatorname{sh} t\, dt$ we get $\operatorname{ch}' x = \operatorname{sh} x$, as $\operatorname{sh} x$ is continuous.

Therefore $(\operatorname{sh} x + \operatorname{ch} x)' = \operatorname{ch} x + \operatorname{sh} x$. And $\operatorname{sh} 0 + \operatorname{ch} 0 = 0 + 1 = 1$. Now by the above Theorem 3.2.3 one gets $\exp x = \operatorname{ch} x + \operatorname{sh} x$.

Other exponential functions. The exponenta as a function inverse to the logarithm transforms sums into products. That is, for all x and y one has

$$\exp(x+y) = \exp x \exp y.$$

A function which has this property (i.e., transform sums into products) is called *exponential*.

THEOREM 3.2.4. *For any positive a there is a unique differentiable function denoted by a^x called the exponential function to base a, such that $a^1 = a$ and $a^{x+y} = a^x a^y$ for any x, y. This function is defined by the formula* $\exp a \ln x$.

PROOF. Consider $l(x) = \ln a^x$. This function has the property $l(x+y) = l(x) + l(y)$. Therefore its derivative at any point is the same: it is equal to $k = \lim_{x\to 0} \frac{l(x)}{x}$. Hence the function $l(x) - kx$ is constant, because its derivative is 0. This constant is equal to $l(0)$, which is 0. Indeed $l(0) = l(0+0) = l(0) + l(0)$. Thus $\ln a^x = kx$. Substituting $x = 1$ one gets $k = \ln a$. Hence $a^x = \exp(x \ln a)$. So if a differentiable exponential function with base a exists, it coincides with $\exp(x \ln a)$. On the other hand it is easy to see that $\exp(x \ln a)$ satisfies all the requirements for an exponential function to base a, that is $\exp(1 \ln a) = a$, $\exp((x+y)\ln a) = \exp(x \ln a)\exp(y \ln a)$; and it is differentiable as composition of differentiable functions. □

Powers. Hence for any positive a and any real b, one defines the number a^b as

$$a^b = \exp(b \ln a)$$

a is called the base, and b is called the exponent. For rational b this definition agrees with the old definition. Indeed if $b = \frac{p}{q}$ then the properties of the exponenta and the logarithm imply $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Earlier, we have defined logarithms to base b as the number c, and called the *logarithm of b to base a*, if $a^c = b$ and denoted $c = \log_a b$.

The basic properties of powers are collected here.

THEOREM 3.2.5.

$$\left(a^b\right)^c = a^{(bc)}, \quad a^{b+c} = a^b a^c, \quad (ab)^c = a^c b^c, \quad \log_a b = \frac{\log b}{\log a}.$$

Power functions. The power operation allows us to define the power function x^α for any real degree α. Now we can prove the equality $(x^\alpha)' = \alpha x^{\alpha-1}$ in its full value. Indeed, $(x^\alpha)' = (\exp(\alpha \ln x))' = \exp'(\alpha \ln x)(\alpha \ln x)' = \exp(\alpha \ln x)\frac{\alpha}{x} = \alpha x^{\alpha-1}$.

Infinite products via the Logarithm.

LEMMA 3.2.6. *Let $f(x)$ be a function continuous at x_0. Then for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ one has $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

PROOF. For any given $\varepsilon > 0$ there is a neighborhood U of x_0 such that $|f(x) - f(x_0)| \leq \varepsilon$ for $x \in U$. As $\lim_{n\to\infty} x_n = x_0$, eventually $x_n \in U$. Hence eventually $|f(x_n) - f(x_0)| < \varepsilon$. □

As we already have remarked, infinite sums and infinite products are limits of partial products.

THEOREM 3.2.7. $\ln \prod_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \ln p_k$.

PROOF.

$$\begin{aligned}\exp(\textstyle\sum_{k=1}^{\infty} \ln p_k) &= \exp(\lim_{n\to\infty} \textstyle\sum_{k=1}^{n} \ln p_k)\\ &= \lim_{n\to\infty} \exp(\textstyle\sum_{k=1}^{n} \ln p_k)\\ &= \lim_{n\to\infty} \textstyle\prod_{k=1}^{n} p_k\\ &= \textstyle\prod_{k=1}^{\infty} p_k.\end{aligned}$$

Now take logarithms of both sides of the equation. □

Symmetric arguments prove the following: $\exp \sum_{k=1}^{\infty} a_k = \prod_{k=1}^{\infty} \exp a_k$.

Irrationality of e. The expansion of the exponenta into a power series gives an expansion into a series for e which is $\exp 1$.

LEMMA 3.2.8. *For any natural n one has $\frac{1}{n+1} < en! - [en!] < \frac{1}{n}$.*

PROOF. $en! = \sum_{k=0}^{\infty} \frac{n!}{k!}$. The partial sum $\sum_{k=0}^{n} \frac{n!}{k!}$ is an integer. The tail $\sum_{k=n+1}^{\infty} \frac{n!}{k!}$ is termwise majorized by the geometric series $\sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n}$. On the other hand the first summand of the tail is $\frac{1}{n+1}$. Consequently the tail has its sum between $\frac{1}{n+1}$ and $\frac{1}{n}$. □

THEOREM 3.2.9. *The number e is irrational.*

PROOF. Suppose $e = \frac{p}{q}$ where p and q are natural. Then $eq!$ is a natural number. But it is not an integer by Lemma 3.2.8. $\square$

Problems.

1. Prove the inequalities $1 + x \leq \exp x \leq \frac{1}{1-x}$.
2. Prove the inequalities $\frac{x}{1+x} \leq \ln(1+x) \leq x$.
3. Evaluate $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.
4. Evaluate $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$.
5. Evaluate $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$.
6. Find the derivative of x^x.
7. Prove: $x > y$ implies $\exp x > \exp y$.
8. Express via e: $\exp 2$, $\exp(1/2)$, $\exp(2/3)$, $\exp(-1)$.
9. Prove that $\exp(m/n) = e^{\frac{m}{n}}$.
10. Prove that $\exp x > 0$ for any x.
11. Prove the addition formulas $\operatorname{ch}(x+y) = \operatorname{ch}(x)\operatorname{ch}(y) + \operatorname{sh}(x)\operatorname{sh}(y)$, $\operatorname{sh}(x+y) = \operatorname{sh}(x)\operatorname{ch}(y) + \operatorname{sh}(y)\operatorname{ch}(x)$.
12. Prove that $\Delta \operatorname{sh}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{ch}(x)$, $\Delta \operatorname{ch}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{sh}(x)$.
13. Prove $\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$.
14. Prove $\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = 1$.
15. Solve the equation $\operatorname{sh} x = 4/5$.
16. Express via e the sum $\sum_{k=1}^{\infty} k/k!$.
17. Express via e the sum $\sum_{k=1}^{\infty} k^2/k!$.
18. Prove that $\{\frac{\exp k}{k^n}\}$ is unbounded.
19. Prove: The product $\prod(1 + p_n)$ converges if and only if the sum $\sum p_n$ $(p_n \geq 0)$ converges.
20. Determine the convergence of $\prod \frac{e^{1/n}}{1+\frac{1}{n}}$.
21. Does $\prod n(e^{1/n} - 1)$ converges?
22. Prove the divergence of $\sum_{k=1}^{\infty} \frac{[k-prime]}{k}$.
23. Expand a^x into a power series.
24. Determine the geometrical sense of $\operatorname{sh} x$ and $\operatorname{ch} x$.
25. Evaluate $\lim_{n\to\infty} \sin \pi e n!$.
26. Does the series $\sum_{k=1}^{\infty} \sin \pi e k!$ converge?
***27.** Prove the irrationality of e^2.

3.3. Euler Formula

On the contents of the lecture. The reader becomes acquainted with the most famous Euler formula. Its special case $e^{i\pi} = -1$ symbolizes the unity of mathematics: here e represents Analysis, i represents Algebra, and π represents Geometry. As a direct consequence of the Euler formula we get power series for sin and cos, which we need to sum up the Euler series.

Complex Newton-Leibniz. For a function of a complex variable $f(z)$ the derivative is defined by the same formula $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We will denote it also by $\frac{df(z)}{dz}$, to distinguish from derivatives of paths: complex valued functions of real variable. For a path $p(t)$ its derivative will be denoted either $p'(t)$ or $\frac{dp(t)}{dt}$. The Newton-Leibniz formula for real functions can be expressed by the equality $\frac{df(t)}{dt}dt = df(t)$. Now we extend this formula to complex functions.

The linearization of a complex function $f(z)$ at z_0 has the same form $f(z_0) + f'(z_0)(z - z_0) + o(z)(z - z_0)$, where $o(z)$ is an infinitesimally small function of complex variable. The same arguments as for real numbers prove the basic rules of differentiation: the derivative of sums, products and compositions.

THEOREM 3.3.1. $\frac{dz^n}{dz} = nz^{n-1}$.

PROOF. $\frac{dz}{dz} = 1$ one gets immediately from the definition of the derivative. Suppose the equality $\frac{dz^n}{dz} = nz^{n-1}$ is proved for n. Then $\frac{dz^{n+1}}{dz} = \frac{dzz^n}{dz} = z\frac{dz^n}{dz} + z^n\frac{dz}{dz} = znz^{n-1} + z^n = (n+1)z^n dz$. And the theorem is proved by induction. □

A *smooth path* is a differentiable mapping $p\colon [a, b] \to \mathbb{C}$ with a continuous bounded derivative. A function $f(z)$ of a complex variable is called *virtually monotone* if for any smooth path $p(t)$ the functions $\operatorname{Re} f(p(t))$ and $\operatorname{Im} f(p(t))$ are virtually monotone.

LEMMA 3.3.2. *If $f'(z)$ is bounded, then $f(z)$ is virtually monotone.*

PROOF. Consider a smooth path p. Then $\frac{df(p(t))}{dt} = f'(p(t))p'(t)$ is bounded by some K. Due to Lemma 3.1.15 one has $|f(p(t)) - f(p(t_0))| \leq K|t - t_0|$. Hence any partial variation of $f(p(t))$ does not exceed $K(b - a)$. Therefore $\operatorname{var}_{f(p(t))}[a, b] \leq K$. □

THEOREM 3.3.3. *If a complex function $f(z)$ has a bounded virtually monotone continuous complex derivative over the image of a smooth path $p\colon [a, b] \to \mathbb{C}$, then $\int_p f'(z)\,dz = f(p(b)) - f(p(a))$.*

PROOF. $\frac{df(p(t))}{dt} = f'(p(t))p'(t) = \frac{d\operatorname{Re} f(p(t))}{dt} + i\frac{d\operatorname{Im} f(p(t))}{dt}$. All functions here are continuous and virtually monotone by hypothesis. Passing to differential forms one gets

$$\begin{aligned}\frac{df(p(t))}{dt}\,dt &= \frac{d\operatorname{Re} f(p(t))}{dt}\,dt + i\,\frac{d\operatorname{Im} f(p(t))}{dt}\,dt\\ &= d(\operatorname{Re} f(p(t))) + i\,d(\operatorname{Im} f(p(t)))\\ &= d(\operatorname{Re} f(p(t)) + i\operatorname{Im} f(p(t)))\\ &= d(f(p(t))).\end{aligned}$$

Hence $\int_p f'(z)\,dz = \int_p df(z)$. □

COROLLARY 3.3.4. *If $f'(z) = 0$ then $f(z)$ is constant.*

PROOF. Consider $p(t) = z_0 + (z - z_0)t$, then $f(z) - f(z_0) = \int_p f'(\zeta)\, d\zeta = 0$. $\square$

Differentiation of series. Let us say that a complex series $\sum_{k=1}^{\infty} a_k$ majorizes (eventually) another such series $\sum_{k=1}^{\infty} b_k$ if $|b_k| \leq |a_k|$ for all k (resp. for k beyond some n).

The series $\sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ is called a *formal derivative* of $\sum_{k=0}^{\infty} c_k(z-z_0)^k$.

LEMMA 3.3.5. *Any power series $\sum_{k=0}^{\infty} c_k(z - z_0)^k$ eventually majorizes its formal derivative $\sum_{k=0}^{\infty} kc_k(z_1 - z_0)^{k-1}$ if $|z_1 - z_0| < |z - z_0|$.*

PROOF. The ratio of the n-th term of the derivative to the n-th term of the series tends to 0 as n tends to infinity. Indeed, this ratio is $\frac{k(z_1-z_0)^k}{(z-z_0)^k} = kq^k$, where $|q| < 1$ since $|z_1 - z_0| < |z - z_0|$. The fact that $\lim_{n\to\infty} nq^n = 0$ follows from the convergence of $\sum_{k=1}^{\infty} kq^k$ which we already have proved before. This series is eventually majorized by any geometric series $\sum_{k=0}^{\infty} AQ^k$ with $Q > q$. $\square$

A path $p(t)$ is called *monotone* if both $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$ are monotone.

LEMMA 3.3.6. *Let $p\colon [a,b] \to \mathbb{C}$ be a smooth monotone path, and let $f(z)$ be virtually monotone. If $|f(p(t))| \leq c$ for $t \in [a,b]$ then $\left|\int_p f(z)\, dz\right| \leq 4c|p(b) - p(a)|$.*

PROOF. Integration of the inequalities $-c \leq \operatorname{Re} f(p(t)) \leq c$ against $d\operatorname{Re} z$ along the path gives $|\int_p \operatorname{Re} f(z)\, d\operatorname{Re} z| \leq c|\operatorname{Re} p(b) - \operatorname{Re} p(a)| \leq c|p(b) - p(a)|$. The same arguments prove $|\int_p \operatorname{Im} f(z)\, d\operatorname{Im} z| \leq c|\operatorname{Im} p(b) - \operatorname{Im} p(a)| \leq c|p(b) - p(a)|$. The sum of these inequalities gives $|\operatorname{Re} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. The same arguments yields $|\operatorname{Im} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. And the addition of the two last inequalities allows us to accomplish the proof of the Lemma because $|\int_p f(z)\, dz| \leq |\operatorname{Re} \int_p f(z)\, dz| + |\int_p f(z)\, dz|$. $\square$

LEMMA 3.3.7. $|z^n - \zeta^n| \leq n|z - \zeta| \max\{|z^{n-1}|, |\zeta^{n-1}|\}$.

PROOF. $(z^n - \zeta^n) = (z - \zeta)\sum_{k=0}^{n-1} z^k \zeta^{n-k-1}$ and $|z^k\zeta^{n-k-1}| \leq \max\{|z^{n-1}|, |\zeta^{n-1}|\}$. $\square$

A *linear path* from z_0 to z_1 is defined as a linear mapping $p\colon [a,b] \to \mathbb{C}$, such that $p(a) = z_0$ and $p(b) = z_1$, that is $p(t) = z_0(t-a) + (z_1 - z_0)(t-a)/(b-a)$.

We denote by $\int_a^b f(z)\, dz$ the integral along the linear path from a to b.

LEMMA 3.3.8. *For any complex z, ζ and natural $n > 0$ one has*

$$|z^n - z_0^n - nz_0^{n-1}(z - z_0)| \leq 2n(n-1)|z - z_0|^2 \max\{|z|^{n-2}, |z_0|^{n-2}\}. \tag{3.3.1}$$

PROOF. By the Newton-Leibniz formula, $z^n - z_0^n = \int_{z_0}^{z} n\zeta^{n-1}\, d\zeta$. Further,

$$\begin{aligned}\int_{z_0}^{z} n\zeta^{n-1}\, d\zeta &= \int_{z_0}^{z} nz_0^{n-1}\, d\zeta + \int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta \\ &= nz_0^{n-1} + \int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta.\end{aligned}$$

Consequently, the left-hand side of (3.3.1) is equal to $\left|\int_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\, d\zeta\right|$. Due to Lemma 3.3.7 the absolute value of the integrand along the linear path does not exceed $(n-1)|z-z_0|\max\{|z^{n-2}|,|z_0^{n-2}|\}$. Now the estimation of the integral by Lemma 3.3.6 gives just the inequality (3.3.1). $\square$

THEOREM 3.3.9. *If $\sum_{k=0}^{\infty} c_k(z_1-z_0)^k$ converges absolutely, then $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ and $\sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ absolutely converge provided by $|z-z_0|<|z_1-z_0|$, and the function $\sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ is the complex derivative of $\sum_{k=0}^{\infty} c_k(z-z_0)^k$.*

PROOF. The series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ and its formal derivative are eventually majorized by $\sum_{k=0}^{\infty} c_k(z_1-z_0)^k$ if $|z-z_0|\leq|z_1-z_0|$ by the Lemma 3.3.5. Hence they absolutely converge in the circle $|z-z_0|\leq|z_1-z_0|$. Consider

$$R(z)=\sum_{k=0}^{\infty} c_k(z-z_0)^k-\sum_{k=0}^{\infty} c_k(\zeta-z_0)^k-(z-\zeta)\sum_{k=1}^{\infty} kc_k(\zeta-z_0)^{k-1}.$$

To prove that the formal derivative is the derivative of $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ at ζ it is sufficient to prove that $R(z)=o(z)(z-\zeta)$, where $o(z)$ is infinitesimally small at ζ. One has $R(z)=\sum_{k=1}^{\infty} c_k\left((z-z_0)^k-(\zeta-z_0)^k-k(\zeta-z_0)^{k-1}\right)$. By Lemma 3.3.8 one gets the following estimate: $|R(z)|\leq\sum_{k=1}^{\infty} 2|c_k|k(k-1)|z-\zeta|^2|z_2-z_0|^{n-2}$, where $|z_2-z_0|=\max\{|z-z_0|,|\zeta-z_0|\}$. Hence all we need now is to prove that $\sum_{k=1}^{\infty} 2k(k-1)|c_k||z_2-z_0|^{k-2}|z-\zeta|$ is infinitesimally small at ζ. And this in its turn follows from the convergence of $\sum_{k=1}^{\infty} 2k(k-1)|c_k||z_2-z_0|^{k-2}$. The latter may be deduced from Lemma 3.3.5. Indeed, consider z_3, such that $|z_2-z_0|<|z_3-z_0|<|z_1-z_0|$. The convergence of $\sum_{k=1}^{\infty} k|c_k||z_3-z_0|^{k-1}$ follows from the convergence of $\sum_{k=0}^{\infty}|c_k||z_1-z_0|^k$ by Lemma 3.3.5. And the convergence of $\sum_{k=2}^{\infty} k(k-1)|c_k||z_2-z_0|^{k-2}$ follows from the convergence of $\sum_{k=1}^{\infty} k|c_k||z_3-z_0|^{k-1}$ by the same lemma. $\square$

COROLLARY 3.3.10. *Let $f(z)=\sum_{k=0}^{\infty} c_k z^k$ converge absolutely for $|z|<r$, and let a,b have absolute values less then r. Then $\int_a^b f(z)\,dz=\sum_{k=0}^{\infty}\frac{c_k}{k+1}(b^{k+1}-a^{k+1})$.*

PROOF. Consider $F(z)=\sum_{k=0}^{\infty}\frac{c_k z^{k+1}}{k+1}$. This series is termwise majorized by the series of $f(z)$, hence it converges absolutely for $|z|<r$. By Theorem 3.3.9 $f(z)$ is its derivative for $|z|<r$. In our case $f(z)$ is differentiable and its derivative is bounded by $\sum_{k=0}^{\infty} k|c_k|r_0^k$, where $r_0=\max\{|a|,|b|\}$. Hence $f(z)$ is continuous and virtually monotone and our result now follows from Theorem 3.3.3. $\square$

Exponenta in $\mathbb{C}$. The exponenta for any complex number z is defined as $\exp z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$. The definition works because the series $\sum_{k=0}^{\infty}\frac{z^k}{k!}$ absolutely converges for any $z\in\mathbb{C}$.

THEOREM 3.3.11. *The exponenta is a differentiable function of a complex variable with derivative* $\exp' z=\exp z$*, such that for all complex z, ζ the following* addition formula *holds:* $\exp(z+\zeta)=\exp z\exp\zeta$.

PROOF. The derivative of the exponenta can be evaluated termwise by Theorem 3.3.9. And this evaluation gives $\exp' z=\exp z$. To prove the addition formula consider the following function $r(z)=\frac{\exp(z+\zeta)}{\exp z}$. Differentiation of the equality $r(z)\exp z=\exp(z+\zeta)$ gives $r'(z)\exp z+r(z)\exp z=\exp(z+\zeta)$. Division by $\exp z$ gives $r'(z)+r(z)=r(z)$. Hence $r(z)$ is constant. This constant is determined by substitution $z=0$ as $r(z)=\exp\zeta$. This proves the addition formula. $\square$

LEMMA 3.3.12. *Let* $p\colon [a,b] \to \mathbb{C}$ *be a smooth path contained in the complement of a neighborhood of* 0. *Then* $\exp \int_p \frac{1}{\zeta}\, d\zeta = \frac{p(b)}{p(a)}$.

PROOF. First consider the case when p is contained in a circle $|z - z_0| < |z_0|$ with center $z_0 \neq 0$. In this circle, $\frac{1}{z}$ expands in a power series:

$$\frac{1}{\zeta} = \frac{1}{z_0 - (z_0 - \zeta)} = \frac{1}{z_0}\frac{1}{1 - \frac{z_0 - \zeta}{z_0}} = \sum_{k=0}^{\infty} \frac{(z_0 - \zeta)^k}{z_0^{k+1}}.$$

Integration of this series is possible to do termwise due to Corollary 3.3.10. Hence the result of the integration does not depend on the path. And Theorem 3.3.9 provides differentiability of the termwise integral and the possibility of its termwise differentiation. Such differentiation simply gives the original series, which represents $\frac{1}{z}$ in this circle.

Consider the function $l(z) = \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta$. Then $l'(z) = \frac{1}{z}$. The derivative of the composition $\exp l(z)$ is $\frac{\exp l(z)}{z}$. Hence the composition satisfies the differential equation $y'z = y$. We search for a solution of this equation in the form $y = zw$. Then $y' = w + w'z$ and our equation turns into $wz + w'z^2 = wz$. Therefore $w' = 0$ and w is constant. To find this constant substitute $z = z_0$ and get $1 = \exp 0 = \exp l(z_0) = wz_0$. Hence $w = \frac{1}{z_0}$ and $\exp l(z) = \frac{z}{z_0}$.

To prove the general case consider a partition $\{x_k\}_{k=0}^{n}$ of $[a,b]$. Denote by p_k the restriction of p over $[x_k, x_{k+1}]$. Choose the partition so small that $|p(x) - p(x_k)| < |p(x_k)|$ for all $x \in [x_k, x_{k+1}]$. Then any p_k satisfies the requirement of the above considered case. Hence $\exp \int_{p_k} \frac{1}{\zeta}\, d\zeta = \frac{p(x_{k+1})}{p(x_k)}$. Further $\exp \int_p \frac{1}{\zeta}\, d\zeta = \exp \sum_{k=0}^{n-1} \int_{p_k} \frac{1}{\zeta}\, d\zeta = \prod_{k=0}^{n-1} \frac{p(x_{k+1})}{p(x_k)} = p(b)/p(a)$. □

THEOREM 3.3.13 (Euler Formula). *For any real* ϕ *one has*

$$\boxed{\exp i\phi = \cos\phi + i \sin\phi}$$

PROOF. In Lecture 2.5 we have evaluated $\int_p \frac{1}{z}\, dz = i\phi$ for $p(t) = \cos t + i \sin t$, $t \in [0, \phi]$. Hence Lemma 3.3.12 applied to $p(t)$ immediately gives the Euler formula. □

Trigonometric functions in $\mathbb{C}$. The Euler formula gives power series expansions for $\sin x$ and $\cos x$:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

These expansions are used to define trigonometric functions for complex variable. On the other hand the Euler formula allows us to express trigonometric functions via the exponenta:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \qquad \cos z = \frac{\exp(iz) + \exp(-iz)}{2}.$$

The other trigonometric functions tan, cot, sec, cosec are defined for complex variables by the usual formulas via sin and cos.

Problems.

1. Evaluate $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

2. Prove the formula of Joh. Bernoulli $\int_0^1 x^x\, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$.

3. Find $\ln(-1)$.

4. Solve the equation $\exp z = i$.

5. Evaluate i^i.

6. Prove $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cos z = \frac{e^{iz}+e^{-iz}}{2}$.

7. Prove the identity $\sin^2 z + \cos^2 z = 1$.

8. Solve the equation $\sin z = 5/3$.

9. Solve the equation $\cos z = 2$.

10. Evaluate $\sum_{k=0}^{\infty} \frac{\cos k}{k!}$.

11. Evaluate $\oint_{|z|=1} \dfrac{dz}{z^2}$.

12. Evaluate $\sum_{k=1}^{\infty} q^k \dfrac{\sin kx}{k}$.

13. Expand into a power series $e^x \cos x$.

3.4. Abel's Theorem

On the contents of the lecture. The expansion of the logarithm into power series will be extended to the complex case. We learn the very important Abel's transformation of sum. This transformation is a discrete analogue of integrations by parts. Abel's theorem on the limit of power series will be applied to the evaluation of trigonometric series related to the logarithm. The concept of Abel's sum of a divergent series will be introduced.

Principal branch of the Logarithm. Since $\exp(x+iy) = e^x(\cos y + i\sin y)$, one gets the following formula for the logarithm: $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$, where $\operatorname{Arg} z = \arg z + 2\pi k$. We see that the logarithm is a multi-valued function, that is why one usually chooses a *branch* of the logarithm to work. For our purposes it is sufficient to consider the *principal* branch of the logarithm:

$$\ln z = \ln|z| + i\arg z, \quad -\pi < \arg z \leq \pi.$$

The principal branch of the logarithm is a differentiable function of a complex variable with derivative $\frac{1}{z}$, inverse to $\exp z$. This branch is not continuous at negative numbers. However its restriction on the upper half-plane is continuous and even differentiable at negative numbers.

Lemma 3.4.1. *For any nonnegative z one has $\int_1^z \frac{1}{\zeta}\,d\zeta = \ln z$.*

Proof. If $\operatorname{Im} z \neq 0$, the segment $[0,z]$ is contained in the circle $|\zeta - z_0| < |z_0|$ for $z_0 = \frac{|z|^2}{\operatorname{Im} z}$. In this circle $\frac{1}{\zeta}$ expands into a power series, which one can integrate termwise. Since for z^k the result of integration depends only on the ends of path of integration, the same is true for power series. Hence, we can change the path of integration without changing the result. Consider the following path: $p(t) = \cos t + i\sin t$, $t \in [0, \arg z]$. We know the integral $\int_p \frac{1}{\zeta}\,d\zeta = i\arg z$. This path terminates at $\frac{z}{|z|}$. Continue this path by the linear path to z. The integral satisfies $\int_{z/|z|}^z \frac{1}{\zeta}\,d\zeta = \int_1^{|z|} \frac{1}{z/|z|t}\,dtz/|z| = \int_1^{|z|} \frac{1}{t}\,dt = \ln|z|$. Therefore $\int_1^z \frac{1}{\zeta}\,d\zeta = \int_p \frac{1}{\zeta}\,d\zeta + \int_{z/|z|}^z \frac{1}{\zeta}\,d\zeta = i\arg z + \ln|z|$. $\square$

Logarithmic series. In particular for $|1-z| < 1$ termwise integration of the series $\frac{1}{\zeta} = \sum_{k=0}^\infty (1-\zeta)^k$ gives the complex Mercator series:

$$\ln(1+z) = \sum_{k=1}^\infty (-1)^{k+1}\frac{z^k}{k}. \tag{3.4.1}$$

Substitute in this series $-z$ for z and subtract the obtained series from (3.4.1) to get the complex Gregory series:

$$\frac{1}{2}\ln\frac{1+z}{1-z} = \sum_{k=0}^\infty (-1)^k \frac{z^{2k+1}}{2k+1}.$$

In particular for $z = ix$, one has $\left|\frac{1+ix}{1-ix}\right| = 1$ and $\arg\frac{1+ix}{1-ix} = 2\operatorname{arctg} x$. Therefore one gets

$$\operatorname{arctg} x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Since $\arg(1+e^{i\phi}) = \operatorname{arctg}\frac{\sin\phi}{1+\cos\phi} = \operatorname{arctg}\tan(\phi/2) = \frac{\phi}{2}$, the substitution of $\exp(i\phi)$ for z in the Mercator series $\ln(1+e^{i\phi}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^{ik\phi}}{k}$ gives for the imaginary parts:

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\sin k\phi}{k} = \frac{\phi}{2}. \tag{3.4.2}$$

However the last substitution is not correct, because $|e^{i\phi}| = 1$ and (3.4.1) is proved only for $|z| < 1$. To justify it we will prove a general theorem, due to Abel.

Summation by parts. Consider two sequences $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$. The difference of their product $\delta a_k b_k = a_{k+1}b_{k+1} - a_k b_k$ can be presented as

$$\delta(a_k b_k) = a_{k+1}\delta b_k + b_k\delta a_k.$$

Summation of these equalities gives

$$a_n b_n - a_1 b_1 = \sum_{k=1}^{n-1} a_{k+1}\delta b_k + \sum_{k=1}^{n-1} b_k \delta a_k.$$

A permutation of the latter equality gives the so-called *Abel's transformation* of sums

$$\sum_{k=1}^{n-1} b_k \Delta a_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}\Delta b_k.$$

Abel's theorem. One writes $x \to a-0$ instead of $x \to a$ and $x < a$, and $x \to a+0$ means $x > a$ and $x \to a$.

THEOREM 3.4.2 (Abel).

$$\textit{If } \sum_{k=0}^{\infty} a_k \textit{ converges, then } \lim_{x\to 1-0}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

PROOF. $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < 1$, because of the boundedness of $\{a_k\}$.

Suppose $\varepsilon > 0$. Set $A(n,m) = \sum_{k=n}^{m} a_k$, $A(n,m)(x) = \sum_{k=n}^{m} a_k x^k$. Choose N so large that

$$|A(0,n) - A(0,\infty)| < \frac{\varepsilon}{9}, \quad \forall n > N. \tag{3.4.3}$$

Applying the Abel transformation for any $m > n$ one gets

$$\begin{aligned} A(n,m) - A(n,m)(x) &= \sum_{k=n}^{m} a_k(1-x^k) \\ &= (1-x)\sum_{k=n}^{m}\delta A(n-1,k-1)\sum_{j=0}^{k-1} x^j \\ &= (1-x)\Big[A(n-1,m)\sum_{j=0}^{m} x^j - A(n-1,n)\sum_{j=0}^{n} x^j - \sum_{k=n}^{m} A(n-1,k)x^k\Big]. \end{aligned}$$

By (3.4.3) for $n > N$, one gets $|A(n-1,m)| = |(A(0,m) - A) + (A - A(0,n))| \leq \varepsilon/9 + \varepsilon/9 = 2\varepsilon/9$. Hence, we can estimate from above by $\frac{2\varepsilon/3}{1-x}$ the absolute value of the expression in the brackets of the previous equation for $A(n,m) - A(n,m)(x)$. As a result we get

$$|A(n,m) - A(n,m)(x)| \leq \frac{2\varepsilon}{3}, \quad \forall m \geq n > N, \forall x. \tag{3.4.4}$$

Since $\lim_{x\to 1-0} A(0,N)(x) = A(0,N)$ one chooses δ so small that for $x > 1-\delta$ the following inequality holds:

$$|A(0,N) - A(0,N)(x)| < \frac{\varepsilon}{3}.$$

Summing up this inequality with (3.4.4) for $n = N+1$ one gets:

$$|A(0,m) - A(0,m)(x)| < \varepsilon, \quad \forall m > N, |1-x| < \delta.$$

Passing to limits as m tends to infinity the latter inequality gives

$$|A(0,\infty) - A(0,\infty)(x)| \leq \varepsilon, \quad \text{for } |1-x| < \delta.$$

□

Leibniz series. As the first application of the Abel Theorem we evaluate the Leibniz series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. This series converges by the Leibniz Theorem 2.4.3. By the Abel Theorem its sum is

$$\lim_{x\to 1-0} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2k+1} = \lim_{x\to 1-0} \operatorname{arctg} x = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

We get the following remarkable equality:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \dots.$$

Abel sum of a series. One defines the *Abel sum* of a series $\sum_{k=0}^{\infty} a_k$ as the limit $\lim_{x\to 1-0} \sum_{k=0}^{\infty} a_k x^k$. The series which have an Abel sum are called *Abel summable*. The Abel Theorem shows that all convergent series have Abel sums coinciding with their usual sums. However there are a lot of series, which have an Abel sum, but do not converge.

Abel's inequality. Consider a series $\sum_{k=1}^{\infty} a_k b_k$, where the partial sums $A_n = \sum_{k=1}^{n-1} a_k$ are bounded by some constant A and the sequence $\{b_k\}$ is monotone. Then $\sum_{k=1}^{n-1} a_k b_k = \sum_{k=1}^{n-1} b_k \delta A_k = A_n b_n - A_1 b_1 + \sum_{k=1}^{n-1} A_{k+1}\delta b_k$. Since $\sum_{k=1}^{n-1} |\delta b_k| = |b_n - b_1|$, one gets the following inequality:

$$\left|\sum_{k=1}^{n-1} a_k b_k\right| \leq 3A \max\{|b_k|\}.$$

Convergence test.

THEOREM 3.4.3. *Let the sequence of partial sums $\sum_{k=1}^{n-1} a_k$ be bounded, and let $\{b_k\}$ be non-increasing and infinitesimally small. Then $\sum_{k=1}^{\infty} a_k b_k$ converges to its Abel sum, if the latter exists.*

PROOF. The difference between a partial sum $\sum_{k=1}^{n-1} a_k b_k$ and the Abel sum is equal to

$$\lim_{x\to 1-0} \sum_{k=1}^{n-1} a_k b_k (1-x^k) + \lim_{x\to 1-0} \sum_{k=n}^{\infty} a_k b_k x^k.$$

The first limit is zero, the second limit can be estimated by Abel's inequality from above by $3Ab_n$. It tends to 0 as n tends to infinity. $\square$

Application. Now we are ready to prove the equality (3.4.2). The series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$ has an Abel sum. Indeed,

$$\begin{aligned}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{q^k\sin kx}{k} &= \operatorname{Im}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(qe^{ix})^k}{k}\\ &= \operatorname{Im}\lim_{q\to 1-0}\ln(1+qe^{ix})\\ &= \operatorname{Im}\ln(1+e^{ix}).\end{aligned}$$

The sums $\sum_{k=1}^{n-1}\sin kx = \operatorname{Im}\sum_{k=1}^{n-1}e^{ikx} = \operatorname{Im}\frac{1-e^{inx}}{1-e^{ix}}$ are bounded. And $\frac{1}{k}$ is decreasing and infinitesimally small. Hence we can apply Theorem 3.4.3.

Problems.

1. Evaluate $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\dots$.

2. Evaluate $\sum_{k=1}^{\infty}\frac{\sin 2k}{k}$.

3. $\sum_{k=1}^{\infty}\frac{\cos k\phi}{k} = -\ln|2\sin\frac{\phi}{2}|$, $(0<|\phi|\le\pi)$.

4. $\sum_{k=1}^{\infty}\frac{\sin k\phi}{k} = \frac{\pi-\phi}{2}$, $(0<\phi<2\pi)$.

5. $\sum_{k=0}^{\infty}\frac{\cos(2k+1)\phi}{2k+1} = \frac{1}{2}\ln|2\cot\frac{\phi}{2}|$, $(0<|\phi|<\pi)$

6. $\sum_{k=0}^{\infty}\frac{\sin(2k+1)\phi}{2k+1} = \frac{\pi}{4}$, $(0<\phi<\pi)$

7. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos k\phi}{k} = \ln\left(2\cos\frac{\phi}{2}\right)$, $(-\pi<\phi<\pi)$

8. Find the Abel sum of $1-1+1-1+\dots$.

9. Find the Abel sum of $1-1+0+1-1+0+\dots$.

10. Prove: A periodic series, such that the sum of the period is zero, has an Abel sum.

11. Telescope $\sum_{k=1}^{\infty}\dfrac{k^2}{2^k}$.

12. Evaluate $\sum_{k=0}^{n-1}k\cos kx$.

13. Estimate from above $\sum_{k=n}^{\infty}\frac{\sin kx}{k^2}$.

***14.** Prove: If $\sum_{k=0}^{\infty}a_k$, $\sum_{k=0}^{\infty}b_k$ and their convolution $\sum_{k=0}^{\infty}c_k$ converge, then $\sum_{k=0}^{\infty}c_k = \sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k$.

3.5. Residue Theory

On the contents of the lecture. At last, the reader learns something, which Euler did not know, and which he would highly appreciate. Residue theory allows one to evaluate a lot of integrals which were not accessible by the Newton-Leibniz formula.

Monotone curve. A *monotone curve* Γ is defined as a subset of the complex plane which is the image of a monotone path. Nonempty intersections of vertical and horizontal lines with a monotone curve are either points or closed intervals.

The points of the monotone curve which have an extremal sum of real and imaginary parts are called its *endpoints*, the other points of the curve are called its *interior* points.

A continuous injective monotone path p whose image coincides with Γ is called a *parametrization* of Γ.

LEMMA 3.5.1. *Let $p_1\colon [a,b] \to \mathbb{C}$ and $p_2\colon [c,d] \to \mathbb{C}$ be two parametrizations of the same monotone curve Γ. Then $p_1^{-1}p_2\colon [c,d] \to [a,b]$ is a continuous monotone bijection.*

PROOF. Set $P_i(t) = \operatorname{Re} p_i(t) + \operatorname{Im} p_i(t)$. Then P_1 and P_2 are continuous and strictly monotone. And $p_1(t) = p_2(\tau)$ if and only if $P_1(t) = P_2(\tau)$. Hence $p_1^{-1}p_2 = P_1^{-1}P_2$. Since P_1^{-1} and P_2 are monotone continuous, the composition $P_1^{-1}P_2$ is monotone continuous. $\square$

Orientation. One says that two parametrizations p_1 and p_2 of a monotone curve Γ have the *same orientation*, if $p_1^{-1}p_2$ is increasing, and one says that they have *opposite orientations*, if $p_1^{-1}p_2$ is decreasing.

Orientation divides all parametrizations of a curve into two classes. All elements of one orientation class have the same orientation and all elements of the other class have the opposite orientation.

An oriented curve is a curve with fixed *circulation direction*. A choice of orientation means distinguishing one of the orientation classes as positive, corresponding to the oriented curve. For a monotone curve, to specify its orientation, it is sufficient to indicate which of its endpoints is its beginning and which is the end. Then all positively oriented parametrizations start with its beginning and finish at its end, and negatively oriented parametrizations do the opposite.

If an oriented curve is denoted by Γ, then its *body*, the curve without orientation, is denoted $|\Gamma|$ and the curve with the same body but with opposite orientation is denoted $-\Gamma$.

If Γ' is a monotone curve which is contained in an oriented curve Γ, then one defines the *induced orientation* on Γ' by Γ as the orientation of a parametrization of Γ' which extends to a positive parametrization of Γ.

Line integral. One defines the integral $\int_\Gamma f(z)\,dg(z)$ along a oriented monotone curve Γ as the integral $\int_p f(z)\,dg(z)$, where p is a positively oriented parametrization of Γ. This definition does not depend on the choice of p, because different parametrizations are obtained from each other by an increasing change of variable (Lemma 3.5.1).

One defines a *partition of a curve* Γ by a point x as a pair of monotone curves Γ_1, Γ_2, such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = x$. And we write in this case $\Gamma = \Gamma_1 + \Gamma_2$.

The Partition Rule for the line integral is

$$\int_{\Gamma_1+\Gamma_2} f(z)\,dg(z) = \int_{\Gamma_1} f(z)\,dg(z) + \int_{\Gamma_2} f(z)\,dg(z), \tag{3.5.1}$$

where the orientations on Γ_i are induced by an orientation of Γ. To prove the Partition Rule consider a positive parametrization $p\colon [a,b] \to \Gamma$. Then the restrictions of p over $[a, p^{-1}(x)]$ and $[p^{-1}(x), b]$ give positive parametrizations of Γ_1 and Γ_2. Hence the equality (3.5.1) follows from $\int_a^{p^{-1}(x)} f(z)\,dg(z) + \int_{p^{-1}(x)}^b f(z)\,dg(z) = \int_a^b f(z)\,dg(z)$.

A sequence of oriented monotone curves $\{\Gamma_k\}_{k=1}^n$ with disjoint interiors is called a *chain* of monotone curves and denoted by $\sum_{k=1}^n \Gamma_k$. The body of a chain $C = \sum_{k=1}^n \Gamma_k$ is defined as $\bigcup_{k=1}^n |\Gamma_k|$ and denoted by $|C|$. The interior of the chain is defined as the union of interiors of its elements.

The integral of a form $f\,dg$ along the chain is defined as $\int_{\sum_{k=1}^n \Gamma_k} f\,dg = \sum_{k=1}^n \int_{\Gamma_k} f\,dg$.

One says that two chains $\sum_{k=1}^n \Gamma_k$ and $\sum_{k=1}^m \Gamma'_k$ have the same orientation, if the orientations induced by Γ_k and Γ'_j on $\Gamma_k \cap \Gamma'_j$ coincide in the case when $\Gamma_k \cap \Gamma'_j$ has a nonempty interior. Two chains with the same body and orientation are called *equivalent*.

LEMMA 3.5.2. *If two chains $C = \sum_{k=1}^n \Gamma_k$ and $C' = \sum_{k=1}^m \Gamma'_k$ are equivalent then the integrals along these chains coincide for any form $f dg$.*

PROOF. For any interior point x of the chain C, one defines the subdivision of C by x as $\Gamma_j^+ + \Gamma_j^- + \sum_{k=1}^n \Gamma_k[k \neq j]$, where Γ_j is the curve containing x and $\Gamma_j^+ + \Gamma_j^-$ is the partition of Γ by x. The subdivision does not change the integral along the chain due to the Partition Rule.

Hence we can subdivide C step by step by endpoints of C' to construct a chain Q whose endpoints include all endpoints of P'. And the integral along Q is the same as along P. Another possibility to construct Q is to subdivide C' by endpoints of C. This construction shows that the integral along Q coincides with the integral along C'. Hence the integrals along C and C' coincide. $\square$

Due to this lemma, one can introduce the integral of a differential form along any oriented piecewise monotone curve Γ. To do this one considers a *monotone partition* of Γ into a sequence $\{\Gamma_k\}_{k=1}^n$ of monotone curves with disjoint interiors and equip all Γ_k with the induced orientation. One gets a chain and the integral along this chain does not depend on the partition.

Contour integral. A *domain* D is defined as a connected bounded part of the plane with piecewise monotone boundary. The boundary of D denoted ∂D is the union of finitely many monotone curves. And we suppose that $\partial D \subset D$, that is we consider a closed domain.

For a monotone curve Γ, which is contained in the boundary of a domain D, one defines the *induced orientation* of Γ by D as the orientation of a parametrization which leaves D on the left during the movement along Γ around D.

One introduces the integral $\oint_{\partial D} f(z)dg(z)$ as the integral along any chain whose body coincides with ∂D and whose orientations of curves are induced by D.

Due to Lemma 3.5.2 the choice of chain does not affect the integral.

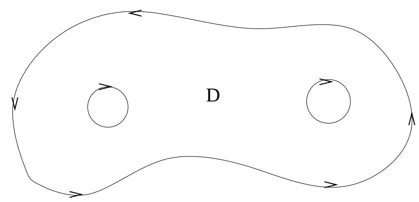

FIGURE 3.5.1. Contour integral

LEMMA 3.5.3. *Let D be a domain and l be either a vertical or a horizontal line, which bisects D into two parts: D' and D'' lying on the different sides of l. Then* $\oint_{\partial D} f(z)dz = \oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz$.

PROOF. The line l intersects the boundary of D in a finite sequence of points and intervals $\{J_k\}_{k=1}^m$.

Set $\partial' D = \partial D \cap \partial D'$ and $\partial'' D = \partial D \cap \partial D''$. The intersection $\partial' D \cap \partial'' D$ consists of finitely many points. Indeed, the interior points of J_k do not belong to this intersection, because their small neighborhoods have points of D only from one side of l. Hence

$$\int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

The boundary of D' consists of $\partial' D$ and some number of intervals. And the boundary of D'' consists of $\partial'' D$ and the same intervals, but with opposite orientation. Therefore

$$L = \int_{l\cap\partial D'} f(z)\,dz = -\int_{l\cap\partial D''} f(z)\,dz.$$

On the other hand

$$\oint_{\partial D'} f(z)dz = \int_{\partial' D} f(z)\,dz + L \text{ and } \oint_{\partial D''} f(z)dz = \int_{\partial'' D} f(z)\,dz - L,$$

hence

$$\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

□

LEMMA 3.5.4 (Estimation). *If $|f(z)| \le B$ for any z from a body of a chain $C = \sum_{k=1}^n \Gamma_k$, then* $\left|\int_C f(z)\,dz\right| \le 4Bn \operatorname{diam}|C|$.

PROOF. By Lemma 3.3.6 for any k one has $\left|\int_{\Gamma_k} f(z)\,dz\right| \le 4B|A_k - B_k| \le 4B \operatorname{diam}|C|$ where A_k and B_k are endpoints of Γ_k. The summation of these inequalities proves the lemma. □

THEOREM 3.5.5 (Cauchy). *If a function f is complex differentiable in a domain D then* $\oint_{\partial D} f(z)dz = 0$.

PROOF. Fix a rectangle R with sides parallel to the coordinate axis which contains D and denote by $|R|$ its area and by P its perimeter.

The proof is by contradiction. Suppose $\left|\oint_{\partial D} f(z)\,dz\right| \neq 0$. Denote by c the ratio of $\left|\oint_{\partial D} f(z)\,dz\right|/|R|$. We will construct a nested sequence of rectangles $\{R_k\}_{k=0}^{\infty}$ such that

- $R_0 = R$, $R_{k+1} \subset R_k$;
- R_{2k} is similar to R;
- $|\oint_{\partial(R_k \cap D)} f(z)\,dz| \geq c|R_k|$, where $|R_k|$ is the area of R_k.

The induction step: Suppose R_k is already constructed. Divide R_k in two equal rectanges R_k' and R_k'' by drawing either a vertical, if k is even, or a horizontal, if k is odd, interval joining the middles of the opposite sides of R_k. Set $D_k = D \cap R_k$, $D' = D \cap R_k'$, $D'' = D \cap R_k''$. We state that at least one of the following inequalities holds:

$$\left|\oint_{\partial D'} f(z)dz\right| \geq c|R_k'|, \qquad \left|\oint_{\partial D''} f(z)dz\right| \geq c|R_k''|. \tag{3.5.2}$$

Indeed, in the opposite case one gets

$$\left|\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz\right| < c|R_k'| + c|R_k'| = c|R_k|.$$

Since $\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \oint_{\partial D_k} f(z)dz$ by Lemma 3.5.3 we get a contradiction with the hypothesis $|\int_{p_k} f(z)\,dz| \geq c|R_k|$. Hence, one of the inequalities (3.5.2) holds. If the first inequality holds we set $R_{k+1} = R_k'$ else we set $R_{k+1} = R_k''$.

After constructing the sequence $\{R_k\}$, consider a point z_0 belonging to $\bigcap_{k=1}^{\infty} R_k$. This point belongs to D, because all its neighborhoods contain points of D. Consider the linearization $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z)(z - z_0)$. Since $\oint_{\partial D_k}(f(z_0) + f'(z_0)(z - z_0))dz = 0$ one gets

$$\left|\oint_{\partial D_k} o(z)(z - z_0)dz\right| = \left|\oint_{\partial D_k} f(z)dz\right| \geq c|R_k|. \tag{3.5.3}$$

The boundary of D_k is contained in the union $\partial R_k \cup R_k \cap \partial D$. Consider a monotone partition $\partial D = \sum_{k=1}^{n} \Gamma_k$. Since the intersection of R_k with a monotone curve is a monotone curve, one concludes that $\partial D \cap R_k$ is a union of at most n monotone curves. As ∂R_k consists of 4 monotone curves we get that ∂D_k is as a body of a chain with at most $4 + n$ monotone curves.

Denote by P_k the perimeter of R_k. And suppose that $o(x)$ is bounded in R_k by a constant o_k. Then $|o(x)(z - z_0)| \leq P_k o_k$ for all $z \in R_k$.

Since $\operatorname{diam} \partial D_k \leq \frac{P_k}{2}$ by the Estimation Lemma 3.5.4, we get the following inequality:

$$\left|\oint_{\partial D_k} o(z)(z - z_0)dz\right| \leq 4(4 + n)P_k o_k \frac{P_k}{2} = 2(4 + n)o_k P_k^2. \tag{3.5.4}$$

The ratio $P_k^2/|R_k|$ is constant for even k. Therefore the inequalities (3.5.3) and (3.5.4) contradict each other for $o_k < \frac{c|R_k|}{2(4+n)P_k^2} = \frac{c|R|}{2(4+n)P^2}$. However the inequality $|o(x)| < \frac{c|R|}{2(4+n)P^2}$ holds for some neighborhood V of z_0 as $o(x)$ is infinitesimally small at z_0. This is a contradiction, because V contains some R_{2k}. $\square$

Residues. By $\oint_{z_0}^{r} f(z)\,dz$ we denote the integral along the boundary of the disk $\{|z-z_0|\le r\}$.

LEMMA 3.5.6. *Suppose a function $f(z)$ is complex differentiable in the domain D with the exception of a finite set of points $\{z_k\}_{k=1}^n$. Then*

$$\oint_{\partial D} f(z)dz = \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz,$$

where r is so small that all disks $|z-z_k|<r$ are contained in D and disjoint.

PROOF. Denote by D' the complement of the union of the disks in D. Then $\partial D'$ is the union of ∂D and the boundary circles of the disks. By the Cauchy Theorem 3.5.5, $\oint_{\partial D'} f(z)dz = 0$. On the other hand this integral is equal to the sum $\oint_{\partial D} f(z)dz$ and the sum of integrals along boundaries of the circles. The orientation induced by D' onto the boundaries of these circles is opposite to the orientation induced from the circles. Hence

$$0 = \oint_{\partial D'} f(z)dz = \oint_{\partial D} f(z)dz - \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz.$$

□

A *singular point* of a complex function is defined as a point where either the function or its derivative are not defined. A singular point is called isolated, if it has a neighborhood, where it is the only singular point. A point is called a *regular point* if it not a singular point.

One defines the *residue* of f at a point z_0 and denotes it as $\operatorname{res}_{z_0} f$ as the limit $\lim_{r\to 0} \frac{1}{2\pi i} \oint_{z_0}^{r} f(z)dz$. The above lemma shows that this limit exists for any isolated singular point and moreover, that all integrals along sufficiently small circumferences in this case are the same.

Since in all regular points the residues are 0 the conclusion of Lemma 3.5.6 for a function with finitely many singular points can be presented in the form:

$$\boxed{\oint_{\partial D} f(z)dz = 2\pi i \sum_{z\in D} \operatorname{res}_z f.} \tag{3.5.5}$$

An isolated singular point z_0 is called a *simple pole* of a function $f(z)$ if there exists a nonzero limit $\lim_{z\to z_0} f(z)(z-z_0)$.

LEMMA 3.5.7. *If z_0 is a simple pole of $f(z)$ then* $\operatorname{res}_{z_0} f = \lim_{z\to z_0}(z-z_0)f(z)$.

PROOF. Set $L = \lim_{z\to z_0}(z-z_0)f(z)$. Then $f(z) = L + \frac{o(z)}{(z-z_0)}$, where $o(z)$ is infinitesimally small at z_0. Hence

$$\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0} = \oint_{z_0}^{r} f(z)\,dz - \oint_{z_0}^{r} \frac{L}{z-z_0}\,dz. \tag{3.5.6}$$

Since the second integral from the right-hand side of (3.5.6) is equal to $2L\pi i$ and the other one is equal to $2\pi i \operatorname{res}_{z_0} f$ for sufficiently small r, we conclude that the integral from the left-hand side also is constant for sufficiently small r. To prove that $L = \operatorname{res}_{z_0} f$ we have to prove that this constant $c = \lim_{r\to 0} \oint_{z_0}^{r} \frac{o(z)}{z-z_0}\,dz$ is 0. Indeed, assume that $|c|>0$. Then there is a neighborhood U of z_0 such that $|o(z)| \le \frac{|c|}{32}$

for all $z \in U$. Then one gets a contradiction by estimation of $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right|$ (which is equal to $|c|$ for sufficiently small r) from above by $\frac{|c|}{\sqrt{2}}$ for r less than the radius of U. Indeed, the integrand is bounded by $\frac{|c|}{32r}$ and the path of integration (the circle) can be divided into four monotone curves of diameter $r\sqrt{2}$: quarters of the circle. Hence by the Estimation Lemma 3.5.4 one gets $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right| \leq 16\sqrt{2}\frac{|c|}{32} = \frac{|c|}{\sqrt{2}}$. □

REMARK 3.5.8. *Denote by $\Gamma(r,\phi,z_0)$ an arc of the circle $|z-z_0| = r$, whose angle measure is ϕ. Under the hypothesis of Lemma 3.5.7 the same arguments prove the following*

$$\lim_{r\to 0}\int_{\Gamma(\phi,r,0z)} f(z)\,dz = i\phi \lim_{z\to z_0} f(z)(z-z_0).$$

Problems.

1. Evaluate $\oint_1^1 \frac{dz}{1+z^4}$.

2. Evaluate $\oint_0^1 \frac{dz}{\sin z}$.

3. Evaluate $\oint_0^1 \frac{dz}{e^z-1}$.

4. Evaluate $\oint_0^1 \frac{dz}{z^2}$.

5. Evaluate $\oint_0^1 \sin\frac{1}{z}\,dz$.

6. Evaluate $\oint_0^1 ze^{\frac{1}{z}}\,dz$.

7. Evaluate $\oint_0^{5/2} z^2\cot\pi z\,dz$.

8. Evaluate $\oint_2^{\frac{1}{2}} \frac{z\,dz}{(z-1)(z-2)^2}$.

9. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{5+3\cos\phi}$.

10. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{(1+\cos^2\phi)^2}$.

11. Evaluate $\int_0^{2\pi} \frac{d\phi}{(1+\cos\phi)^2}$.

12. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$.

13. Evaluate $\int_0^{+\infty} \frac{dx}{(1+x^2)(4+x^2)}$.

14. Evaluate $\int_{-\infty}^{+\infty} \frac{1+x^2}{1+x^4}$.

15. Evaluate $\int_{-\infty}^{+\infty} \frac{x^3}{1+x^6}\,dx$.

3.6. Analytic Functions

On the contents of the lecture. This lecture introduces the reader into the phantastically beautiful world of analytic functions. Integral Cauchy formula, Taylor series, Fundamental Theorem of Algebra. The reader will see all of these treasures in a single lecture.

THEOREM 3.6.1 (Integral Cauchy Formula). *If function f is complex differentiable in the domain D, then for any interior point $z \in D$ one has:*

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(\zeta)\,dz}{\zeta - z}}$$

PROOF. The function $\frac{f(z)}{z-z_0}$ has its only singular point inside the circle. This is z_0, which is a simple pole. The residue of $\frac{f(z)}{z-z_0}$ by Lemma 3.5.7 is equal to $\lim_{z\to z_0}(z-z_0)\frac{f(z)}{z-z_0} = \lim_{z\to z_0} f(z) = f(z_0)$. And by the formula (3.5.5) the integral is equal to $2\pi i f(z_0)$. □

LEMMA 3.6.2. *Let $\sum_{k=1}^{\infty} f_k$ be a series of virtually monotone complex functions, which is termwise majorized by a convergent positive series $\sum_{k=1}^{\infty} c_k$ on a monotone curve Γ (that is $|f_k(z)| \leq c_k$ for natural k and $z \in \Gamma$) and such that $F(z) = \sum_{k=1}^{\infty} f_k(z)$ is virtually monotone. Then*

$$\sum_{k=1}^{\infty}\int_{\Gamma} f_k(z)\,dz = \int_{\Gamma}\sum_{k=1}^{\infty} f_k(z)\,dz. \tag{3.6.1}$$

PROOF. By the Estimation Lemma 3.5.4 one has the following inequalities:

$$\left|\int_{\Gamma} f_k(z)\,dz\right| \leq 4c_k \operatorname{diam}\Gamma, \qquad \left|\int_{\Gamma}\sum_{k=n}^{\infty} f_k(z)\,dz\right| \leq 4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k. \tag{3.6.2}$$

Set $F_n(z) = \sum_{k=1}^{n-1} f_k(z)$. By the left inequality of (3.6.2), the module of difference between $\int_\Gamma F_n(z)\,dz = \sum_{k=1}^{n-1}\int_\Gamma f_k(z)\,dz$ and the left-hand side of (3.6.1) does not exceed $4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k$. Hence this module is infinitesimally small as n tends to infinity. On the other hand, by the right inequality of (3.6.2) one gets $\left|\int_\Gamma F_n(z)\,dz - \int_\Gamma F(z)\,dz\right| \leq 4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k$. This implies that the difference between the left-hand and right-hand sides of (3.6.1) is infinitesimally small as n tends to infinity. But this difference does not depend on n. Hence it is zero. □

LEMMA 3.6.3. *If a real function f defined over an interval $[a,b]$ is locally bounded, then it is bounded.*

PROOF. The proof is by contradiction. Suppose that f is unbounded. Divide the interval $[a,b]$ in half. Then the function has to be unbounded at least on one of the halves. Consider this half and divide it in half. Choose the half where the function is unbounded. So we construct a nested infinite sequence of intervals converging to a point, which coincides with the intersection of all the intervals. And f is obviously not locally bounded at this point. □

COROLLARY 3.6.4. *A complex function $f(z)$ continuous on the boundary of a domain D is bounded on ∂D.*

PROOF. Consider a path $p\colon [a,b] \to \partial D$. Then $|f(p(t))|$ is continuous on $[a,b]$, hence it is locally bounded, hence it is bounded. Since ∂D can be covered by images of finitely many paths this implies boundedness of f over ∂D. □

THEOREM 3.6.5. *If a function $f(z)$ is complex differentiable in the disk $|z-z_0| \leq R$, then for $|z-z_0| < R$*

$$f(z) = \sum_{k=0}^{\infty} (z-z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta,$$

where the series on the right-hand side absolutely converges for $|z-z_0| < R$.

PROOF. Fix a point z such that $|z-z_0| < R$ and consider ζ as a variable. For $|\zeta - z_0| > |z - z_0|$ one has

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(\zeta - z_0)^{k+1}}. \tag{3.6.3}$$

On the circle $|\zeta - z_0| = R$ the series on the right-hand side is majorized by the convergent series $\sum_{k=0}^{\infty} \frac{|z-z_0|^k}{R^{k+1}}$ for $r > |z - z_0|$. The function $f(\zeta)$ is bounded on $|\zeta - z_0| = R$ by Corollary 3.6.4. Therefore after multiplication of (3.6.3) by $f(\zeta)$ all the conditions of Lemma 3.6.2 are satisfied. Termwise integration gives:

$$f(z) = \oint_{z_0}^{R} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=0}^{\infty} (z-z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{k+1}}.$$

□

Analytic functions. A function $f(z)$ of complex variable is called an *analytic function* in a point z_0 if there is a positive ε such that $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ for all z from a disk $|z - z_0| \leq \varepsilon$ and the series absolutely converges. Since one can differentiate power series termwise (Theorem 3.3.9), any function which is analytic at z is also complex differentiable at z. Theorem 3.6.5 gives a converse. Thus, we get the following:

COROLLARY 3.6.6. *A function $f(z)$ is analytic at z if and only if it is complex differentiable in some neighborhood of z.*

THEOREM 3.6.7. *If f is analytic at z then f' is analytic at z. If f and g are analytic at z then $f+g$, $f-g$, fg are analytic at z. If f is analytic at z and g is analytic at $f(z)$ then $g(f(z))$ is analytic at z.*

PROOF. Termwise differentiation of the power series representing f in a neighborhood of z gives the power series for its derivative. Hence f' is analytic. The differentiability of $f \pm g$, fg and $g(f(z))$ follow from corresponding differentiation rules. □

LEMMA 3.6.8 (Isolated Zeroes). *If $f(z)$ is analytic and is not identically equal to 0 in some neighborhood of z_0, then $f(z) \neq 0$ for all $z \neq z_0$ sufficiently close to z_0.*

PROOF. Let $f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ in a neighborhood U of z_0. Let c_m be the first nonzero coefficient. Then $\sum_{k=m}^{\infty} c_k (z-z_0)^{k-m}$ converges in U to a differentiable function $g(z)$ by Theorem 3.3.9. Since $g(z_0) = c_m \neq 0$ and $g(z)$ is continuous at z_0, the inequality $g(z) \neq 0$ holds for all z sufficiently close to z_0. As $f(z) = g(z)(z - z_0)^m$, the same is true for $f(z)$. $\square$

THEOREM 3.6.9 (Uniqueness Theorem). *If two power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_0)^k$ converge in a neighborhood of z_0 and their sums coincide for some infinite sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \neq z_0$ for all k and $\lim_{k\to\infty} z_k = z_0$, then $a_k = b_k$ for all k.*

PROOF. Set $c_k = a_k - b_k$. Then $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ has a non-isolated zero at z_0. Hence $f(z) = 0$ in a neighborhood of z_0. We get a contradiction by considering the function $g(z) = \sum_{k=m}^{\infty} c_k(z - z_0)^{k-m}$, which is nonzero for all z sufficiently close to z_0 (cf. the proof of the Isolated Zeroes Lemma 3.6.8), and satisfies the equation $f(z) = g(z)(z - z_0)^m$. $\square$

Taylor series. Set $f^{(0)} = f$ and by induction define the $(k + 1)$-th derivative $f^{(k+1)}$ of f as the derivative of its k-th derivative $f^{(k)}$. For the first and the second derivatives one prefers the notation f' and f''. For example, the k-th derivative of z^n is $n^{\underline{k}} z^{n-k}$. (Recall that $n^{\underline{k}} = n(n - 1) \dots (n - k + 1)$.)

The following series is called the *Taylor series* of a function f at point z_0:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

The Taylor series is defined for any analytic function, because an analytic function has derivative of any order due to Theorem 3.6.7.

THEOREM 3.6.10. *If a function f is analytic in the disk $|z - z_0| < r$ then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ for any z from the disk.*

PROOF. By Theorem 3.6.5, $f(z)$ is presented in the disk by a convergent power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$. To prove our theorem we prove that

$$a_k = \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)}{k!}. \tag{3.6.4}$$

Indeed, $a_0 = f(z_0)$ and termwise differentiatiion of $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ applied n times gives $f^{(n)}(z) = \sum_{k=n}^{\infty} k^{\underline{n}} a_k(z - z_0)^k$. Putting $z = z_0$, one gets $f^{(n)}(z_0) = n^{\underline{n}} a_n = a_n n!$. $\square$

THEOREM 3.6.11 (Liouville). *If a function f is analytic and bounded on the whole complex plane, then f is constant.*

PROOF. If f is analytic on the whole plane then $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where a_k is defined by (3.6.4). If $|f(z)| \leq B$ by the Estimation Lemma 3.5.4 one gets

$$|a_k| = \left| \oint_0^R \frac{f(\zeta)}{z^{k+1}}\, d\zeta \right| \leq 4 \cdot 4 \frac{B}{R^{k+1}} \frac{R}{\sqrt{2}} = \frac{C}{R^k}. \tag{3.6.5}$$

Consequently a_k for $k > 0$ is infinitesimally small as R tends to infinity. But a_k does not depend on R, hence it is 0. Therefore $f(z) = a_0$. $\square$

THEOREM 3.6.12 (Fundamental Theorem of Algebra). *Any nonconstant polynomial $P(z)$ has a complex root.*

PROOF. If $P(z)$ has no roots the function $f(z) = \frac{1}{P(z)}$ is analytic on the whole plane. Since $\lim_{z\to\infty} f(z) = 0$ the inequality $|f(z)| < 1$ holds for $|z| = R$ if R is sufficiently large. Therefore the estimation (3.6.5) for the k-th coefficient of f holds with $B = 1$ for sufficiently large R. Hence the same arguments as in proof of the Liouville Theorem 3.6.11 show that $f(z)$ is constant. This is a contradiction. □

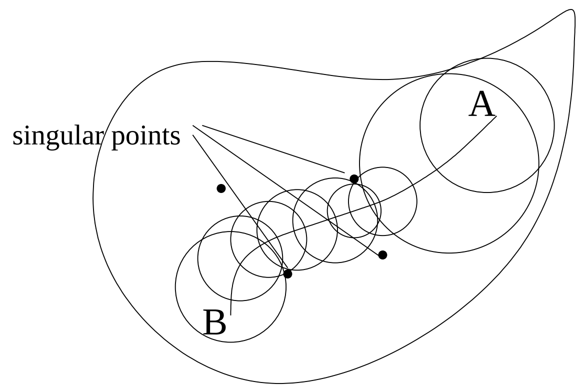

FIGURE 3.6.1. Analytic continuation

Analytic continuation.

LEMMA 3.6.13. *If an analytic function $f(z)$ has finitely many singular points in a domain D and a non isolated zero at a point $z_0 \in D$ then $f(z) = 0$ for all regular $z \in D$.*

PROOF. For any nonsingular point $z \in D$, we construct a sequence of sufficiently small disks $D_0, D_1, D_2, \ldots, D_n$ without singular points with the following properties: 1) $z_0 \in D_0 \subset U$; 2) $z \in D_n$; 3) z_k, the center of D_k, belongs to D_{k-1} for all $k > 0$. Then by induction we prove that $f(D_k) = 0$. First step: if z_0 is a non-isolated zero of f, then the Taylor series of f vanishes at z_0 by the Uniqueness Theorem 3.6.9. But this series represents $f(z)$ on D_0 due to Theorem 3.6.10, since D_0 does not contain singular points. Hence, $f(D_0) = 0$. Suppose we have proved already that $f(D_k) = 0$. Then z_{k+1} is a non-isolated zero of f by the third property of the sequence $\{D_k\}_{k=0}^n$. Consequently, the same arguments as above for $k = 0$ prove that $f(D_{k+1}) = 0$. And finally we get $f(z) = 0$. □

Consider any formula which you know from school about trigonometric functions. For example, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. The above lemma implies that this formula remains true for complex x and y. Indeed, consider the function $T(x, y) = \tan(x + y) - \frac{\tan x + \tan y}{1 - \tan x \tan y}$. For a fixed x the function $T(x, y)$ is analytic and has finitely many singular points in any disk. This function has non-isolated zeroes in all real points, hence this function is zero in any disk intersecting the real line. This implies that $T(x, y)$ is zero for all y. The same arguments applied to $T(x, y)$ with fixed y and variable x prove that $T(x, y)$ is zero for all complex x, y.

The same arguments prove the following theorem.

THEOREM 3.6.14. *If some analytic relation between analytic functions holds on a curve* Γ, *it holds for any* $z \in \mathbb{C}$, *which can be connected with* Γ *by a paths avoiding singular points of the functions.*

LEMMA 3.6.15. $\sin t \geq \frac{2t}{\pi}$ *for* $t \in [0, \pi/2]$.

PROOF. Let $f(t) = \sin t - \frac{2t}{\pi}$. Then $f'(x) = \cos t - \frac{2}{\pi}$. Set $y = \arccos \frac{2}{\pi}$. Then $f'(x) \geq 0$ for $x \in [0, y]$. Therefore f is nondecreasing on $[0, y]$, and nonnegative, because $f(0) = 0$. On the interval $[y, \pi/2]$ the derivative of f is negative. Hence $f(x)$ is non-increasing and nonnegative, because its value on the end of the interval is 0. $\square$

LEMMA 3.6.16 (Jordan). *Let* $f(z)$ *be an analytic function in the upper half-plane such that* $\lim_{z\to\infty} f(z) = 0$. *Denote by* Γ_R *the upper half of the circle* $|z| = R$. *Then for any natural* m

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) \exp(miz)\, dz = 0. \tag{3.6.6}$$

PROOF. Consider the parametrization $z(t) = R\cos t + Ri \sin t$, $t \in [0, \pi]$ of Γ_R. Then the integral (3.6.6) turns into

$$\begin{aligned} &\int_0^\pi f(z) \exp(iRm \cos t - Rm \sin t)\, d(R\cos t + Ri \sin t) \\ &\qquad = \int_0^\pi Rf(z) \exp(iRm\cos t) \exp(-Rm \sin t)(-\sin t + i \cos t)\, dt. \end{aligned} \tag{3.6.7}$$

If $|f(z)| \leq B$ on Γ_R, then $|f(z) \exp(iRm\cos t)(-\sin t + i\cos t)| \leq B$ on Γ_R. And the module of the integral (3.6.7) can be estimated from above by

$$BR \int_0^\pi \exp(-Rm \sin t)\, dt.$$

Since $\sin(\pi - t) = \sin t$, the latter integral is equal to $2BR \int_0^{\pi/2} \exp(-Rm \sin t)\, dt$. Since $\sin t \geq \frac{2t}{\pi}$, the latter integral does not exceed

$$2BR \int_0^{\pi/2} \exp(-2Rmt/\pi)\, dt = 2BR \frac{1 - \exp(-Rm)}{2Rm} \leq \frac{B}{m}.$$

Since B can be chosen arbitrarily small for sufficiently large R, this proves the lemma. $\square$

Evaluation of $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \lim_{N\to\infty} \int_{-N}^{N} \frac{\sin x}{x}\, dx$**.** Since $\sin x = \operatorname{Im} e^{ix}$ our integral is equal to $\operatorname{Im} \int_{-\infty}^{+\infty} \frac{e^{iz}}{z}\, dz$. Set $\Gamma(r) = \{z \mid |z| = r, \operatorname{Im} z \geq 0\}$. This is a semicircle. Let us orient it counter-clockwise, so that its initial point is r.

Consider the domain $D(R)$ bounded by the semicircles $-\Gamma(r)$, $\Gamma(R)$ and the intervals $[-R, -r]$, $[r, R]$, where $r = \frac{1}{R}$ and $R > 1$. The function $\frac{e^{iz}}{z}$ has no singular points inside $D(R)$. Hence $\oint_{\partial D(R)} \frac{e^{iz}}{z} dz = 0$. Hence for any R

$$\int_{-r}^{-R} \frac{e^{iz}}{z}\, dz + \int_r^R \frac{e^{iz}}{z}\, dz = \int_{\Gamma(r)} \frac{e^{iz}}{z}\, dz - \int_{\Gamma(R)} \frac{e^{iz}}{z}\, dz. \tag{3.6.8}$$

The second integral on the right-hand side tends to 0 as R tends to infinity due to Jordan's Lemma 3.6.16. The function $\frac{e^{iz}}{z}$ has a simple pole at 0, hence the first

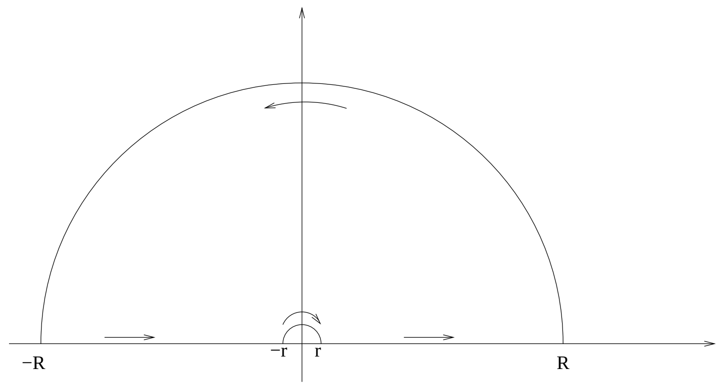

FIGURE 3.6.2. The domain $D(R)$

integral on the right-hand side of (3.6.8) tends to $\pi i \operatorname{res} \frac{e^{iz}}{z} = \pi i$ due to Remark 3.5.8. As a result, the right-hand side of (3.6.8) tends to πi as R tends to infinity. Consequently the left-hand side of (3.6.8) also tends to πi as $R \to \infty$. The imaginary part of left-hand side of (3.6.8) is equal to $\int_{-R}^{R} \frac{\sin x}{x}\,dx - \int_{-r}^{r} \frac{\sin x}{x}\,dx$. The last integral tends to 0 as $r \to 0$, because $|\frac{\sin x}{x}| \leq 1$. Hence $\int_{-R}^{R} \frac{\sin x}{x}\,dx$ tends to π as $R \to \infty$. Finally $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi$.

Problems.

1. Prove that an even analytic function f, i.e., a function such that $f(z) = f(-z)$, has a Taylor series at 0 consisting only of even powers.

2. Prove that analytic function which has a Taylor series only with even powers is an even function.

3. Prove: If an analytic function $f(z)$ takes real values on $[0, 1]$, then $f(x)$ is real for any real x.

4. Evaluate $\int_{-\infty}^{+\infty} \frac{1}{1+x^4}dx$.

5. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{5+3\cos\phi}$.

6. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx$ $(a > 0)$.

7. Evaluate $\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+4x+20}\,dx$.

8. Evaluate $\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$ $(a, b > 0)$.