CHAPTER 1

Series

1.1. Autorecursion of Infinite Expressions

On the contents of the lecture. The lecture presents a romantic style of early analytics. The motto of the lecture could be "infinity, equality and no definitions!". *Infinity* is the main personage we will play with today. We demonstrate how infinite expressions (i.e., infinite sums, products, fractions) arise in solutions of simple equations, how it is possible to calculate them, and how the results of such calculations apply to finite mathematics. In particular, we will deduce the Euler-Binet formula for Fibonacci numbers, the first Euler's formula of the course. We become acquainted with geometric series and the golden section.

Achilles and the turtle. The ancient Greek philosopher Zeno claimed that Achilles pursuing a turtle could never pass it by, in spite of the fact that his velocity was much greater than the velocity of the turtle. His arguments adopted to our purposes are the following.

First Zeno proposed a pursuing algorithm for Achilles:

Initialization. Assign to the variable *goal* the original position of the turtle.

Action. Reach the *goal*.

Correction. If the current turtle's position is *goal*, then stop, else reassign to the variable *goal* the current position of the turtle and go to **Action**.

Secondly, Zeno remarks that this algorithm never stops if the turtle constantly moves in one direction.

And finally, he notes that Achilles has to follow his algorithm if he want pass the turtle by. He may be not aware of this algorithm, but unconsciously he must perform it. Because he cannot run the turtle down without reaching the original position of the turtle and then all positions of the turtle which the variable *goal* takes.

Zeno's algorithm generates a sequence of times $\{t_k\}$, where t_k is the time of execution of the k-th action of the algorithm. And the whole time of work of the algorithm is the infinite sum $\sum_{k=1}^{\infty} t_k$; and this sum expresses the time Achilles needs to run the turtle down. (The corrections take zero time, because Achilles really does not think about them.) Let us name this sum the *Zeno series*.

Assume that both Achilles and the turtle run with constant velocities v and w, respectively. Denote the initial distance between Achilles and the turtle by d_0. Then $t_1 = \frac{d_0}{v}$. The turtle in this time moves by the distance $d_1 = t_1 w = \frac{w}{v} d_0$. By his second action Achilles overcomes this distance in time $t_2 = \frac{d_1}{v} = \frac{w}{v} t_1$, while the turtle moves away by the distance $d_2 = t_2 w = \frac{w}{v} d_1$. So we see that the sequences of times $\{t_k\}$ and distances $\{d_k\}$ satisfy the following *recurrence relations*: $t_k = \frac{w}{v} t_{k-1}$, $d_k = \frac{w}{v} d_{k-1}$.

Hence $\{t_k\}$ as well as $\{d_k\}$ are *geometric progressions* with ratio $\frac{w}{v}$. And the time t which Achilles needs to run the turtle down is

$$t = t_1 + t_2 + t_3 + \cdots = t_1 + \frac{w}{v} t_1 + \frac{w^2}{v^2} t_1 + \cdots = t_1 \left(1 + \frac{w}{v} + \frac{w^2}{v^2} + \cdots\right).$$

In spite of Zeno, we know that Achilles does catch up with the turtle. And one easily gets the time t he needs to do it by the following argument: the distance between Achilles and the turtle permanently decreases with the velocity $v - w$. Consequently it becomes 0 in the time $t = \frac{d_0}{v-w} = t_1 \frac{v}{v-w}$. Comparing the results we come to the following conclusion

$$\frac{v}{v-w} = 1 + \frac{w}{v} + \frac{w^2}{v^2} + \frac{w^3}{v^3} + \cdots. \tag{1.1.1}$$

Infinite substitution. We see that some infinite expressions represent finite values. The fraction in the left-hand side of (1.1.1) expands into the infinite series on the right-hand side. Infinite expressions play a key rôle in mathematics and physics. Solutions of equations quite often are presented as infinite expressions.

For example let us consider the following simple equation

$$t = 1 + qt. \tag{1.1.2}$$

Substituting on the right-hand side $1 + qt$ instead of t, one gets a new equation $t = 1 + q(1 + qt) = 1 + q + q^2 t$. Any solution of the original equation satisfies this one. Repeating this trick, one gets $t = 1 + q(1 + q(1 + qt)) = 1 + q + q^2 + q^3 t$. Repeating this infinitely many times, one eliminates t on the right hand side and gets a solution of (1.1.2) in an infinite form

$$t = 1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{\infty} q^k.$$

On the other hand, the equation (1.1.2) solved in the usual way gives $t = \frac{1}{1-q}$. As a result, we obtain the following formula

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \cdots = \sum_{k=0}^{\infty} q^k. \tag{1.1.3}$$

which represents a special case of (1.1.1) for $v = 1$, $w = q$.

Autorecursion. An infinite expression of the form $a_1 + a_2 + a_3 + \dots$ is called a *series* and is concisely denoted by $\sum_{k=1}^{\infty} a_k$. Now we consider a summation method for series which is inverse to the above method of infinite substitution. To find the sum of a series we shall construct an equation which is satisfied by its sum. We name this method *autorecursion*. Recursion means "return to something known". Autorecursion is "return to oneself".

The series $a_2 + a_3 + \cdots = \sum_{k=2}^{\infty} a_k$ obtained from $\sum_{k=1}^{\infty} a_k$ by dropping its first term is called the *shift* of $\sum_{k=1}^{\infty} a_k$.

We will call the following equality the *shift formula*:

$$\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} a_k.$$

Another basic formula we need is the following *multiplication formula*:

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

These two formulas are all one needs to find the sum of geometric series $\sum_{k=0}^{\infty} q^k$. To be exact, the multiplication formula gives the equality $\sum_{k=1}^{\infty} q^k = q \sum_{k=0}^{\infty} q^k$. Hence the shift formula turns into equation $x = 1 + qx$, where x is $\sum_{k=0}^{\infty} q^k$. The solution of this equation gives us the formula (1.1.3) for the sum of the geometric series again.

From this formula, one can deduce the formula for the sum of a finite geometric progression. By $\sum_{k=0}^{n} a_k$ is denoted the sum $a_0 + a_1 + a_2 + \cdots + a_n$. One has

$$\sum_{k=0}^{n-1} q^k = \sum_{k=0}^{\infty} q^k - \sum_{k=n}^{\infty} q^k = \frac{1}{1-q} - \frac{q^n}{1-q} = \frac{1-q^n}{1-q}.$$

This is an important formula which was traditionally studied in school.

The series $\sum_{k=0}^{\infty} kx^k$. To find the sum of $\sum_{k=1}^{\infty} kx^k$ we have to apply additionally the following *addition formula*,

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

which is the last general formula for series we introduce in the first lecture.

Reindexing the shift $\sum_{k=2}^{\infty} kx^k$ we give it the form $\sum_{k=1}^{\infty}(k+1)x^{k+1}$. Further it splits into two parts

$$x\sum_{k=1}^{\infty}(k+1)x^k = x\sum_{k=1}^{\infty} kx^k + x\sum_{k=1}^{\infty} x^k = x\sum_{k=1}^{\infty} kx^k + x\frac{x}{1-x}$$

by the addition formula. The first summand is the original sum multiplied by x. The second is a geometric series. We already know its sum. Now the shift formula for the sum $s(x)$ of the original series turns into the equation $s(x) = x + x\frac{x}{1-x} + xs(x)$. Its solution is $s(x) = \frac{x}{(1-x)^2}$.

Fibonacci Numbers. Starting with $\phi_0 = 0$, $\phi_1 = 1$ and applying the recurrence relation

$$\phi_{n+1} = \phi_n + \phi_{n-1},$$

one constructs an infinite sequence of numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$ called *Fibonacci numbers*. We are going to get a formula for ϕ_n.

To do this let us consider the following function $\Phi(x) = \sum_{k=0}^{\infty} \phi_k x^k$, which is called the *generating function* for the sequence $\{\phi_k\}$. Since $\phi_0 = 0$, the sum $\Phi(x) + x\Phi(x)$ transforms in the following way:

$$\sum_{k=1}^{\infty} \phi_k x^k + \sum_{k=1}^{\infty} \phi_{k-1} x^k = \sum_{k=1}^{\infty} \phi_{k+1} x^k = \frac{\Phi(x) - x}{x}.$$

Multiplying both sides of the above equation by x and collecting all terms containing $\Phi(x)$ on the right-hand side, one gets $x = \Phi(x) - x\Phi(x) - x^2\Phi(x) = x$. It leads to

$$\Phi(x) = \frac{x}{1 - x - x^2}.$$

The roots of the equation $1 - x - x^2 = 0$ are $\frac{-1\pm\sqrt{5}}{2}$. More famous is the pair of their inverses $\frac{1\pm\sqrt{5}}{2}$. The number $\phi = \frac{-1+\sqrt{5}}{2}$ is the so-called *golden section* or *golden mean*. It plays a significant rôle in mathematics, architecture and biology. Its dual is $\hat{\phi} = \frac{-1-\sqrt{5}}{2}$. Then $\phi\hat{\phi} = -1$, and $\phi + \hat{\phi} = 1$. Hence $(1 - x\phi)(1 - x\hat{\phi}) = 1 - x - x^2$, which in turn leads to the following decomposition:

$$\frac{x}{x^2 + x - 1} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi}x}\right).$$

We expand both fractions on the right hand side into geometric series:

$$\frac{1}{1 - \phi x} = \sum_{k=0}^{\infty} \phi^k x^k, \qquad \frac{1}{1 - \hat{\phi}x} = \sum_{k=0}^{\infty} \hat{\phi}^k x^k.$$

This gives the following representation for the generating function

$$\Phi(x) = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\phi^k - \hat{\phi}^k) x^k.$$

On the other hand the coefficient at x^k in the original presentation of $\Phi(x)$ is ϕ_k. Hence

$$\phi_k = \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k) = \frac{(\sqrt{5}+1)^k + (-1)^k(\sqrt{5}-1)^k}{2^k\sqrt{5}}. \tag{1.1.4}$$

This is called the *Euler-Binet* formula. It is possible to check it for small k and then prove it by induction using Fibonacci recurrence.

Continued fractions. The application of the method of infinite substitution to the solution of quadratic equation leads us to a new type of infinite expressions, the so-called *continued fractions*. Let us consider the golden mean equation $x^2 - x - 1 = 0$. Rewrite it as $x = 1 + \frac{1}{x}$. Substituting $1 + \frac{1}{x}$ instead of x on the right-hand side we get $x = 1 + \frac{1}{1+\frac{1}{x}}$. Repeating the substitution infinitely many times we obtain a solution in the form of the *continued fraction*:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1+\dots}}} \tag{1.1.5}$$

As this fraction seems to represent a positive number and the golden mean is the unique positive root of the golden mean equation, it is natural to conclude that this fraction is equal to $\phi = \frac{1+\sqrt{5}}{2}$. This is true and this representation allows one to calculate the golden mean and $\sqrt{5}$ effectively with great precision.

To be precise, consider the sequence

$$1, \quad 1+\tfrac{1}{1}, \quad 1+\cfrac{1}{1+\frac{1}{1}}, \quad 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}, \quad \dots \tag{1.1.6}$$

of so-called *convergents* of the continued fraction (1.1.5). Let us remark that all odd convergents are less than ϕ and all even convergents are greater than ϕ. To see this, compare the n-th convergent with the corresponding term of the following sequence of fractions:

$$1+\tfrac{1}{x}, \quad 1+\cfrac{1}{1+\frac{1}{x}}, \quad 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{x}}}, \quad \dots. \tag{1.1.7}$$

We know that for $x = \phi$ all terms of the above sequence are equal to ϕ. Hence all we need is to observe how the removal of $\frac{1}{x}$ affects the value of the considered fraction. The value of the first fraction of the sequence decreases, the value of the second fraction increases. If we denote the value of n-th fraction by f_n, then the value of the next fraction is given by the following recurrence relation:

$$f_{n+1} = 1 + \frac{1}{f_n}. \tag{1.1.8}$$

Hence increasing f_n decreases f_{n+1} and decreasing f_n increases f_{n+1}. Consequently in general all odd fractions of the sequence (1.1.7) are less than the corresponding convergent, and all even are greater. The recurrence relation (1.1.8) is valid for the golden mean convergent. By this recurrence relation one can quickly calculate the first ten convergents $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$. The golden mean lies between last two fractions, which have the difference $\frac{1}{34\cdot 55}$. This allows us to determine the first four decimal digits after the decimal point of it and of $\sqrt{5}$.

Problems.

1. Evaluate $\sum_{k=0}^{\infty} \frac{2^{2k}}{3^{3k}}$.
2. Evaluate $1 - 1 + 1 - 1 + \cdots$.
3. Evaluate $1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + \cdots$.
4. Evaluate $\sum_{k=1}^{\infty} \frac{k}{3^k}$.
5. Evaluate $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$.
6. Decompose the fraction $\frac{1}{a+x}$ into a power series.
7. Find the generating function for the sequence $\{2^k\}$.
8. Find sum the $\sum_{k=1}^{\infty} \phi_k 3^{-k}$.
9. Prove by induction the Euler-Binet formula.
***10.** Evaluate $1 - 2 + 1 + 1 - 2 + 1 + \cdots$.
11. Approximate $\sqrt{2}$ by a rational with precision 0.0001.
12. Find the value of $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}$.
13. Find the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$.
14. By infinite substitution, solve the equation $x^2 - 2x - 1 = 0$, and represent $\sqrt{2}$ by a continued fraction.
15. Find the value of the infinite product $2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdots$.
16. Find a formula for n-th term of the recurrent sequence $x_{n+1} = 2x_n + x_{n-1}$, $x_0 = x_1 = 1$.
17. Find the sum of the Fibonacci numbers $\sum_{k=1}^{\infty} \phi_k$.
18. Find sum $1 + 0 - 1 + 1 + 0 - 1 + \cdots$.
19. Decompose into the sum of partial fractions $\frac{1}{x^2-3x+2}$.

1.2. Positive Series

On the contents of the lecture. Infinity is pregnant with paradoxes. Paradoxes throw us down from the heavens to the earth. We leave the poetry for prose, and rationalize the *infinity and equality* by working with *finiteness and inequality.* We shall lay a solid foundation for a summation theory for positive series. And the reader will find out what $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ precisely means.

Divergent series paradox. Let us consider the series $\sum_{k=0}^{\infty} 2^k$. This is a geometric series. We know how to sum it up by autorecursion. The autorecursion equation is $s = 1 + 2s$. The only number satisfying this equation is -1. The sum of positive numbers turns to be negative!? Something is wrong!

A way to save the situation is to admit infinity as a feasible solution. Infinity is an obvious solution of $s = 1+2s$. The sum of any geometric series $\sum_{k=0}^{\infty} q^k$ with denominator $q \geq 1$ is obviously infinite, isn't it?

Indeed, this sum is greater than $1+1+1+1+\dots$, which symbolizes infinity. (The autorecursion equation for $1+1+1+\dots$ is $s = s+1$. Infinity is the unique solution of this equation.)

The series $\sum_{k=0}^{\infty} 2^k$ represents Zeno's series in the case of the *Mighty Turtle*, which is faster than Achilles. To be precise, this series arises if $v = d_0 = 1$ and $w = 2$. As the velocity of the turtle is greater than the velocity of Achilles he never reaches it. So the infinity is right answer for this problem. But the negative solution -1 also makes sense. One could interpret it as an event in the past. Just the point in time when *the turtle passed Achilles.*

Oscillating series paradoxes. The philosopher Gvido Grandy in 1703 attracted public attention to the series $1-1+1-1+\dots$. He claimed this series symbolized the Creation of Universe from Nothing. Namely, insertion of brackets in one way gives Nothing (that is 0), in another way, gives 1.

$$(1-1)+(1-1)+(1-1)+\dots = 0+0+0+\dots = 0,$$
$$1-(1-1)-(1-1)-(1-1)-\dots = 1-0-0-0-\dots = 1.$$

On the other hand, this series $1-1+1-1+1-1+\dots$ is geometric with negative ratio $q=-1$. Its autorecursion equation $s = 1-s$ has the unique solution $s = \frac{1}{2}$. Neither $+\infty$ nor $-\infty$ satisfy it. So $\frac{1}{2}$ seems to be its true sum.

Hence we see the Associativity Law dethroned by $1-1+1-1+\dots$. The next victim is the Commutativity Law. The sum $-1+1-1+1-1+\dots$ is equal to $-\frac{1}{2}$. But the last series is obtained from $1-1+1-1+\dots$ by transposition of odd and even terms.

And the third amazing thing: diluting it by zeroes changes its sum. The sum $1+0-1+1+0-1+1+0-1+\dots$ by no means is $\frac{1}{2}$. It is $\frac{2}{3}$. Indeed, if we denote this sum by s then by shift formulas one gets

$$\begin{aligned}
s &= 1+0-1+1+0-1+1+0-1+1+0-1+\dots,\\
s-1 &= 0-1+1+0-1+1+0-1+1+0-1+1+\dots,\\
s-1-0 &= -1+1+0-1+1+0-1+1+0-1+1+0+\dots.
\end{aligned}$$

Summing the numbers column-wise (i.e., by the Termwise Addition Formula), we get

$$\begin{aligned} s + (s-1) + (s-1-0) &= (1+0-1) + (0-1+1) + (-1+1+0) \\ &\quad + (1+0-1) + (0-1+1) + (-1+1+0) + \dots. \end{aligned}$$

The left-hand side is $3s - 2$. The right-hand side is the zero series. That is why $s = \frac{2}{3}$.

The series $1 - 1 + 1 - 1 + \dots$ arises as Zeno's series in the case of a blind Achilles directed by a cruel Zeno, who is interested, as always, only in proving his claim, and a foolish, but merciful turtle. The blind Achilles is not fast, his velocity equals the velocity of the turtle. At the first moment Zeno tells the blind Achilles where the turtle is. Achilles starts the rally. But the merciful turtle wishing to help him goes towards him instead of running away. Achilles meets the turtle half-way. But he misses it, being busy to perform the first step of the algorithm. When he accomplishes this step, Zeno orders: "Turn about!" and surprises Achilles by saying that the turtle is on Achilles' initial position. The turtle discovers that Achilles turns about and does the same. The situation repeats ad infinitum. Now we see that assigning the sum $\frac{1}{2}$ to the series $1 - 1 + 1 - 1 + \dots$ makes sense. It predicts accurately the time of the first meeting of Achilles and turtle.

Positivity. The paradoxes discussed above are discouraging. Our intuition based on handling finite sums fails when we turn to infinite ones. Observe that all paradoxes above involve negative numbers. And to eliminate the evil in its root, let us consider only nonnegative numbers.

We return to the ancient Greeks. They simply did not know what a negative number is. But in contrast to the Greeks, we will retain zero. A series with nonnegative terms will be called a *positive* series. We will show that for positive series all familiar laws, including associativity and commutativity, hold true and zero terms do not affect the sum.

Definition of Infinite Sum. Let us consider what Euler's equality could mean:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The natural answer is: the *partial sums* $\sum_{k=1}^{n} \frac{1}{k^2}$, which contain more and more reciprocal squares, approach closer and closer the value $\frac{\pi^2}{6}$. Consequently, all partial sums have to be less than $\frac{\pi^2}{6}$, its *ultimate sum*. Indeed, if some partial sum exceeds or coincides with $\frac{\pi^2}{6}$ then all subsequent sums will move away from $\frac{\pi^2}{6}$. Furthermore, any number c which is less than $\frac{\pi^2}{6}$ has to be surpassed by partial sums eventually, when they approach $\frac{\pi^2}{6}$ closer than by $\frac{\pi^2}{6} - c$. Hence the ultimate sum majorizes all partial ones, and any lesser number does not. This means that the ultimate sum is the smallest number which majorizes all partial sums.

Geometric motivation. Imagine a sequence $[a_{i-1}, a_i]$ of intervals of the real line. Denote by l_i the length of i-th interval. Let $a_0 = 0$ be the left end point of the first interval. Let $[0, A]$ be the smallest interval containing the whole sequence. Its length is naturally interpreted as the sum $\sum_{i=1}^{\infty} l_i$

This motivates the following definition.

DEFINITION. *If the partial sums of the positive series* $\sum_{k=1}^{\infty} a_k$ *increase without bound, its sum is defined to be* ∞ *and the series is called* divergent. *In the opposite case the series called* convergent, *and its sum is defined as the smallest number* A *such that* $A \geq \sum_{k=1}^{n} a_k$ *for all* n.

This Definition is equivalent to the following couple of principles. The first principle limits the ultimate sum from below:

PRINCIPLE (One-for-All). *The ultimate sum of a positive series majorizes all partial sums.*

And the second principle limits the ultimate sum from above:

PRINCIPLE (All-for-One). *If all partial sums of a positive series do not exceed a number, then the ultimate sum also does not exceed it.*

THEOREM 1.2.1 (Termwise Addition Formula).

$$\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k).$$

PROOF. The inequality $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k)$ is equivalent to $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k$. By All-for-One, the last is equivalent to the system of inequalities

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k \quad N = 1, 2, \ldots.$$

This system is equivalent to the following system

$$\sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad N = 1, 2, \ldots.$$

Each inequality of the last system, in its turn, is equivalent to the system of inequalities

$$\sum_{k=1}^{M} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad M = 1, 2, \ldots.$$

But these inequalities are true for all N and M, as the following computations show.

$$\sum_{k=1}^{M} b_k + \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{M+N} b_k + \sum_{k=1}^{M+N} a_k = \sum_{k=1}^{M+N} (a_k + b_k) \leq \sum_{k=1}^{\infty}(a_k + b_k).$$

In the opposite direction, we see that any partial sum on the right-hand side $\sum_{k=1}^{n}(a_k + b_k)$ splits into $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. And by virtue of the One-for-All principle, this does not exceed $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$. Now, the All-for-One principle provides the inequality in the opposite direction. □

THEOREM 1.2.2 (Shift Formula).

$$\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k.$$

PROOF. The Shift Formula immediately follows from the Termwise Addition formula. To be precise, immediately from the definition, one gets the following: $a_0 + 0 + 0 + 0 + 0 + \cdots = a_0$ and that $0 + a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$. Termwise Addition of these series gives

$$a_0 + \sum_{k=1}^{\infty} a_k = (a_0 + 0) + (0 + a_1) + (0 + a_2) + (0 + a_3) + \cdots = \sum_{k=0}^{\infty} a_k.$$

□

THEOREM 1.2.3 (Termwise Multiplication Formula).

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

PROOF. For any partial sum from the right-hand side one has

$$\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k \leq \lambda \sum_{k=1}^{\infty} a_k$$

by the Distributivity Law for finite sums and One-for-All. This implies the inequality $\lambda \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} \lambda a_k$ by All-for-One. The opposite inequality is equivalent to $\sum_{k=1}^{\infty} a_k \geq \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$. As any partial sum $\sum_{k=1}^{n} a_k$ is equal to $\frac{1}{\lambda} \sum_{k=1}^{n} \lambda a_k$, which does not exceed $\frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$, one gets the opposite inequality. □

Geometric series. We have to return to the geometric series, because the autorecursion equation produced by shift and multiplication formulas says nothing about convergence. So we have to prove convergence for $\sum_{k=0}^{\infty} q^k$ with positive $q < 1$. It is sufficient to prove the following inequality for all n

$$1 + q + q^2 + q^3 + \cdots + q^n < \tfrac{1}{1-q}.$$

Multiplying both sides by $1 - q$ one gets on the left-hand side

$$\begin{aligned} &(1-q) + (q - q^2) + (q^2 - q^3) + \cdots + (q^{n-1} - q^n) + (q^n - q^{n+1}) \\ &\quad = 1 - q + q - q^2 + q^2 - q^3 + q^3 - \cdots - q^n + q^n - q^{n+1} \\ &\quad = 1 - q^{n+1} \end{aligned}$$

and 1 on the right-hand side. The inequality $1 - q^{n+1} < 1$ is obvious. Hence we have proved the convergence. Now the autorecursion equation $x = 1 + qx$ for $\sum_{k=0}^{\infty} q^k$ is constructed in usual way by the shift formula and termwise multiplication. It leaves only two possibilities for $\sum_{k=0}^{\infty} q^k$, either $\frac{1}{q-1}$ or ∞. For $q < 1$ we have proved convergence, and for $q \geq 1$ infinity is the true answer.

Let us pay special attention to the case $q = 0$. We adopt a common convention:

$$0^0 = 1.$$

This means that the series $\sum_{k=0}^{\infty} 0^k$ satisfies the common formula for a convergent geometric series $\sum_{k=0}^{\infty} 0^k = \frac{1}{1-0} = 1$. Finally we state the theorem, which is essentially due to Eudoxus, who proved the convergence of the geometric series with ratio $q < 1$.

THEOREM 1.2.4 (Eudoxus). *For every nonnegative q one has*

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \text{for } q < 1, \text{ and} \quad \sum_{k=0}^{\infty} q^k = \infty \quad \text{for } q \geq 1.$$

Comparison of series. Quite often exact summation of series is too difficult, and for practical purposes it is enough to know the sum approximatively. In this case one usually compares the series with another one whose sum is known. Such a comparison is based on the following *Termwise Comparison Principle*, which immediately follows from the definition of a sum.

PRINCIPLE (Termwise Comparison). *If $a_k \leq b_k$ for k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

The only series we have so far to compare with are the geometric ones. The following lemma is very useful for this purposes.

LEMMA 1.2.5 (Ratio Test). *If $a_{k+1} \leq qa_k$ for k holds for some $q < 1$ then*

$$\sum_{k=0}^{\infty} a_k \leq \frac{a_0}{1-q}.$$

PROOF. By induction one proves the inequality $a_k \leq a_0 q^k$. Now by Termwise Comparison one estimates $\sum_{k=0}^{\infty} a_k$ from above by the geometric series $\sum_{k=0}^{\infty} a_0 q^k = \frac{a_0}{1-q}$ □

If the series under consideration satisfies an autorecursion equation, to prove its convergence usually means to evaluate it exactly. For proving convergence, the Termwise Comparison Principle can be strengthened. Let us say that the series $\sum_{k=1}^{\infty} a_k$ is *eventually* majorized by the series $\sum_{k=1}^{\infty} b_k$, if the inequality $b_k \geq a_k$ holds for each k starting from $k = n$ for some n. The following lemma is very useful to prove convergence.

PRINCIPLE (Eventual Comparison). *A series $\sum_{k=1}^{\infty} a_k$, which is eventually majorized by a convergent series $\sum_{k=1}^{\infty} b_k$, is convergent.*

PROOF. Consider a tail $\sum_{k=n}^{\infty} b_k$, which termwise majorizes $\sum_{k=n}^{\infty} a_k$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} a_k \\
&\leq \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} b_k \\
&\leq \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{\infty} b_k \\
&< \infty.
\end{aligned}$$

□

Consider the series $\sum_{k=1}^{\infty} k2^{-k}$. The ratio of two successive terms $\frac{a_{k+1}}{a_k}$ of the series is $\frac{k+1}{2k}$. This ratio is less or equal to $\frac{2}{3}$ starting with $k = 3$. Hence this series is eventually majorizes by the geometric series $\sum_{k=0}^{\infty} a_3 \frac{2^k}{3^k}$, $(a_3 = \frac{2}{3})$. This proves its convergence. And now by autorecursion equation one gets its sum.

Harmonic series paradox. Now we have a solid background to evaluate positive series. Nevertheless, we must be careful about infinity! Consider the following calculation:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} &= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{2k} \right) \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}.
\end{aligned}$$

We get that the sum $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k}$ satisfies the equation $s = \frac{s}{2}$. This equation has two roots 0 and ∞. But s satisfies the inequalities $\frac{1}{2} < s < \frac{\pi^2}{6}$. What is wrong?

Problems.

1. Prove $\sum_{k=1}^{\infty} 0 = 0$.
2. Prove $\sum_{k=1}^{\infty} 0^k = 1$.
3. Prove $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}$.
4. Prove $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$ for convergent series.
5. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.
6. Prove $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots = 1 - [(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + \cdots$.
7. Prove the convergence of $\sum_{k=0}^{\infty} \frac{2^k}{k!}$.
8. Prove the convergence of $\sum_{k=1}^{\infty} \frac{1000^k}{k!}$.
9. Prove the convergence of $\sum_{k=1}^{\infty} \frac{k^{1000}}{2^k}$.
10. Prove that $q^n < \frac{1}{n(1-q)}$ for $0 < q < 1$.
11. Prove that for any positive $q < 1$ there is an n that $q^n < \frac{1}{2}$.
12. Prove $\sum_{k=1}^{\infty} \frac{1}{k!} \leq 2$.
13. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.
14. Prove the convergence of the Euler series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.
*__15.__ Prove that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ for $a_{ij} \geq 0$.

1.3. Unordered Sums

On the contents of the lecture. Our summation theory culminates in the Sum Partition Theorem. This lecture will contribute towards evaluation of the Euler series in two ways: we prove its convergence, and even estimate its sum by 2. On the other hand, we will realize that evaluation of the Euler series with Euler's accuracy (10^{-18}) seems to be beyond a human being's strength.

Consider a family $\{a_i\}_{i\in I}$ of nonnegative numbers indexed by elements of an arbitrary set I. An important special case of I is the set of pairs of natural numbers $\mathbb{N}\times\mathbb{N}$. Families indexed by $\mathbb{N}\times\mathbb{N}$ are called double series. They arise when one multiplies one series by another one.

Any sum of the type $\sum_{i\in K} a_i$, where K is a finite subset of I is called a *subsum* of $\{a_i\}_{i\in I}$ over K.

DEFINITION. *The least number majorizing all subsums of $\{a_i\}_{i\in I}$ over finite subsets is called its (ultimate) sum and denoted by $\sum_{i\in I} a_i$*

The One-for-All and All-for-One principles for non-ordered sums are obtained from the corresponding principles for ordered sums by replacing "partial sums" by "finite subsums".

Commutativity. In case $I = \mathbb{N}$ we have a definition which apparently is new. But fortunately this definition is equivalent to the old one. Indeed, as any finite subsum of positive series does not exceed its ultimate (ordered) sum, the non-ordered sum also does not exceed it. On the other hand, any partial sum of the series is a finite subsum. This implies the opposite inequality. Therefore we have established the equality.

$$\sum_{k=1}^{\infty} a_k = \sum_{k\in\mathbb{N}} a_k$$

This means that positive series obey the Commutativity Law. Because the non-ordered sum obviously does not depend on the order of summands.

Partitions. A family of subsets $\{I_k\}_{k\in K}$ of a set I is called a *partition* of I and is written $\bigsqcup_{k\in K} I_k$ if $I = \bigcup_{k\in K} I_k$ and $I_k \cap I_j = \varnothing$ for all $k \neq j$.

THEOREM 1.3.1 (Sum Partition Theorem). *For any partition $I = \bigsqcup_{j\in J} I_j$ of the indexing set and any family $\{a_i\}_{i\in I}$ of nonnegative numbers,*

$$\sum_{i\in I} a_i = \sum_{j\in J}\sum_{i\in I_j} a_i. \tag{1.3.1}$$

Iverson notation. We will apply the following notation: a statement included into [ ] takes value 1, if the statement is true, and 0, if it is false. Prove the following simple lemmas to adjust to this notation. In both lemmas one has $K \subset I$.

LEMMA 1.3.2. $\sum_{i\in K} a_i = \sum_{i\in I} a_i[i\in K]$.

In particular, for $K = I$, Lemma 1.3.2 turns into

LEMMA 1.3.3. $\sum_{i\in I} a_i = \sum_{i\in I} a_i[i\in I]$.

LEMMA 1.3.4. $\sum_{k\in K}[i\in I_k] = [i\in I_K]$ *for all* $i\in I$ *iff* $I_K = \bigsqcup_{k\in K} I_k$.

Proof of Sum Partition Theorem. At first we prove the following Sum Transposition formula for finite J,

$$\sum_{i\in I}\sum_{j\in J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}. \tag{1.3.2}$$

Indeed, if J contains just two elements, this formula turns into the Termwise Addition formula. The proof of this formula is the same as for series. Suppose the formula is proved for any set which contains fewer elements than J does. Decompose J into a union of two nonempty subsets $J_1 \sqcup J_2$. Then applying only Termwise Addition and Lemmas 1.3.2, 1.3.3, 1.3.4, we get

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in J} a_{ij} &= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J]\\
&= \sum_{i\in I}\sum_{j\in J}(a_{ij}[j\in J_1] + a_{ij}[j\in J_2])\\
&= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_1] + \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_2]\\
&= \sum_{i\in I}\sum_{j\in J_1} a_{ij} + \sum_{i\in I}\sum_{j\in J_2} a_{ij}.
\end{aligned}$$

But the last two sums can be transposed by the induction hypothesis. After such a transposition one gets

$$\begin{aligned}
\sum_{j\in J_1}\sum_{i\in I} a_{ij} + \sum_{j\in J_2}\sum_{i\in I} a_{ij} &= \sum_{j\in J}[j\in J_1]\sum_{i\in I} a_{ij} + \sum_{j\in J}[j\in J_2]\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}([j\in J_1] + [j\in J_2])\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}[j\in J]\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}\sum_{i\in I} a_{ij}
\end{aligned}$$

and the Sum Transposition formula for finite J is proved. Consider the general case. To prove $\leq$ in (1.3.2), consider a finite $K \subset I$. By the finite Sum Transposition formula the subsum $\sum_{i\in K}\sum_{j\in J} a_{ij}$ is equal to $\sum_{j\in J}\sum_{i\in K} a_{ij}$. But this sum is termwise majorized by the right-hand side sum in (1.3.2). Therefore the left-hand side does not exceed the right-hand side by All-for-One principle.

To derive the Sum Partition Theorem from the Sum Transposition formula, pose $a_{ij} = a_i[i \in I_j]$. Then $a_i = \sum_{j\in J} a_{ij}$ and (1.3.1) turns into (1.3.2). This completes the proof of the Sum Partition Theorem.

Blocking. For a given a series $\sum_{k=0}^{\infty} a_k$ and an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ starting with $n_0 = 0$ one defines a new series $\sum_{k=0}^{\infty} A_k$ by the rule $A_k = \sum_{i=n_k}^{n_{k+1}-1} a_i$. The series $\sum_{k=0}^{\infty} A_k$ is called *blocking of* $\sum_{k=0}^{\infty} a_k$ *by* $\{n_k\}$.

The Sum Partition Theorem implies that the sums of blocked and unblocked series coincide. Blocking formalizes putting of brackets. Therefore the Sum Partition Theorem implies the *Sequential Associativity Law*: *Placing brackets does not change the sum of series.*

Estimation of the Euler series. Let us compare the Euler series with the series $\sum_{k=0}^{\infty}\frac{1}{2^k}$, blocked by $\{2^n\}$ to $\sum_{k=1}^{\infty}a_k$. The sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ consists of 2^n summands, all of which are less then the first one, which is $\frac{1}{2^{2n}}$. As $2^n\frac{1}{2^{2n}}=\frac{1}{2^n}$, it follows that $a_n\leq\frac{1}{2^n}$ for each n. Summing these inequalities, one gets $\sum_{k=1}^{\infty}a_k\leq 2$.

Now let us estimate how many terms of Euler's series one needs to take into account to find its sum up to the eighteenth digit. To do this, we need to estimate its tail. The arguments above give $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\leq\frac{1}{2^{n-1}}$. To obtain a lower estimate, let us remark that all terms of sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ exceed $\frac{1}{2^{2(n+1)}}$. As the number of summands is 2^n, one gets $a_n\geq\frac{1}{4\cdot 2^n}$. Hence $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\geq\frac{1}{2^{n+1}}$. Since $2^{10}=1024\simeq 10^3$, one gets $2^{60}\simeq 10^{18}$. So, to provide an accuracy of 10^{-18} one needs to sum approximately 10^{18} terms. This task is inaccessible even for a modern computer. How did Euler manage to do this? He invented a summation formula (Euler-MacLaurin formula) and transformed this slowly convergent series into non-positive divergent (!) one, whose partial sum containing as few as ten terms gave eighteen digit accuracy. The whole calculation took him an evening. To introduce this formula, one needs to know integrals and derivatives. We will do this later.

Problems.

1. Find $\sum_{k=1}^{\infty}\frac{1}{(2k)^2}$ and $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, assuming $\sum_{k=1}^{\infty}\frac{1}{k^2}=\pi^2/6$.
2. Prove the convergence of $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}}$.
3. Estimate how many terms of the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are necessary for calculation of its sum with precision 10^{-3}.
4. Estimate the value of $\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{k}$.
5. Prove the equality $\sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k=\sum_{j,k\in\mathbb{N}}a_jb_k$.
6. Estimate how many terms of the Harmonic series give the sum surpassing 1000.
7. Prove the Dirichlet formula $\sum_{k=1}^{n}\sum_{i=1}^{k}a_{ki}=\sum_{i=1}^{n}\sum_{k=i}^{n}a_{ki}$.
8. Evaluate $\sum_{i,j\in N}\frac{1}{2^i3^j}$.
9. Evaluate $\sum_{i,j\in N}\frac{i+j}{2^i3^j}$.
10. Represent an unordered sum $\sum_{i+j<n}a_{ij}$ as a double sum.
11. Evaluate $\sum_{i,j\in N}\frac{ij}{2^i3^j}$.
12. Change the summation order in $\sum_{i=0}^{\infty}\sum_{j=0}^{2i}a_{ij}$.
13. Define by Iverson notation the following functions:
    - $[x]$ (integral part),
    - $|x|$ (module),
    - $\operatorname{sgn} x$ (signum),
    - $n!$ (factorial).
14. Define only by formulas the expression $[p \text{ is prime}]$.

1.4. Infinite Products

On the contents of the lecture. In this lecture we become acquainted with infinite products. The famous Euler Identity will be proved. We will find out that $\zeta(2)$ is another name for the Euler series. And we will see how Euler's decomposition of the sine function into a product works to sum up the Euler Series.

DEFINITION. *The product of an infinite sequence of numbers* $\{a_k\}$, *such that* $a_k \geq 1$ *for all* k, *is defined as the least number majorizing all partial products* $\prod_{k=1}^{n} a_k = a_1 a_2 \dots a_n$.

A sequence of natural numbers is called *essentially finite* if all but finitely many of its elements are equal to zero. Denote by $\mathbb{N}^\infty$ the set of all essentially finite sequences of natural numbers.

THEOREM 1.4.1. *For any given sequence of positive series* $\sum_{k=0}^{\infty} a_k^j$, $j = 1, 2, \dots$ *such that* $a_0^j = 1$ *for all* j *one has*

$$\prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^\infty} \prod_{j=1}^{\infty} a_{k_j}^j. \tag{1.4.1}$$

The summands on the right-hand side of (1.4.1) usually contain factors which are less than one. But each of the summands contains only finitely many factors different from 1. So the summands are in fact finite products.

PROOF. For a sequence $\{k_j\} \in \mathbb{N}^\infty$ define its *length* as maximal j for which $k_j \neq 0$ and its *maximum* as the value of its maximal term. The length of the zero sequence is defined as 0.

Consider a finite subset $S \subset \mathbb{N}^\infty$. Consider the partial sum

$$\sum_{\{k_j\} \in S} \prod_{k=1}^{\infty} a_{k_j}^j.$$

To estimate it, denote by L the maximal length of elements of S and denote by M the greatest of maxima of $\{k_j\} \in S$. In this case

$$\sum_{\{k_j\} \in S} \prod_{j=1}^{\infty} a_{k_j}^j = \sum_{\{k_j\} \in S} \prod_{j=1}^{L} a_{k_j}^j \leq \sum_{\{k_j\} \in \mathbb{N}_M^L} \prod_{j=1}^{L} a_{k_j}^j = \prod_{j=1}^{L} \sum_{k=0}^{M} a_k^j \leq \prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j,$$

where $\mathbb{N}_M^L$ denotes the set of all finite sequences $\{k_1, k_2, \dots, k_L\}$ of natural numbers such that $k_i \leq M$. By All-for-One this implies one of the required inequalities, namely, $\geq$.

To prove the opposite inequality, we prove that for any natural L one has

$$\prod_{j=1}^{L} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^L} \prod_{j=1}^{L} a_{k_j}^j, \tag{1.4.2}$$

where $\mathbb{N}^L$ denotes the set of all finite sequences $\{k_1, \dots k_L\}$ of natural numbers. The proof is by induction on L.

LEMMA 1.4.2. *For any families* $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$ *of nonnegative numbers, one has*

$$\sum_{i\in I} a_i \sum_{j\in J} b_j = \sum_{(i,j)\in I\times J} a_i b_j.$$

PROOF OF LEMMA 1.4.2. Since $I \times J = \bigsqcup_{i\in I}\{i\} \times J$ by the Sum Partition Theorem one gets:

$$\begin{aligned}\sum_{(i,j)\in I\times J} a_i b_j &= \sum_{i\in I} \sum_{(i,j)\in\{i\}\times J} a_i b_j \\ &= \sum_{i\in I}\sum_{j\in J} a_i b_j \\ &= \sum_{i\in I} a_i \sum_{j\in J} b_j \\ &= \sum_{j\in J} b_j \sum_{i\in I} a_i.\end{aligned}$$

□

Case $L = 2$ follows from Lemma 1.4.2, because $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. The induction step is done as follows

$$\begin{aligned}\prod_{j=1}^{L+1}\sum_{k=0}^{\infty} a_k^j &= \sum_{k=0}^{\infty} a_k^{L+1} \prod_{j=1}^{L}\sum_{k=0}^{\infty} a_k^j \\ &= \sum_{k\in\mathbb{N}} a_k^{L+1} \sum_{\{k_j\}\in\mathbb{N}^L} \prod_{j=1}^{L} a_{k_j}^j \\ &= \sum_{\{k_j\}\in\mathbb{N}^{L+1}} \prod_{j=1}^{L+1} a_{k_j}^j.\end{aligned}$$

The left-hand side of (1.4.2) is a partial product for the left-hand side of (1.4.1) and the right-hand side of (1.4.2) is a subsum of the right-hand side of (1.4.1). Consequently, all partial products of the right-hand side in (1.4.1) do not exceed its left-hand side. This proves the inequality $\leq$. □

Euler's Identity. Our next goal is to prove the *Euler Identity.*

$$\boxed{\sum_{k=1}^{\infty}\frac{1}{k^\alpha} = \prod_{p=1}^{\infty}\left(1-\frac{1}{p^\alpha}\right)^{-[p\text{ is prime}]}}$$

Here α is any rational (or even irrational) positive number.

The product on the right-hand side is called the *Euler Product.* The series on the left-hand side is called the *Dirichlet series.* Each factor of the Euler Product expands into the geometric series $\sum_{k=0}^{\infty}\frac{1}{p^{k\alpha}}$. By Theorem 1.4.1, the product of these geometric series is equal to the sum of products of the type $p_1^{-k_1\alpha}p_2^{-k_2\alpha}\dots p_n^{-k_n\alpha}$ $= N^{-\alpha}$. Here $\{p_i\}$ are different prime numbers, $\{k_i\}$ are positive natural numbers and $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$. But each product $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$ is a natural number, different products represent different numbers and any natural number has a unique representation of this sort. This is exactly what is called Principal Theorem of

Arithmetic. That is, the above decomposition of the Euler product expands in the Dirichlet series.

Convergence of the Dirichlet series.

THEOREM 1.4.3. *The Dirichlet series* $\sum_{n=1}^{\infty}\frac{1}{n^s}$ *converges if and only if* $s>1$.

PROOF. Consider a $\{2^k\}$ packing of the series. Then the n-th term of the packed series one estimates from above as

$$\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^s}\le\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{(2^n)^s}=2^n\frac{1}{2^{ns}}=2^{n-ns}=(2^{1-s})^n.$$

If $s>1$ then $2^{1-s}<1$ and the packed series is termwise majorized by a convergent geometric progression. Hence it converges. In the case of the Harmonic series $(s=1)$ the n-th term of its packing one estimates from below as

$$\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k}\ge\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{2^{n+1}}=2^n\frac{1}{2^{n+1}}=\frac{1}{2}.$$

That is why the harmonic series diverges. A Dirichlet series for $s<1$ termwise majorizes the Harmonic series and so diverges. □

The Riemann ζ-function. The function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

is called the *Riemann ζ-function*. It is of great importance in number theory.

The simplest application of Euler's Identity represents Euler's proof of the infinity of the set of primes. The divergence of the *harmonic series* $\sum_{k=1}^{\infty}\frac{1}{k}$ implies the Euler Product has to contain infinitely many factors to diverge.

Euler proved an essentially more exact result: the series of reciprocal primes diverges $\sum\frac{1}{p}=\infty$.

Summing via multiplication. Multiplication of series gives rise to a new approach to evaluating their sums. Consider the geometric series $\sum_{k=0}^{\infty}x^k$. Then

$$\left(\sum_{k=0}^{\infty}x^k\right)^2=\sum_{j,k\in\mathbb{N}^2}x^jx^k=\sum_{m=0}^{\infty}\sum_{j+k=m}x^jx^k=\sum_{m=0}^{\infty}(m+1)x^m.$$

As $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$ one gets $\sum_{k=0}^{\infty}(k+1)x^k=\frac{1}{(1-x)^2}$.

Sine-product. Now we are ready to understand how two formulas

$$\frac{\sin x}{x}=\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right),\qquad \sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \tag{1.4.3}$$

which appeared in the Legends, yield an evaluation of the Euler Series. Since at the moment we do not know how to multiply infinite sequences of numbers which are less than one, we invert the product in the first formula. We get

$$\frac{x}{\sin x}=\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)^{-1}=\prod_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{x^{2j}}{k^{2j}\pi^{2j}}. \tag{1.4.4}$$

To avoid negative numbers, we interpret the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

in the second formula of (1.4.3) as the difference

$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$

Substituting this expression for $\sin x$ in $\frac{x}{\sin x}$ and cancelling out x, we get

$$\frac{x}{\sin x} = \frac{1}{1 - \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}} = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}\right)^j.$$

All terms on the right-hand side starting with $j = 2$ are divisible by x^4. Consequently the only summand with x^2 on the right-hand side is $\frac{x^2}{6}$. On the other hand in (1.4.4) after an expansion into a sum by Theorem 1.4.1, the terms with x^2 give the series $\sum_{k=1}^{\infty}\frac{x^2}{k^2\pi^2}$. Comparing these results, one gets $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

Problems.

1. Prove $\prod_{n=1}^{\infty} 1.1 = \infty$.
2. Prove the identity $\prod_{n=1}^{\infty} a_n^2 = (\prod_{n=1}^{\infty} a_n)^2$ $(a_n \geq 1)$.
3. Does $\prod_{n=1}^{\infty}(1+\frac{1}{n})$ converge?
4. Evaluate $\prod_{n=2}^{\infty}\frac{n^2}{n^2-1}$.
5. Prove the divergence of $\prod_1^{\infty}(1+\frac{1}{k})^{[k \text{ is prime}]}$.
6. Evaluate $\prod_{n=3}^{\infty}\frac{n(n+1)}{(n-2)(n+3)}$.
7. Evaluate $\prod_{n=3}^{\infty}\frac{n^2-1}{n^2-4}$.
8. Evaluate $\prod_{n=1}^{\infty}(1+\frac{1}{n(n+2)})$.
9. Evaluate $\prod_{n=1}^{\infty}\frac{(2n+1)(2n+7)}{(2n+3)(2n+5)}$.
10. Evaluate $\prod_{n=2}^{\infty}\frac{n^3+1}{n^3-1}$.
11. Prove the inequality $\prod_{k=2}^{\infty}(1+\frac{1}{k^2}) \geq \sum_{k=2}^{\infty}\frac{1}{k^2}$.
12. Prove the convergence of the Wallis product $\prod\frac{4k^2}{4k^2-1}$.
13. Evaluate $\sum_{k=1}^{\infty}\frac{1}{k^4}$ by applying (1.4.3).
14. Prove $\prod_{n=2}^{\infty}\frac{n^2+1}{n^2} < \infty$.
15. Multiply a geometric series by itself and get a power series expansion for $(1-x)^{-2}$.
16. Define $\tau(n)$ as the number of divisors of n. Prove $\zeta^2(x) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^x}$.
17. Define $\phi(n)$ as the number of numbers which are less than n are relatively prime to n. Prove $\frac{\zeta(x-1)}{\zeta(x)} = \sum_{n=1}^{\infty}\frac{\phi(n)}{n^x}$.
18. Define $\mu(n)$ (Möbius function) as follows: $\mu(1) = 1$, $\mu(n) = 0$, if n is divisible by the square of a prime number, $\mu(n) = (-1)^k$, if n is the product of k different prime numbers. Prove $\frac{1}{\zeta(x)} = \sum_{k=1}^{\infty}\frac{\mu(n)}{n^x}$.
***19.** Prove $\sum_{k=1}^{\infty}\frac{[k \text{ is prime}]}{k} = \infty$.
***20.** Prove the identity $\prod_{n=0}^{\infty}(1+x^{2^n}) = \frac{1}{1-x}$.

1.5. Telescopic Sums

On the content of this lecture. In this lecture we learn the main secret of elementary summation theory. We will evaluate series via their partial sums. We introduce *factorial powers*, which are easy to sum. Following Stirling we expand $\frac{1}{1+x^2}$ into a series of negative factorial powers and apply this expansion to evaluate the Euler series with Stirling's accuracy of 10^{-8}.

The series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. In the first lecture we calculated infinite sums directly without invoking partial sums. Now we present a dual approach to summing series. According to this approach, at first one finds a formula for the n-th partial sum and then substitutes in this formula infinity instead of n. The series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ gives a simple example for this method. The key to sum it up is the following identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Because of this identity $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ turns into the sum of differences

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\dots. \tag{1.5.1}$$

Its n-th partial sum is equal to $1-\frac{1}{n+1}$. Substituting in this formula $n=+\infty$, one gets 1 as its ultimate sum.

Telescopic sums. The sum (1.5.1) represents a *telescopic sum*. This name is used for sums of the form $\sum_{k=0}^{n}(a_k - a_{k+1})$. The value of such a telescopic sum is determined by the values of the first and the last of a_k, similarly to a telescope, whose thickness is determined by the radii of the external and internal rings. Indeed,

$$\sum_{k=0}^{n}(a_k - a_{k+1}) = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} a_{k+1} = a_0 + \sum_{k=1}^{n} a_k - \sum_{k=0}^{n-1} a_{k+1} - a_{n+1} = a_0 - a_{n+1}.$$

The same arguments for infinite telescopic sums give

$$\sum_{k=0}^{\infty}(a_k - a_{k+1}) = a_0. \tag{1.5.2}$$

But this proof works only if $\sum_{k=0}^{\infty} a_k < \infty$. This is untrue for $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, owing to the divergence of the Harmonic series. But the equality (1.5.2) holds also if a_k tends to 0 as k tends to infinity. Indeed, in this case a_0 is the least number majorizing all $a_0 - a_n$, the n-th partial sums of $\sum_{k=0}^{\infty} a_k$.

Differences. For a given sequence $\{a_k\}$ one denotes by $\{\Delta a_k\}$ the sequence of differences $\Delta a_k = a_{k+1} - a_k$ and calls the latter sequence the *difference* of $\{a_k\}$. This is the main formula of elementary summation theory.

$$\boxed{\sum_{k=0}^{n-1} \Delta a_k = a_n - a_0}$$

To telescope a series $\sum_{k=0}^{\infty} a_k$ it is sufficient to find a sequence $\{A_k\}$ such that $\Delta A_k = a_k$. On the other hand the sequence of sums $A_n = \sum_{k=0}^{n-1} a_k$ has difference $\Delta A_n = a_n$. Therefore, we see that to telescope a sum is equivalent to find a formula for partial sums. This lead to concept of a *telescopic function.* For a function $f(x)$ we introduce its difference $\Delta f(x)$ as $f(x+1)-f(x)$. A function $f(x)$ telescopes $\sum a_k$ if $\Delta f(k)=a_k$ for all k.

Often the sequence $\{a_k\}$ that we would like to telescope has the form $a_k=f(k)$ for some function. Then we are searching for a *telescopic function* $F(x)$ for $f(x)$, i.e., a function such that $\Delta F(x)=f(x)$.

To evaluate the difference of a function is usually much easier than to telescope it. For this reason one has evaluated the differences of all basic functions and organized a *table of differences.* In order to telescope a given function, look in this table to find a table function whose difference coincides with or is close to given function.

For example, the differences of x^n for $n\leq 3$ are $\Delta x=1$, $\Delta x^2=2x+1$, $\Delta x^3=3x^2+3x+1$. To telescope $\sum_{k=1}^{\infty}k^2$ we choose in this table x^3. Then $\frac{\Delta x^3}{3}-x^2=x+\frac{1}{3}=\frac{\Delta x^2}{2}-\Delta\frac{x}{6}$. Therefore, $x^2=\Delta\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}\right)$. This immediately implies the following formula for sums of squares:

$$\sum_{k=1}^{n-1}k^2=\frac{2n^3-3n^2+n}{6}. \tag{1.5.3}$$

Factorial powers. The usual powers x^n have complicated differences. The so-called *factorial powers* $x^{\underline{k}}$ have simpler differences. For any number x and any natural number k, let $x^{\underline{k}}$ denote $x(x-1)(x-2)\dots(x-k+1)$, and by $x^{\underline{-k}}$ we denote $\frac{1}{(x+1)(x+2)\dots(x+k)}$. At last we define $x^{\underline{0}}=1$. The factorial power satisfies the following *addition law.*

$$\boxed{x^{\underline{k+m}}=x^{\underline{k}}(x-k)^{\underline{m}}}$$

We leave to the reader to check this rule for all integers m, k. The power $n^{\underline{n}}$ for a natural n coincides with the factorial $n!=1\cdot 2\cdot 3\cdots n$. The main property of factorial powers is given by:

$$\boxed{\Delta x^{\underline{n}}=nx^{\underline{n-1}}}$$

The proof is straightforward:

$$\begin{aligned}(x+1)^{\underline{k}}-x^{\underline{k}}&=(x+1)^{\underline{1+(k-1)}}-x^{\underline{(k-1)+1}}\\&=(x+1)x^{\underline{k-1}}-x^{\underline{k-1}}(x-k+1)\\&=kx^{\underline{k-1}}.\end{aligned}$$

Applying this formula one can easily telescope any *factorial polynomial*, i.e., an expression of the form

$$a_0+a_1x^{\underline{1}}+a_2x^{\underline{2}}+a_3x^{\underline{3}}+\cdots+a_nx^{\underline{n}}.$$

Indeed, the explicit formula for the telescoping function is

$$a_0x^{\underline{1}}+\tfrac{a_1}{2}x^{\underline{2}}+\tfrac{a_2}{3}x^{\underline{3}}+\tfrac{a_3}{4}x^{\underline{4}}+\cdots+\tfrac{a_n}{n+1}x^{\underline{n+1}}.$$

Therefore, another strategy to telescope x^k is to represent it as a factorial polynomial.

For example, to represent x^2 as factorial polynomial, consider $a+bx+cx^{\underline{2}}$, a general factorial polynomial of degree 2. We are looking for $x^2=a+bx+cx^{\underline{2}}$. Substituting $x=0$ in this equality one gets $a=0$. Substituting $x=1$, one gets

$1 = b$, and finally for $x = 2$ one has $4 = 2 + 2c$. Hence $c = 1$. As result $x^2 = x + x^{\underline{2}}$. And the telescoping function is given by

$$\tfrac{1}{2}x^{\underline{2}} + \tfrac{1}{3}x^{\underline{3}} = \tfrac{1}{2}(x^2 - x) + \tfrac{1}{3}(x(x^2 - 3x + 2)) = \tfrac{1}{6}(2x^3 - 3x^2 + x).$$

And we have once again proved the formula (1.5.3).

Stirling Estimation of the Euler series. We will expand $\frac{1}{(1+x)^2}$ into a series of negative factorial powers in order to telescope it. A natural first approximation to $\frac{1}{(1+x^2)}$ is $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$. We represent $\frac{1}{(1+x)^2}$ as $x^{\underline{-2}} + R_1(x)$, where

$$R_1(x) = \frac{1}{(1+x)^2} - x^{\underline{-2}} = \frac{1}{(x+1)^2(x+2)}.$$

The remainder $R_1(x)$ is in a natural way approximated by $x^{\underline{-3}}$. If $R_1(x) = x^{\underline{-3}} + R_2(x)$ then $R_2(x) = \frac{2}{(x+1)^2(x+2)(x+3)}$. Further, $R_2(x) = 2x^{\underline{-4}} + R_3(x)$, where

$$R_3(x) = \frac{2 \cdot 3}{(x+1)^2(x+2)(x+3)(x+4)} = \frac{3!}{x+1}x^{\underline{-4}}.$$

The above calculations lead to the conjecture

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{n-1} k!x^{\underline{-k-2}} + \frac{n!}{x+1}x^{\underline{-n-1}}. \tag{1.5.4}$$

This conjecture is easily proved by induction. The remainder $R_n(x) = \frac{n!}{x+1}x^{\underline{-n-1}}$ represents the difference $\frac{1}{(1+x)^2} - \sum_{k=0}^{n-1} k!x^{\underline{-2-k}}$. Owing to the inequality $x^{\underline{-1-n}} \le \frac{1}{(n+1)!}$, which is valid for all $x \ge 0$, the remainder decreases to 0 as n increases to infinity. This implies

THEOREM 1.5.1. *For all $x \ge 0$ one has*

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} k!x^{\underline{-2-k}}.$$

To calculate $\sum_{k=p}^{\infty} \frac{1}{(1+k)^2}$, replace all summands by the expressions (1.5.4). We will get

$$\sum_{k=p}^{\infty}\left(\sum_{m=0}^{n-1} m!k^{\underline{-2-m}} + \frac{n!}{k+1}k^{\underline{-1-n}}\right).$$

Changing the order of summation we have

$$\sum_{m=0}^{n-1} m! \sum_{k=p}^{\infty} k^{\underline{-2-m}} + \sum_{k=p}^{\infty} \frac{n!}{k+1}k^{\underline{-1-n}}.$$

Since $\frac{1}{1+m}x^{\underline{-1-m}}$ telescopes the sequence $\{k^{\underline{-2-m}}\}$, $\sum_{k=p}^{\infty} k^{\underline{-2-m}} = \frac{1}{1+m}p^{\underline{-1-m}}$, Denote the sum of remainders $\sum_{k=p}^{\infty} \frac{n!}{k+1}k^{\underline{-1-n}}$ by $R(n,p)$. Then for all natural p and n one has

$$\sum_{k=p}^{\infty} \frac{1}{(1+k)^2} = \sum_{m=0}^{n-1} \frac{m!}{1+m}p^{\underline{-1-m}} + R(n,p)$$

For $p = 0$ and $n = +\infty$, the right-hand side turns into the Euler series, and one could get a false impression that we get nothing new. But $k^{\underline{-2-n}} \leq \frac{1}{k+1} k^{\underline{-1-n}} \leq (k-1)^{\underline{-2-n}}$, hence

$$\frac{n!}{1+n} p^{\underline{-1-n}} = \sum_{k=p}^{\infty} n! k^{\underline{-2-n}} \leq R(n,p) \leq \sum_{k=p}^{\infty} n!(k-1)^{\underline{-2-n}} = \frac{n!}{1+n}(p-1)^{\underline{-1-n}}.$$

Since $(p-1)^{\underline{-1-n}} - p^{\underline{-1-n}} = (1+n)(p-1)^{\underline{-2-n}}$, there is a $\theta \in (0,1)$ such that

$$R(n,p) = \frac{n!}{1+n} p^{\underline{-1-n}} + \theta n!(p-1)^{\underline{-2-n}}.$$

Finally we get:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{p-1} \frac{1}{(1+k)^2} + \sum_{k=0}^{n-1} \frac{k!}{1+k} p^{\underline{-1-k}} + \theta n!(p-1)^{\underline{-2-n}}.$$

For $p = n = 3$ this formula turns into

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{40} + \frac{1}{180} + \frac{\theta}{420}.$$

For $p = n = 10$ one gets $R(10,10) \leq 10!9^{\underline{-12}}$. After cancellations one has $\frac{1}{2\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 17\cdot 19}$. This is approximately $2 \cdot 10^{-8}$. Therefore

$$\sum_{k=0}^{10-1} \frac{1}{(k+1)^2} + \sum_{k=0}^{10-1} \frac{k!}{1+k} 10^{\underline{-1-k}}$$

is less than the sum of the Euler series by only $2 \cdot 10^{-8}$. In such a way one can in one hour calculate eight digits of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ after the decimal point. It is not a bad result, but it is still far from Euler's eighteen digits. For $p = 10$, to provide eighteen digits one has to sum essentially more than one hundred terms of the series. This is a bit too much for a person, but is possible for a computer.

Problems.

1. Telescope $\sum k^3$.
2. Represent x^4 as a factorial polynomial.
3. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.
4. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}$.
5. Prove: If $\Delta a_k \geq \Delta b_k$ for all k and $a_1 \geq b_1$ then $a_k \geq b_k$ for all k.
6. $\Delta(x+a)^{\underline{n}} = n(x+a)^{\underline{n-1}}$.
7. Prove Archimedes's inequality $\frac{n^3}{3} \leq \sum_{k=1}^{n-1} k^2 \leq \frac{(n+1)^3}{3}$.
8. Telescope $\sum_{k=1}^{\infty} \frac{k}{2^k}$.
9. Prove the inequalities $\frac{1}{n} \geq \sum_{k=n+1}^{\infty} \frac{1}{k^2} \geq \frac{1}{n+1}$.
10. Prove that the degree of $\Delta P(x)$ is less than the degree of $P(x)$ for any polynomial $P(x)$.
11. Relying on $\Delta 2^n = 2^n$, prove that $P(n) < 2^n$ eventually for any polynomial $P(x)$.
12. Prove $\sum_{k=0}^{\infty} k!(x-1)^{\underline{-1-k}} = \frac{1}{x}$.

1.6. Complex Series

On the contents of the lecture. Complex numbers hide the key to the Euler Series. The summation theory developed for positive series now extends to complex series. We will see that complex series can help to sum real series.

Cubic equation. Complex numbers arise in connection with the solution of the cubic equation. The substitution $x = y - \frac{a}{3}$ reduces the general cubic equation $x^3 + ax^2 + bx + c = 0$ to

$$y^3 + py + q = 0.$$

The reduced equation one solves by the following trick. One looks for a root in the form $y = \alpha+\beta$. Then $(\alpha+\beta)^3+p(\alpha+\beta)+q = 0$ or $\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)+p(\alpha+\beta)+q = 0$. The latter equality one reduces to the system

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 3\alpha\beta &= -p. \end{aligned} \tag{1.6.1}$$

Raising the second equation into a cube one gets

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 27\alpha^3\beta^3 &= -p^3. \end{aligned}$$

Now α^3, β^3 are roots of the quadratic equation

$$x^2 + qx - \tfrac{p^3}{27},$$

called the *resolution* of the original cubic equation. Sometimes the resolution has no roots, while the cubic equation always has a root. Nevertheless one can evaluate a root of the cubic equation with the help of its resolution. To do this one simply ignores that the numbers under the square roots are negative.

For example consider the following cubic equation

$$x^3 - \tfrac{3}{2}x - \tfrac{1}{2} = 0. \tag{1.6.2}$$

Then (1.6.1) turns into

$$\begin{aligned} \alpha^3 + \beta^3 &= \tfrac{1}{2}, \\ \alpha^3\beta^3 &= \tfrac{1}{8}, \end{aligned}$$

The corresponding resolution is $t^2 - \frac{t}{2} + \frac{1}{8} = 0$ and its roots are

$$t_{1,2} = \tfrac{1}{4} \pm \sqrt{\tfrac{1}{16} - \tfrac{1}{8}} = \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{-1}.$$

Then the desired root of the cubic equation is given by

$$\sqrt[3]{\tfrac{1}{4}(1+\sqrt{-1})} + \sqrt[3]{\tfrac{1}{4}(1-\sqrt{-1})} = \tfrac{1}{\sqrt[3]{4}}\left(\sqrt[3]{1+\sqrt{-1}} + \sqrt[3]{1-\sqrt{-1}}\right). \tag{1.6.3}$$

It turns out that the latter expression one uniquely interprets as a real number which is a root of the equation (1.6.2). To evaluate it consider the following expression

$$\sqrt[3]{(1+\sqrt{-1})^2} - \sqrt[3]{(1+\sqrt{-1})}\sqrt[3]{(1-\sqrt{-1})} + \sqrt[3]{(1-\sqrt{-1})^2}. \tag{1.6.4}$$

Since

$$(1+\sqrt{-1})^2 = 1^2 + 2\sqrt{-1} + \sqrt{-1}^2 = 1 + 2\sqrt{-1} - 1 = 2\sqrt{-1},$$

the left summand of (1.6.4) is equal to

$$\sqrt[3]{2\sqrt{-1}} = \sqrt[3]{2}\sqrt[3]{\sqrt{-1}} = \sqrt[3]{2}\sqrt{\sqrt[3]{-1}} = \sqrt[3]{2}\,\sqrt{-1}.$$

Similarly $(1-\sqrt{-1})^2 = -2\sqrt{-1}$, and the right summand of (1.6.4) turns into $-\sqrt[3]{2}\sqrt{-1}$. Finally $(1+\sqrt{-1})(1-\sqrt{-1}) = 1^2 - \sqrt{-1}^2 = 2$ and the central one is $-\sqrt[3]{2}$. As a result the whole expression (1.6.4) is evaluated as $-\sqrt[3]{2}$.

On the other hand one evaluates the product of (1.6.3) and (1.6.4) by the usual formula as the sum of cubes

$$\tfrac{1}{\sqrt[3]{4}}((1+\sqrt{-1})+(1-\sqrt{-1})) = \tfrac{1}{\sqrt[3]{4}}((1+1)+(\sqrt{-1})-\sqrt{-1})) = \tfrac{1}{\sqrt[3]{4}}(2+0) = \sqrt[3]{2}.$$

Consequently (1.6.3) is equal to $\frac{\sqrt[3]{2}}{-\sqrt[3]{2}} = -1$. And -1 is a true root of (1.6.2).

Arithmetic of complex numbers. In the sequel we use i instead of $\sqrt{-1}$. There are two basic ways to represent a complex number. The representation $z = a + ib$, where a and b are real numbers we call the *Cartesian form* of z. The numbers a and b are called respectively the *real* and the *imaginary* parts of z and are denoted by $\operatorname{Re} z$ and by $\operatorname{Im} z$ respectively. Addition and multiplication of complex numbers are defined via their real and imaginary parts as follows

$$\begin{aligned}
\operatorname{Re}(z_1+z_2) &= \operatorname{Re} z_1 + \operatorname{Re} z_2,\\
\operatorname{Im}(z_1+z_2) &= \operatorname{Im} z_1 + \operatorname{Im} z_2,\\
\operatorname{Re}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2,\\
\operatorname{Im}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Im} z_1 \operatorname{Re} z_2.
\end{aligned}$$

The *trigonometric form* of a complex number is $z = \rho(\cos\phi + i\sin\phi)$, where $\rho \geq 0$ is called the *module* or the *absolute value* of a complex number z and is denoted $|z|$, and ϕ is called its *argument*. The argument of a complex number is defined modulo 2π. We denote by $\operatorname{Arg} z$ the set of all arguments of z, and by $\arg z$ the element of $\operatorname{Arg} z$ which satisfies the inequalities $-\pi < \arg z \leq \pi$. So $\arg z$ is uniquely defined for all complex numbers. $\arg z$ is called the *principal argument* of z.

The number $a - bi$ is called the *conjugate* to $z = a + bi$ and denoted $\overline{z}$. One has $z\overline{z} = |z|^2$. This allows us to express z^{-1} as $\frac{\overline{z}}{|z|^2}$.

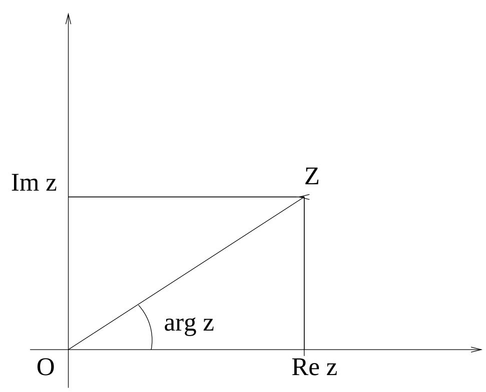

FIGURE 1.6.1. The representation of a complex number

If $z = a+ib$ then $|z| = \sqrt{a^2+b^2}$ and $\arg z = \operatorname{arctg}\frac{b}{a}$. One represents a complex number $z = a+bi$ as a point Z of the plane with coordinates (a,b). Then $|z|$ is equal to the distance from Z to the origin O. And $\arg z$ represents the angle between the axis of abscises and the ray $\overrightarrow{OZ}$. Addition of complex numbers corresponds to usual vector addition. And the usual triangle inequality turns into the *module inequality*:

$$|z+\zeta| \leq |z| + |\zeta|.$$

The multiplication formula for complex numbers in the trigonometric form is especially simple:

$$\begin{aligned} r(\cos\phi + i\sin\phi)r'(\cos\psi + i\sin\psi) \\ = rr'(\cos(\phi+\psi) + i\sin(\phi+\psi)). \end{aligned} \tag{1.6.5}$$

Indeed, the left-hand side and the right-hand side of (1.6.5) transform to

$$rr'(\cos\phi\cos\psi - \sin\phi\sin\psi) + irr'(\sin\phi\cos\psi + \sin\psi\cos\phi).$$

That is, the module of the product is equal to the product of modules and the argument of product is equal to the sum of arguments:

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 \oplus \operatorname{Arg} z_2.$$

Any complex number is uniquely defined by its module and argument.

The multiplication formula allows us to prove by induction the following:

$$(\cos\phi + i\sin\phi)^n = (\cos n\phi + i\sin n\phi). \tag{Moivre Formula}$$

Sum of a complex series. Now is the time to extend our summation theory to series made of complex numbers. We extend the whole theory without any losses to so-called absolutely convergent series. The series $\sum_{k=1}^{\infty} z_k$ with arbitrary complex terms is called *absolutely convergent*, if the series $\sum_{k=1}^{\infty} |z_k|$ of absolute values converges.

For any real number x one defines two nonnegative numbers: its *positive x^+ and negative x^- parts* as $x^+ = x[x \geq 0]$ and $x^- = -x[x < 0]$. The following identities characterize the positive and negative parts of x

$$x^+ + x^- = |x|, \qquad\qquad x^+ - x^- = x.$$

Now the sum of an absolutely convergent series of real numbers is defined as follows:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-. \tag{1.6.6}$$

That is, from the sum of all positive summands one subtracts the sum of modules of all negative summands. The two series on the right-hand side converge, because $a_k^+ \leq |a_k|$, $a_k^- \leq |a_k|$ and $\sum_{k=1}^{\infty} |a_k| < \infty$.

For an absolutely convergent complex series $\sum_{k=1}^{\infty} z_k$ we define the real and imaginary parts of its sum separately by the formulas

$$\operatorname{Re}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Re} z_k, \qquad\qquad \operatorname{Im}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Im} z_k. \tag{1.6.7}$$

The series in the right-hand sides of these formulas are absolutely convergent, since $|\operatorname{Re} z_k| \leq |z_k|$ and $|\operatorname{Im} z_k| \leq |z_k|$.

THEOREM 1.6.1. *For any pair of absolutely convergent series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *its termwise sum* $\sum_{k=1}^{\infty}(a_k+b_k)$ *absolutely converges and*

$$\sum_{k=1}^{\infty}(a_k+b_k)=\sum_{k=1}^{\infty} a_k+\sum_{k=1}^{\infty} b_k. \tag{1.6.8}$$

PROOF. First, remark that the absolute convergence of the series on the left-hand side follows from the Module Inequality $|a_k+b_k| \le |a_k|+|b_k|$ and the absolute convergence of the series on the right-hand side.

Now consider the case of real numbers. Representing all sums in (1.6.8) as differences of their positive and negative parts and separating positive and negative terms in different sides one transforms (1.6.8) into

$$\sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} b_k^+ + \sum_{k=1}^{\infty}(a_k+b_k)^- = \sum_{k=1}^{\infty} a_k^- + \sum_{k=1}^{\infty} b_k^- + \sum_{k=1}^{\infty}(a_k+b_k)^+.$$

But this equality is true due to termwise addition for positive series and the following identity,

$$x^- + y^- + (x+y)^+ = x^+ + y^+ + (x+y)^-.$$

Moving terms around turns this identity into

$$(x+y)^+ - (x+y)^- = (x^+ - x^-) + (y^+ - y^-),$$

which is true due to the identity $x^+ - +x^- = x$.

In the complex case the equality (1.6.8) splits into two equalities, one for real parts and another for imaginary parts. As for real series the termwise addition is already proved, we can write the following chain of equalities,

$$\begin{aligned}
\operatorname{Re}\left(\textstyle\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k\right) &= \operatorname{Re}\textstyle\sum_{k=1}^{\infty} a_k + \operatorname{Re}\sum_{k=1}^{\infty} b_k\\
&= \textstyle\sum_{k=1}^{\infty} \operatorname{Re} a_k + \sum_{k=1}^{\infty} \operatorname{Re} b_k\\
&= \textstyle\sum_{k=1}^{\infty}(\operatorname{Re} a_k + \operatorname{Re} b_k)\\
&= \textstyle\sum_{k=1}^{\infty} \operatorname{Re}(a_k+b_k)\\
&= \operatorname{Re}\textstyle\sum_{k=1}^{\infty}(a_k+b_k),
\end{aligned}$$

which proves the equality of real parts in (1.6.8). The same proof works for the imaginary parts. □

Sum Partition Theorem. An unordered sum of a family of complex numbers is defined by the same formulas (1.6.6) and (1.6.7). Since for positive series non-ordered sums coincide with the ordered sums, we get the same coincidence for all absolutely convergent series. Hence the commutativity law holds for all absolutely convergence series.

THEOREM 1.6.2. *If* $I = \bigsqcup_{j\in J} I_j$ *and* $\sum_{k=1}^{\infty} |a_k| < \infty$ *then* $\sum_{j\in J}\left|\sum_{i\in I_j} a_i\right| < \infty$ *and* $\sum_{j\in J}\sum_{i\in I_j} a_i = \sum_{i\in I} a_i$.

PROOF. At first consider the case of real summands. By definition $\sum_{i\in I} a_i = \sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^-$. By Sum Partition Theorem positive series one transforms the original sum into

$$\textstyle\sum_{j\in J}\sum_{i\in I_j} a_i^+ - \sum_{j\in J}\sum_{i\in I_j} a_i^-.$$

Now by the Termwise Addition applied at first to external and after to internal sums one gets

$$\sum_{j\in J}\left(\sum_{i\in I_j} a_i^+ - \sum_{i\in I_j} a_i^-\right) = \sum_{j\in J}\sum_{i\in I_j}(a_i^+ - a_i^-) = \sum_{j\in J}\sum_{i\in I_j} a_i.$$

So the Sum Partition Theorem is proved for all absolutely convergent real series. And it immediately extends to absolutely convergent complex series by its splitting into real and imaginary parts. □

THEOREM 1.6.3 (Termwise Multiplication). *If $\sum_{k=1}^\infty |z_k| < \infty$ then for any (complex) c, $\sum_{k=1}^\infty |cz_k| < \infty$ and $\sum_{k=1}^\infty cz_k = c\sum_{k=1}^\infty z_k$.*

PROOF. Termwise Multiplication for positive numbers gives the first statement of the theorem $\sum_{k=1}^\infty |cz_k| = \sum_{k=1}^\infty |c||z_k| = |c|\sum_{k=1}^\infty |z_k|$. The further proof is divided into five cases.

At first suppose c is positive and z_k real. Then $cz_k^+ = cz_k^+$ and by virtue of Termwise Multiplication for positive series we get

$$\begin{aligned}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty cz_k^+ - \sum_{k=1}^\infty cz_k^- \\ &= c\sum_{k=1}^\infty z_k^+ - c\sum_{k=1}^\infty z_k^- \\ &= c\left(\sum_{k=1}^\infty z_k^+ - \sum_{k=1}^\infty z_k^-\right) \\ &= c\sum_{k=1}^\infty z_k.\end{aligned}$$

The second case. Let $c = -1$ and z_k be real. In this case

$$\sum_{k=1}^\infty -z_k = \sum_{k=1}^\infty(-z_k)^+ - \sum_{k=1}^\infty(-z_k)^- = \sum_{k=1}^\infty z_k^- - \sum_{k=1}^\infty z_k^+ = -\sum_{k=1}^\infty z_k.$$

The third case. Let c be real and z_k complex. In this case $\operatorname{Re} cz_k = c\operatorname{Re} z_k$ and the two cases above imply the Termwise Multiplication for any real c. Hence

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty \operatorname{Re} cz_k \\ &= \sum_{k=1}^\infty c\operatorname{Re} z_k \\ &= c\sum_{k=1}^\infty \operatorname{Re} z_k \\ &= c\operatorname{Re}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} c\sum_{k=1}^\infty z_k.\end{aligned}$$

The same is true for imaginary parts.

The fourth case. Let $c = i$ and z_k be complex. Then $\operatorname{Re} iz_k = -\operatorname{Im} z_k$ and $\operatorname{Im} iz_k = \operatorname{Re} z_k$. So one gets for real parts

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty iz_k &= \sum_{k=1}^\infty \operatorname{Re}(iz_k) \\ &= \sum_{k=1}^\infty -\operatorname{Im} z_k \\ &= -\sum_{k=1}^\infty \operatorname{Im} z_k \\ &= -\operatorname{Im}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} i\sum_{k=1}^\infty z_k.\end{aligned}$$

The general case. Let $c = a + bi$ with real a, b. Then

$$\begin{aligned} c\sum_{k=1}^{\infty} z_k &= a\sum_{k=1}^{\infty} z_k + ib\sum_{k=1}^{\infty} z_k \\ &= \sum_{k=1}^{\infty} az_k + \sum_{k=1}^{\infty} ibz_k \\ &= \sum_{k=1}^{\infty}(az_k + ibz_k) \\ &= \sum_{k=1}^{\infty} cz_k. \end{aligned}$$

□

Multiplication of Series. For two given series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, one defines their *convolution* as a series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

THEOREM 1.6.4 (Cauchy). *For any pair of absolutely convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ their convolution $\sum_{k=0}^{\infty} c_k$ absolutely converges and*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k.$$

PROOF. Consider the double series $\sum_{i,j} a_i b_j$. Then by the Sum Partition Theorem its sum is equal to

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_i b_j\right) = \sum_{j=0}^{\infty} b_j \left(\sum_{i=0}^{\infty} a_i\right) = \left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right).$$

On the other hand, $\sum_{i,j} a_i b_j = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1-1} a_k b_{n-k}$. But the last sum is just the convolution.

This proof goes through for positive series. In the generalcase we have to prove absolute convergence of the double series. But this follows from

$$\left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{k=0}^{\infty} |b_k|\right) = \sum_{k=0}^{\infty} |c_k|.$$

□

Module Inequality.

$$\left|\sum_{k=1}^{\infty} z_k\right| \leq \sum_{k=1}^{\infty} |z_k|. \tag{1.6.9}$$

Let $z_k = x_k + iy_k$. Summation of the inequalities $-|x_k| \leq x_k \leq |x_k|$ gives $-\sum_{k=1}^{\infty} |x_k| \leq \sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} |x_k|$, which means $|\sum_{k=1}^{\infty} x_k| \leq \sum_{k=1}^{\infty} |x_k|$. The same inequality is true for y_k. Consider $z'_k = |x_k| + i|y_k|$. Then $|z_k| = |z'_k|$ and $|\sum_{k=1}^{\infty} z_k| \leq |\sum_{k=1}^{\infty} z'_k|$. Therefore it is sufficient to prove the inequality (1.6.9) for z'_k, that is, for numbers with non-negative real and imaginary parts. Now supposing x_k, y_k to be nonnegative one gets the following chain of equivalent transformations of (1.6.9):

$$\begin{aligned} \left(\sum_{k=1}^{\infty} x_k\right)^2 + \left(\sum_{k=1}^{\infty} y_k\right)^2 &\leq \left(\sum_{k=1}^{\infty} |z_k|\right)^2 \\ \sum_{k=1}^{\infty} x_k &\leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{\infty} y_k\right)^2} \\ \sum_{k=1}^{n} x_k &\leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{\infty} y_k\right)^2}, \quad \forall n = 1, 2, \dots \\ \sum_{k=1}^{\infty} y_k &\leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\operatorname{Re}\sum_{k=1}^{n} x_k\right)^2}, \quad \forall n = 1, 2, \dots \\ \sum_{k=1}^{m} y_k &\leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{n} x_k\right)^2}, \quad \forall n, m = 1, 2, \dots \\ \left(\sum_{k=1}^{n} x_k\right)^2 + \left(\sum_{k=1}^{m} y_k\right)^2 &\leq \left(\sum_{k=1}^{\infty} |z_k|\right)^2, \quad \forall m, n = 1, 2, \dots \end{aligned}$$

$$\sqrt{\left(\sum_{k=1}^N x_k\right)^2 + \left(\sum_{k=1}^N y_k\right)^2} \le \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$
$$\left|\sum_{k=1}^N z_k\right| \le \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$

The inequalities of the last system hold because $\left|\sum_{k=1}^N z_k\right| \le \sum_{k=1}^N |z_k| \le \sum_{k=1}^\infty |z_k|$.

Complex geometric progressions. The sum of a geometric progression with a complex ratio is given by the same formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}. \tag{1.6.10}$$

And the proof is the same as in the case of real numbers. But the meaning of this formula is different. Any complex formula is in fact a pair of formulas. Any complex equation is in fact a pair of equations.

In particular, for $z = q(\sin\phi + i\cos\phi)$ the real part of the left-hand side of (1.6.10) owing to the Moivre Formula turns into $\sum_{k=0}^{n-1} q^k \sin k\phi$ and the right-hand side turns into $\sum_{k=0}^{n-1} q^k \cos k\phi$. So the formula for a geometric progression splits into two formulas which allow us to telescope some trigonometric series.

Especially interesting is the case with the ratio $\varepsilon_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. In this case the geometric progression cyclically takes the same values, because $\varepsilon_n^n = 1$. The terms of this sequence are called the *roots of unity*, because they satisfy the equation $z^n - 1 = 0$.

LEMMA 1.6.5. $(z^n - 1) = \prod_{k=1}^n (z - \varepsilon_n^k)$.

PROOF. Denote by $P(z)$ the right-hand side product. This polynomial has degree n, has major coefficient 1 and has all ε_n^k as its roots. Then the difference $(z^n - 1) - P(z)$ is a polynomial of degree $< n$ which has n different roots. Such a polynomial has to be 0 by virtue of the following general theorem. □

THEOREM 1.6.6. *The number of roots of any nonzero complex polynomial does not exceed its degree.*

PROOF. The proof is by induction on the degree of $P(z)$. A polynomial of degree 1 has the form $az+b$ and the only root is $-\frac{b}{a}$. Suppose our theorem is proved for any polynomial of degree $< n$. Consider a polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ of degree n, where the coefficients are complex numbers. Suppose it has at least n roots $z_1, \dots, z_n$. Consider the polynomial $P^*(z) = a_n \prod_{k=1}^n (z - z_k)$. The difference $P(z) - P^*(z)$ has degree $< n$ and has at least n roots (all z_k). By the induction hypothesis this difference is zero. Hence, $P(z) = P^*(z)$. But $P^*(z)$ has only n roots. Indeed, for any z different from all z_k one has $|z - z_k| > 0$. Therefore $|P^*(z)| = |a_n| \prod_{k=1}^n |z - z_k| > 0$. □

By blocking conjugated roots one gets a pure real formula:

$$z^n - 1 = (z-1) \prod_{k=1}^{(n-1)/2} \left(z^2 - 2z\cos\frac{2k\pi}{n} + 1\right).$$

Complexification of series. Complex numbers are effectively applied to sum up so-called *trigonometric series*, i.e., series of the type $\sum_{k=0}^{\infty} a_k \cos kx$ and $\sum_{k=0}^{\infty} a_k \sin kx$. For example, to sum the series $\sum_{k=1}^{\infty} q^k \sin k\phi$ one couples it with its dual $\sum_{k=0}^{\infty} q^k \cos k\phi$ to form a complex series $\sum_{k=0}^{\infty} q^k(\cos k\phi + i \sin k\phi)$. The last is a complex geometric series. Its sum is $\frac{1}{1-z}$, where $z = \cos\phi + i \sin\phi$. Now the sum of the sine series $\sum_{k=1}^{\infty} q^k \sin k\phi$ is equal to $\operatorname{Im} \frac{1}{1-z}$, the imaginary part of the complex series, and the real part of the complex series coincides with the cosine series. In particular, for $q = 1$, one has $\frac{1}{1-z} = \frac{1}{1+\cos\phi + i \sin\phi}$. To evaluate the real and imaginary parts one multiplies both numerator and denominator by $1 + \cos\phi - i \sin\phi$. Then one gets $(1 - \cos\phi)^2 + \sin^2\phi = 1 - 2\cos^2\phi + \cos^2\phi + \sin^2\phi = 2 - 2\cos\phi$ as the denominator. Hence $\frac{1}{1-z} = \frac{1-\cos\phi+i\sin\phi}{2-2\cos\phi} = \frac{1}{2} + \frac{1}{2}\cot\frac{\phi}{2}$. And we get two remarkable formulas for the sum of the divergent series

$$\sum_{k=0}^{\infty} \cos k\phi = \frac{1}{2}, \qquad \sum_{k=1}^{\infty} \sin k\phi = \frac{1}{2}\cot\frac{\phi}{2}.$$

For $\phi = 0$ the left series turns into $\sum_{k=0}^{\infty}(-1)^k$. The evaluation of the Euler series via this cosine series is remarkably short, it takes one line. But one has to know integrals and a something else to justify this evaluation.

Problems.

1. Find real and imaginary parts for $\frac{1}{1-i}$, $(\frac{1-i}{1+i})^3$, $\frac{i^5+2}{i^{19}+1}$, $\frac{(1+i)^5}{(1-i)^3}$.
2. Find trigonometric form for -1, $1+i$, $\sqrt{3}+i$.
3. Prove that $z_1 z_2 = 0$ implies either $z_1 = 0$ or $z_2 = 0$.
4. Prove the distributivity law for complex numbers.
5. Analytically prove the inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.
6. Evaluate $\sum_{k=1}^{n-1} \frac{1}{z_k(z_k+1)}$, where $z_k = 1 + kz$.
7. Evaluate $\sum_{k=1}^{n-1} z_k^2$, where $z_k = 1 + kz$.
8. Evaluate $\sum_{k=1}^{n-1} \dfrac{\sin k}{2^k}$.
9. Solve $z^2 = i$.
10. Solve $z^2 = 3 - 4i$.
11. Telescope $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
12. Prove that the conjugated to a root of polynomial with real coefficient is the root of the polynomial.
13. Prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
14. Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
***15.** Solve $8x^3 - 6x - 1 = 0$.
16. Evaluate $\sum_{k=1}^{\infty} \dfrac{\sin k}{2^k}$.
17. Evaluate $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
18. Prove absolute convergence of $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ for any z.
19. For which z the series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ absolutely converges?
20. Multiply a geometric series onto itself several times applying Cauchy formula.
21. Find series for $\sqrt{1+x}$ by method of indefinite coefficients.
22. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ absolutely converge?
23. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ absolutely converge?