Complements to the textbook ”Elementary Analysis” by Kenneth A.Ross

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Preface

The author of any textbook makes choices. In the textbook by Kenneth A.Ross, which we use in this course (MAT 319: Foundations of Analysis), the choices made by the author are quite traditional. Some of them seems old-fashioned. It is natural, because the book was written about 35 years ago. The goal of these notes is to make remarks which complement the textbook.

Analysis of what? The main meaning of the word *analysis*, the word, which gives the name to our subject, is an investigation of the component parts of a whole and their relations in making up the whole and also the abstract separation of a whole into its constituent parts in order to study the parts and their relations. Its antonym is *synthesis* which means a combination of parts into a complex whole.

In the textbook I could not find an answer to the question of what is analyzed. The answer is well-known from other sources: here analysis stays for analysis of functions. Functions are analyzed. More specifically, this initial, most elementary, part of Analysis analyzes real valued functions of a single real variable. A function of this type is a map defined on a subset S of the set $\mathbb{R}$ of all real numbers and taking values in $\mathbb{R}$.

The stage. All the actions take place on the real line, that is the set $\mathbb{R}$ of all real numbers. This is a very rich environment, with lots of structures intertwined. One can do many things with real numbers: perform the arithmetic operations (addition, subtraction, multiplications, division), compare real numbers (i.e., say which of two numbers is greater) consider distances between real numbers (for real numbers x and y, the distance between them is $|x-y|$), etc. The inequalities and distances allow us to define a collection of properties, which a function may have and which help to analyze a function.

Choices. The collection of the tools used in Analysis for analyzing a function is pretty standard. It does not change from book to book. However, due to a great number of structures on $\mathbb{R}$, they can be introduced in many ways.

A choice of a specific set of definitions depends on many reasons. One may intend to optimize the length of proofs, facilitate understanding or applications (that is to make the exposition more conceptual and motivated or directed towards calculation algorithms). The tastes of the author and the way how the author studied the subject also influence the choice.

When we pass to analysis of more general functions, some of the structures disappear, others stay available, but change their appearance. A desire to facilitate understanding in the forthcoming parts of Analysis also are taken into consideration.

The choices made by K.A.Ross in his textbook have allowed him to give short simple proofs, however some of the definitions are left unmotivated and cannot survive a transition to a more general environment.

Below I try to put the theory into the context of more natural and flexible systems of notions. This is not original, but rather well known. I hope that it makes the theory easier to understand and, on the other hand, prepare the reader to forthcoming generalizations.

1 How to invent topological structures

One of the major directions in the analysis of a function is concentrating on its local properties. This simplifies the task. Instead of dealing with complicated global picture, we concentrate on a comparatively simple local picture. Locally a real valued function of a real variable is quite simple usually. If the function is continuous and differentiable (which are already its local properties), then the values of its derivative and (if a more detailed information is needed) the values of its high derivatives characterize the local behavior of the function.

The idea of localization can be applied in more complicated situations, when the function to be studied is a function of several variables, the variables may be non real, but complex, or have some other nature (say, may be matrices, or functions). The development of the idea of localization, which was motivated by these generalizations, also helped to understand better what is happening in the original situation. It gave rise to important mathematical structures. Often these structures are presented abstractly and dogmatically. It makes them easier to study, but leaves aside questions about motivation.

In this section I try to show a genesis of the notion of topological space. I do not try to follow the real history, but rather explain the genesis of ideas. The section is not necessary for what follows. It may be more difficult to read than the next sections. It is not required for success in exams. An non-patient reader may want to jump to the next section 2 where a more traditional formal text starts.

1.1 Local means "in a neighborhood". The core of local consideration is the notion of *neighborhood*. Informally speaking, a neighborhood of a point is a set surrounding the point. It may be narrow or broad, but it must contain *all the points that are sufficiently close* to the point under consideration.

1.2 Neighborhoods on the line. Consider this notion in the standard environment of the real line. For a point $p \in \mathbb{R}$ (aka a real number), a neighborhood of p is any set which contains all points that are sufficiently close to p.

What does it mean "*sufficiently close*"? These words mean nothing, unless we know the answer to the question: "*Sufficiently for what?*" Depending on the answer, the meaning may vary.

But the words *close to* p (without the adjective *sufficiently*) mean *at a small distance from* p. Therefore, independently of the purpose for being close to p, the words *sufficiently close to* p may be characterized by the distance from p. We may fix some positive distance ε (the Greek letter ε, epsilon, is traditional for this context, although you are free to use any other letter) and say that *sufficiently close* means *at a distance less than* ε.

As soon as the choice of ε is made, a neighborhood should contain all the points whose distance from p is less than ε. In other words, it should contain the set $\{x \in \mathbb{R} : |x-p| < \varepsilon\}$ that is the interval $(p-\varepsilon, p+\varepsilon) = \{x \in \mathbb{R} : p-\varepsilon < x < p+\varepsilon\}$. Whether it contains more remote points or not, is not essential, it will be called a neighborhood of p, anyway.

The choice of ε cannot be done once and for all. If we want to specify ε, we speak about a ε-neighborhood, but often it is enough that a set is an ε-neighborhood for whatever $\varepsilon > 0$. In other words, a set $N \subset \mathbb{R}$ is a neighborhood of $p \in \mathbb{R}$ if there exists an interval $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ such that $p \in (a,b) \subset N$.

1.3 Three main properties of neighborhoods Since informally a neighborhood of p means the set which includes all points that are sufficiently close to p, the following two properties seems to be natural:

1. *The intersection of any two neighborhoods of p is a neighborhood of p.*
2. *Any set which contains a neighborhood of p is a neighborhood of p.*

It would be natural to expect that *a neighborhood is shared by neighbors.* In other words, *a neighborhood N of a point p is a neighborhood for points sufficiently close to p.* "Sufficiently close" means "belonging to a neighborhood". This time it is probably a new neighborhood, more narrow neighborhood, because the original one could contain quite remote points. Therefore the statement: "a neighborhood is shared by neighbors" is transformed to a more complicated and formal statement:

3. *For any neighborhood N of p there is a neighborhood $M \subset N$ such that N is a neighborhood of each $q \in M$.*

It happens that most arguments about neighborhoods are based on the properties 1-3 formulated above. So far our consideration of neighborhoods was based on a vague intuitive idea of close points and one example, a real line $\mathbb{R}$, in which it is realized rigorously. The same considerations stay valid as long as a notion of distance is available. Furthermore, a reasonable notion of neighborhood can be extended to situations without distances. The three properties stated above hold true in all these situations and inspire to use the same intuition and terminology.

Topological spaces could be introduced as follows: one starts with an arbitrary set X. Elements of X are called *points* of this space. For each point $p \in X$, there is a distinguished collection of subsets of X containing p. These subsets are called *neighborhoods* of p. It is required that these collections satisfy the three properties stated above. That's it. The set X equipped with all this stuff is a topological space.

In literature (in particular, in most of textbooks) topological spaces introduced a little bit differently. The difference is not important: the standard definition of topological space which is presented below in Section 2.1, is equivalent to the one that was just outlined. The only difference is that instead of all the neighborhoods, only especially nice neighborhoods are distinguished. These especially nice neighborhoods are called *open sets*. In the approach outlined above they can be described as follows: a set is open if it is a neighborhood for each of its points.

As a primary notion used in the definition of topology, open sets are more convenient than neighborhoods. Neighborhoods can be defined in terms of open sets: a set N, which contains a point p, is a neighborhood of p if there exists an open set U such that $p \in U \in N$. Of course, open sets are required to have properties which implies the three properties of neighborhoods discussed above.

Now, after this informal introduction, we have to pass to formal definitions. If for whatever reasons you had difficulties in this section, please, do not try to overcome them now. Go ahead. A formal theory is easier. Revisit this section later.

2 Spaces

One of the main notions in Analysis is the notion of continuous function. It can be considered in the three levels of generality: on the real line $\mathbb{R}$, on metric spaces, and on topological spaces. It belongs to the third level, i.e, to the most general setup of topological spaces. In this setup it has the simplest form. On the other hand, this level is the most abstract. Many mathematicians believe that it is difficult for a beginner. I think it depends on the beginner and on the teacher.

In section 3 we will present continuity from all the three perspectives in the most concise form. In this section, we will study the underlying spaces.

2.1 Topological spaces

Let X be a set. Let Ω be a collection of its subsets such that:

1. the union of any collection of sets that are elements of Ω belongs to Ω;
2. the intersection of any finite collection of sets that are elements of Ω belongs to Ω;
3. the empty set $\varnothing$ and the whole X belong to Ω.

Then

- Ω is called a *topological structure* or just a *topology* on X;
- the pair (X, Ω) is called a *topological space*;
- elements of X are called *points* of this topological space;
- elements of Ω are called *open sets* of the topological space (X, Ω).

The conditions in the definition above are the *axioms of topological structure*. Let us reformulate the axioms of topological structure using the words *open set* wherever possible.

1. The union of any collection of open sets is open.
2. The intersection of any finite collection of open sets is open.
3. The empty set and the whole space are open.

There is a large collection of notions, which can be used in a topological space. The notion of a neighborhood was already mentioned in section 1 in a preliminary informal discussion of topological spaces.

Here is how they appear if we start with the notion of topological space. Let X be a topological space, $a \in X$. A set $N \subset X$ is said to be a *neighborhood* of a, if there is an open set U such that $a \in U \subset N$.

Obviously, any open set, which contains a point, is a neighborhood of this point. A set N is a neighborhood of a point a if it contains a smaller open neighborhood U of a.

Any open set is a neighborhood of each of its points.

Often one uses more narrow notion of neighborhood. Namely, by neighborhoods one means only open neighborhoods.

Let X be a topological space, $A \subset X$ and $b \in X$. Then

- b is called an *interior point* of A if it has a neighborhood, which is contained in A;
- b is called an *exterior point* of A if it has a neighborhood, which is disjoint with A (i.e., contained in $X \smallsetminus A$);
- b is called a *boundary point* of A if every of its neighborhoods has a non-empty intersection both with A and $X \smallsetminus A$.

A set $F \subset X$ is said to be *closed* in the space (X, Ω) if its complement $X \smallsetminus F$ is open (i.e., $X \smallsetminus F \in \Omega$).

2.2 The simplest examples of topological spaces

A *discrete topological space* is a set with the topological structure which consists of all the subsets of this set.

Let us check that this is a topological space, i.e., all axioms of topological structure hold true.

What should we check? The first axiom reads here that the union of any collection of subsets of X is a subset of X. Well, this is true, of course. If $A \subset X$ for each $A \in \Gamma$, then, obviously, $\bigcup_{A \in \Gamma} A \subset X$. Exactly in the same way we check the second axiom. Finally, we obviously have $\varnothing \subset X$ and $X \subset X$. $\square$

An *indiscrete topological space* is the opposite example, in which the topological structure is the most meager. (It is also called *trivial topology*.) It consists only of X and $\varnothing$.

This is a topological structure, is it not?

Yes, it is. If one of the united sets is X, then the union is X, otherwise the union is empty. If one of the sets to intersect is $\varnothing$, then the intersection is $\varnothing$. Otherwise, the intersection equals X.

2.3 The real line

Let X be the set $\mathbb{R}$ of all real numbers, Ω the set of arbitrary unions of open intervals (a, b) with $a, b \in \mathbb{R}$.

Exercise 2.1. Check whether Ω satisfies the axioms of topological structure.

First, show that $\bigcup_{A\in\Gamma} A \cap \bigcup_{B\in\Sigma} B = \bigcup_{A\in\Gamma, B\in\Sigma}(A \cap B)$. Therefore, if A and B are intervals, then the right-hand side is a union of intervals. $\square$

This is the topological structure which is always meant when $\mathbb{R}$ is considered as a topological space (unless another topological structure is explicitly specified). This space is usually called the ***real line***, and the structure is referred to as the ***canonical*** or ***standard*** topology on $\mathbb{R}$.

2.4 Metric spaces

A function $d : X \times X \to \mathbb{R}_{\geq 0} = \{\, x \in \mathbb{R} : x \geq 0 \,\}$ is called a ***metric*** (or ***distance function***) on X if

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ for any $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$.

The pair (X, d), where d is called a metric on X, is a ***metric space***. Condition (3) is called the ***triangle inequality***.

The notion of metric originated in geometry, and it brings geometric notions and intuition into other contexts. It particular, elements of X are called ***points***, no matter what is their original nature.

Exercise 2.2. Prove that the function

$$d : X \times X \to \mathbb{R}_{\geq 0} : \ (x, y) \mapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric for any set X.

Exercise 2.3. Prove that $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0} : (x, y) \mapsto |x - y|$ is a metric.

Let us solve this exercise. The triangle inequality in this case takes the form $|x-y| \le |x-z|+|z-y|$. Putting $a = x-z$ and $b = z-y$, we transform the triangle inequality into the well-known inequality $|a+b| \le |a| + |b|$.

Exercise 2.4. Prove that the plane with the usual distance between points is a metric space.

The metrics of Exercise 2.3 is always meant when $\mathbb{R}$ is considered as a metric space, unless another metric is specified explicitly.

Let (X, d) be a metric space, $a \in X$ a point, r a positive real number. Then the sets

$$B_r(a) = \{\, x \in X : d(a,x) < r \,\}, \tag{1}$$

$$D_r(a) = \{\, x \in X : d(a,x) \le r \,\}, \tag{2}$$

$$S_r(a) = \{\, x \in X : d(a,x) = r \,\} \tag{3}$$

are called, respectively, the ***open ball***, ***closed ball*** (or ***disk***), and ***sphere*** of the space (X, d) with center a and radius r.

The word ***ball*** was borrowed from a specific metric space, the three dimensional Euclidean space, but then it is used in all other metric spaces. There it may have quite different look. Say, on the line $\mathbb{R}$, an open ball $B_r(a)$ is the interval $(a-r, a+r)$. The common definition and properties justify the common name.

For a metric space (X, d), denote by Ω_d the collection of sets which contain each point together with a ball centered at it. In other words, a set $U \subset X$ belongs to Ω_d if for any $a \in U$ there exists $r \in \mathbb{R}_+$ such that $B_r(a) \subset U$.

Theorem 2.1. *Ω_d is a topological structure.*

The topological structure Ω_d is called the ***metric topology***. We also say that it is ***generated*** by the metric d. This topological structure is always meant whenever the metric space is regarded as a topological space (for instance, when we speak about open and closed sets, neighborhoods, etc. in this space).

Exercise 2.5. Prove that any open ball $B_r(a)$ is open in the metric topology.

Exercise 2.6. Prove that a set is open in metric topology if and only if it is a union of open balls.

Exercise 2.7. Prove that the standard topological structure in $\mathbb{R}$ introduced in Section 2.3 is generated by the metric $(x, y) \mapsto |x - y|$.

Exercise 2.8. Prove that in a metric space a set N is a neighborhood of a if and only if there exists a ball with center at a, which is contained in N.

Exercise 2.9. Prove that in a metric space a set N is a neighborhood of a if and only if there exists a real number $\varepsilon > 0$ such that $B_\varepsilon(a) \subset N$.

Yet another reformulation:

Exercise 2.10. In a metric space, N is a neighborhood of a if and only if there exists a real number $\varepsilon > 0$ such that $x \in N$ for any $x \in X$ with $\rho(x, a) < \varepsilon$.

2.5 Subspace topology

Let (X, Ω) be a topological space, $A \subset X$. Denote by Ω_A the collection of sets $A \cap V$, where $V \in \Omega$: $\Omega_A = \{A \cap V : V \in \Omega\}$.

Theorem 2.2. *The collection Ω_A is a topological structure in A.*

Proof. We must check that Ω_A satisfies the axioms of topological structure. Consider the first axiom. Let $\Gamma \subset \Omega_A$ be a collection of sets in Ω_A. We must prove that $\bigcup_{U \in \Gamma} U \in \Omega_A$. For each $U \in \Gamma$, find $U_X \in \Omega$ such that $U = A \cap U_X$. This is possible due to the definition of Ω_A. Transform the union under consideration: $\bigcup_{U \in \Gamma} U = \bigcup_{U \in \Gamma}(A \cap U_X) = A \cap \bigcup_{U \in \Gamma} U_X$. The union $\bigcup_{U \in \Gamma} U_X$ belongs to Ω (i.e., is open in X) as the union of sets open in X. (Here we use the fact that Ω, being a topology on X, satisfies the first axiom of topological structure.) Therefore, $A \cap \bigcup_{U \in \Gamma} U_X$ belongs to Ω_A. Similarly we can check the second axiom. The third axiom: $A = A \cap X$, and $\varnothing = A \cap \varnothing$. $\square$

The pair (A, Ω_A) is a ***subspace*** of the space (X, Ω). The collection Ω_A is the ***subspace topology***, the ***relative topology***, or the topology ***induced*** on A by Ω, and its elements are said to be sets ***open*** in A.

Exercise 2.11. Prove that the canonical topology on $\mathbb{R}^1$ coincides with the topology induced on $\mathbb{R}^1$ as on a subspace of the plane.

Proof. Let us prove that a subset of $\mathbb{R}^1$ is open in the relative topology iff it is open in the canonical topology.

The intersection of an open disk with $\mathbb{R}^1$ is either an open interval or the empty set. Any open set in the plane is a union of open disks. Therefore, the intersection of any open set of the plane with $\mathbb{R}^1$ is a union of open intervals. Thus, it is open in $\mathbb{R}^1$.

Conversely, an open set in $\mathbb{R}^1$ is a union of open intervals, each open interval is the intersection with $\mathbb{R}^1$ of the open disk with the same center and radius. $\square$

Theorem 2.3. *A set F is closed in a subspace $A \subset X$ iff F is the intersection of A and a closed subset of X.*

Proof. Assume that F is closed in A. Then the complement $A \smallsetminus F$ is open in A, i.e., $A \smallsetminus F = A \cap U$, where U is open in X. What closed set cuts F on A? It is cut by $X \smallsetminus U$. Indeed, we have $A \cap (X \smallsetminus U) = A \smallsetminus (A \cap U) = A \smallsetminus (A \smallsetminus F) = F$. The converse is proved similarly. $\square$

Exercise 2.12. If a subset of a subspace is open (respectively, closed) in the ambient space, then it is also open (respectively, closed) in the subspace.

Sets that are open in a subspace are not necessarily open in the ambient space.

Theorem 2.4. *The unique open set in $\mathbb{R}^1$ which is also open in $\mathbb{R}^2$ is $\varnothing$.*

Proof. No disk of $\mathbb{R}^2$ is contained in $\mathbb{R}$. $\square$

However, the following is true.

Theorem 2.5. *An open set of an open subspace is open in the ambient space, i.e., if $A \in \Omega$, then $\Omega_A \subset \Omega$.*

Proof. If $A \in \Omega$ and $B \in \Omega_A$, then $B = A \cap U$, where $U \in \Omega$. Therefore, $B \in \Omega$ is the intersection of two sets, A and U, belonging to Ω. $\square$

The same relation holds true for closed sets. Sets that are closed in the subspace are not necessarily closed in the ambient space. However, the following is true.

Theorem 2.6. *Closed sets of a closed subspace are closed in the ambient space.*

Proof. Act as in the proof of Theorem 2.5, but use Theorem 2.3 instead of the definition of relative topology. $\square$

Theorem 2.7 (Transitivity of induced topology). *Let (X, Ω) be a topological space, $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.*

Proof. The core of the proof is the equality $(U \cap A) \cap B = U \cap B$. It holds true because $B \subset A$, and we apply it to $U \in \Omega$. When U runs through Ω, the right-hand side of the equality $(U \cap A) \cap B = U \cap B$ runs through Ω_B, while the left-hand side runs through $(\Omega_A)_B$. Indeed, elements of Ω_B are intersections $U \cap B$ with $U \in \Omega$, and elements of $(\Omega_A)_B$ are intersections $V \cap B$ with $V \in \Omega_A$, but V, in turn, being an element of Ω_A, is the intersection $U \cap A$ with $U \in \Omega$. $\square$

3 Continuous maps

3.1 Definition and basic properties Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if the preimage of each open subset of Y is an open subset of X.

Recall that the *preimage* of a subset $B \subset Y$ under a map $f : X \to Y$ is $\{a \in X : f(a) \in B\}$ (in words: this is the set of all the elements of X which are mapped by f to elements of B). The preimage of B under f is denoted by $f^{-1}(B)$.

Theorem 3.1. *A map is continuous iff the preimage of each closed set is closed.*

Proof. Let $f : X \to Y$ be a map. If $f : X \to Y$ is continuous, then, for each closed set $F \subset Y$, the set $X \smallsetminus f^{-1}(F) = f^{-1}(Y \smallsetminus F)$ is open, and therefore $f^{-1}(F)$ is closed. To prove the converse statement, exchange the words *open* and *closed* in the above argument. $\square$

Theorem 3.2. *The identity map of any topological space is continuous.*

Theorem 3.3. *Any constant map (i.e., a map with one-point image) is continuous.*

Proof. The preimage of any set under a constant map either is empty or coincides with the whole space. $\square$

Exercise 3.1. Let Ω_1 and Ω_2 be two topological structures in a space X. Prove that the identity map

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous iff $\Omega_2 \subset \Omega_1$.

Exercise 3.2. Let $f : X \to Y$ be a continuous map. Find out whether or not it is continuous with respect to

1. a larger topology on X and the same topology on Y,
2. a smaller topology on X and the same topology on Y,
3. a larger topology on Y and the same topology on X,
4. a smaller topology on Y and the same topology on X.

Exercise 3.3. Let X be a discrete space, Y an arbitrary space.
1) Which maps $X \to Y$ are continuous?
2) Which maps $Y \to X$ are continuous for each topology on Y?

Exercise 3.4. Let X be an indiscrete space, Y an arbitrary space.
2) Which maps $Y \to X$ are continuous?
1) Which maps $X \to Y$ are continuous for each topology on Y?

Theorem 3.4. *Let A be a subspace of X. Then the inclusion* $\mathrm{in} : A \to X$ *is continuous.*

Proof. If a set U is open in X, then its preimage $\mathrm{in}^{-1}(U) = U \cap A$ is open in A by the definition of the induced topology. $\square$

Exercise 3.5. The topology Ω_A induced on $A \subset X$ by the topology of X is the smallest topology on A with respect to which the inclusion $\mathrm{in} : A \to X$ is continuous.

Theorem 3.5. *A composition of continuous maps is continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. We must show that for every $U \subset Z$ that is open in Z its preimage $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. The set $g^{-1}(U)$ is open in Y by continuity of g. In turn, its preimage $f^{-1}(g^{-1}(U))$ is open in X by the continuity of f. $\square$

Recall that the restriction of a map $f : X \to Y$ to $A \subset X$ is the map $f|_A : A \to Y$ defined by formula $(f|A)(x) = x$ for $x \in A$.

Theorem 3.6. *A restriction of a continuous map is continuous.*

Proof. Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $A \subset X$. Then $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$. $\square$

3.2 Local Continuity A map f from a topological space X to a topological space Y is said to be ***continuous at a point*** $a \in X$ if the preimage of every neighborhood of $f(a)$ is a neighborhood of a.

Theorem 3.7. *A map $f : X \to Y$ is continuous iff it is continuous at each point of X.*

Proof. Assume that f is continuous. Let us prove that f is continuous at every $a \in X$. Let N be a neighborhood of $f(a)$. By the definition of neighborhood, it contains an open neighborhood: there exists an open set U such that $f(a) \in U \subset N$. Then $a \in f^{-1}(U) \subset f^{-1}(N)$. The set $f^{-1}(U)$ is open in X, because U is open in Y and f is continuous. Thus $f^{-1}(N)$ contains open set $f^{-1}(U)$ which contains a. Therefore $f^{-1}(N)$ is a neighborhood of a.

Now let us assume that f is continuous at every point $a \in X$ and prove that f is continuous. We must check that the preimage of each open set is open. Let $V \subset Y$ be an open set in Y. Take $a \in f^{-1}(V)$. By continuity of f at a, the set $f^{-1}(V)$ is a neighborhood of a. Hence, there exists an open set U_a such that $a \in U_a \subset f^{-1}(V)$. Take such U_a for each $a \in f^{-1}(V)$ and unite all of them. The union $U = \cup_{a \in f^{-1}(V)} U_a$ is an open set (as a union of open sets), it is contained in $f^{-1}(V)$ as each U_a is contained, and it contains $f^{-1}(V)$, as each $a \in f^{-1}(V)$ belongs to its U_a. Thus $f^{-1}(V) = U$ and is an open set. $\square$

Theorem 3.8. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff each ball centered at $f(a)$ contains the image of a ball centered at a.*

Proof. Let as assume that f is continuous at a point $a \in X$. Let $B_\varepsilon(f(a))$ be a ball centered at $f(a)$. As an open set, it is a neighborhood of $f(a)$ in Y. By local continuity of f at a, its preimage $f^{-1}B_\varepsilon(f(a))$ is a neighborhood of a. Hence, there exists a ball $B_\delta(a) \subset f^{-1}B_\varepsilon(f(a))$. Applying f to both sides of this inclusion formula, we obtain $f(B_\delta(a)) \subset B_\varepsilon(f(a))$.

Let us assume that each ball $B_\varepsilon(f(a))$ contains the image of a ball $B_\delta(a)$ and prove that then f is continuous at a. Let N be a neighborhood of $f(a)$. In a metric space Y this means that there exists a ball $B_\varepsilon(f(a)) \subset N$. By our assumption, $B_\varepsilon(f(a))$ contains the image of some ball $B_\delta(a)$. Therefore, $f^{-1}(N) \supset B_\delta(a)$ is a neighborhood of a. □

Theorem 3.9. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Proof. The condition "for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\left(f(x), f(a)\right) < \varepsilon$" means that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. Now, apply the preceding Theorem 3.8. □

Theorem 3.9 means that the definition of continuity usually studied in Calculus, when applicable, is equivalent to the definition of continuity at a point stated in terms of topological structures.

3.3 Sequential continuity

Let X be a topological space, $s_n \in X$ a sequence of its points, $s \in X$. The sequence s_n is said to ***converge*** to s, if for any neighborhood U of s there exists $N \in \mathbb{N}$ such that $s_n \in U$ for any $n > N$. The convergence of s_n to s is denoted by $s_n \to_{n\to\infty} s$ or $s = \lim_n s_n$.

A map $f : X \to Y$ is said to ***sequentially continuous*** at $a \in X$ if for any sequence $a_n \in X$, which converges to a, the sequence $f(a_n)$ converges to $f(a)$.

A map $f : X \to Y$ is said to be ***sequentially continuous*** if for each $b \in X$ and each sequence $a_n \in X$ converging to b the sequence $f(a_n)$ converges to $f(b)$. In other words, $f : X \to Y$ is sequentially continuous if it is sequentially continuous at each point.

Theorem 3.10. *Any map $f : X \to Y$ continuous at $a \in X$ is sequentially continuous at a.*

Proof. let $f : X \to Y$ be a continuous map, let $a \in X$, and let $a_n \to a$ in X. We must prove that $f(a_n) \to f(a)$ in Y. Let $V \subset Y$ be a neighborhood of $f(a)$. Since f is continuous, $f^{-1}(V) \subset X$ is a neighborhood of a, and since $a_n \to a$, we have there exists $N \in \mathbb{N}$ such that $a_n \in f^{-1}(V)$ for $n > N$. Then also $f(a_n) \in V$ for $n > N$, as required. $\square$

Corollary 1. *Any continuous map is sequentially continuous.* $\square$

Theorem 3.11. *Let X be a metric space, Y a topological space and $a \in X$. If $f : X \to Y$ is sequentially continuous at a then f is continuous at a.*

Proof. Assume the contrary, that f is *not* continuous at a. Then there exists a neighborhood V of $f(a)$ such that its preimage $f^{-1}(V)$ is not a neighborhood for a. Then $B_\delta(a) \not\subset f^{-1}(V)$ for each $\delta > 0$. In particular, $B_{\frac{1}{n}}(a) \not\subset f^{-1}(V)$ for each $n \in \mathbb{N}$. Hence, there exists $a_n \in B_{\frac{1}{n}}(a) \smallsetminus f^{-1}(V)$ for each $n \in \mathbb{N}$. The sequence a_n converges to a, but $f(a_n) \notin V$ for any n. Hence $f(a_n)$ does not converge to $f(a)$. This contradicts the assumption of sequential continuity of f at a. $\square$

Corollary 2. *For a map $X \to Y$ of a metric space X to an arbitrary topological space Y continuity and sequential continuity are equivalent.*