Lecture 8

Structure of mathematical text. Proof techniques

Structures in a mathematical text

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Next come examples and **detailed** reading of proofs.

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What is this? Promises, planning.

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What is this? Definition. Definition of what?

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What is this? Theorem. Probably very simple, because it is not called Theorem.

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What is this? This is a promise, planning.

Let us read!

The number $\sqrt{2}$ is algebraic over $\mathbb{Q}$ because it is zero of the polynomial $f(x) = x^2 - 2$, which is nonzero and has coefficients in $\mathbb{Q}$.

In order to show that a number is algebraic, we look for a suitable polynomial having that number as zero. Try to prove that $1 + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

It is useful to be able to recognize the definition of "algebraic over a field $\mathbb{F}$" when it appears in different guises: a number $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ if and only if there is a positive integer n such that $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n\}$ are linearly dependent over $\mathbb{F}$.

Indeed, if $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ then there exists a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients $a_0, a_1, \ldots, a_n$ all belong to $\mathbb{F}$, at least one of these coefficients is nonzero, and $f(\alpha) = 0$, that is

$$a_0 + a_1\alpha + a_2\alpha^2 \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n = 0. \qquad (*)$$

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What is this? *Example.*

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What is this? Explanation, advice.

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What is this? Motivation

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In order to show that a number is algebraic, we look for a suitable polynomial having that number as zero. Try to prove that $1 + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

It is useful to be able to recognize the definition of "algebraic over a field $\mathbb{F}$" when it appears in different guises: a number $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ if and only if there is a positive integer n such that $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}, \alpha^n\}$ are linearly dependent over $\mathbb{F}$.
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Indeed, if $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ then there exists a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients $a_0, a_1, \ldots, a_n$ all belong to $\mathbb{F}$, at least one of these coefficients is nonzero, and $f(\alpha) = 0$, that is

$$a_0 + a_1\alpha + a_2\alpha^2 \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n = 0. \qquad (*)$$

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The coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$, on the other hand, are all in $\mathbb{F}$ so we can regard them as scalars. Thus, the equality $(*)$ can be interpreted as a linear dependence of vectors $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n$ in $\mathbb{C}$.

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We formalize this as follows.

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Proofs

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(quod erat demonstrandum)

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Discussion. We have to prove that $\forall\, a, b \in \mathbb{R}\, (a, b \geq 0 \implies \frac{a+b}{2} \geq \sqrt{ab})$.

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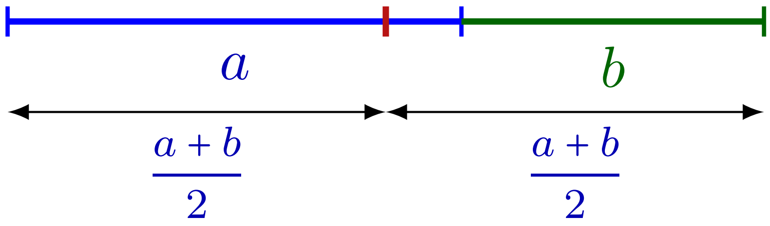

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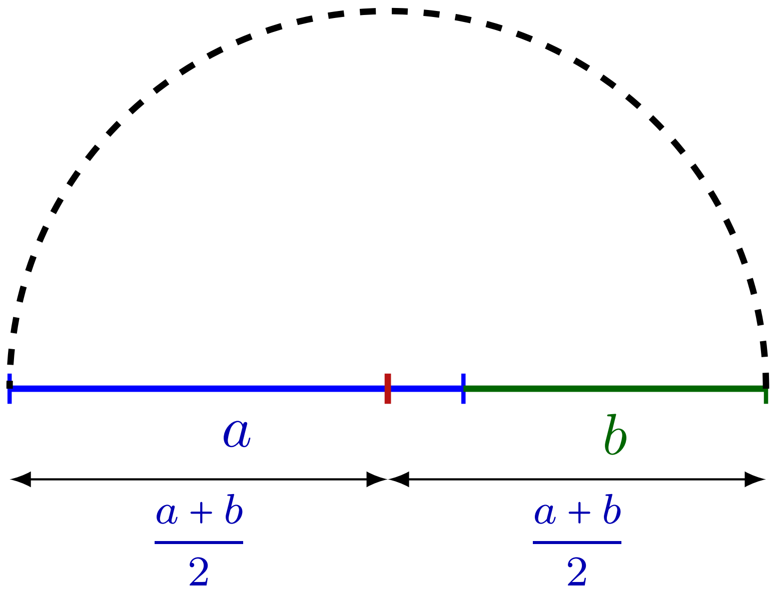

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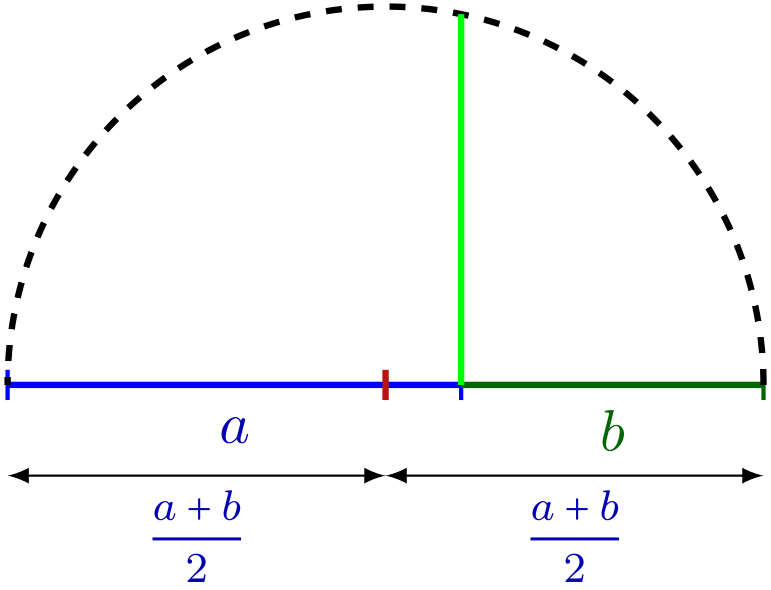

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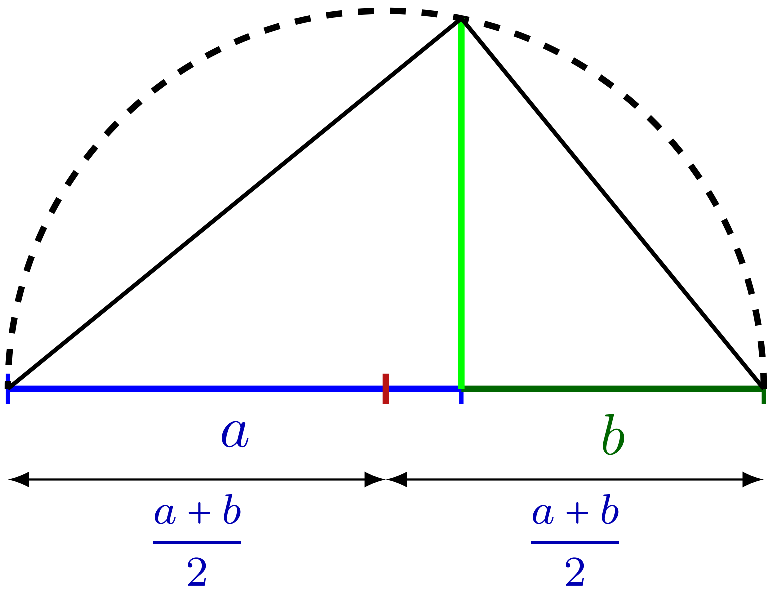

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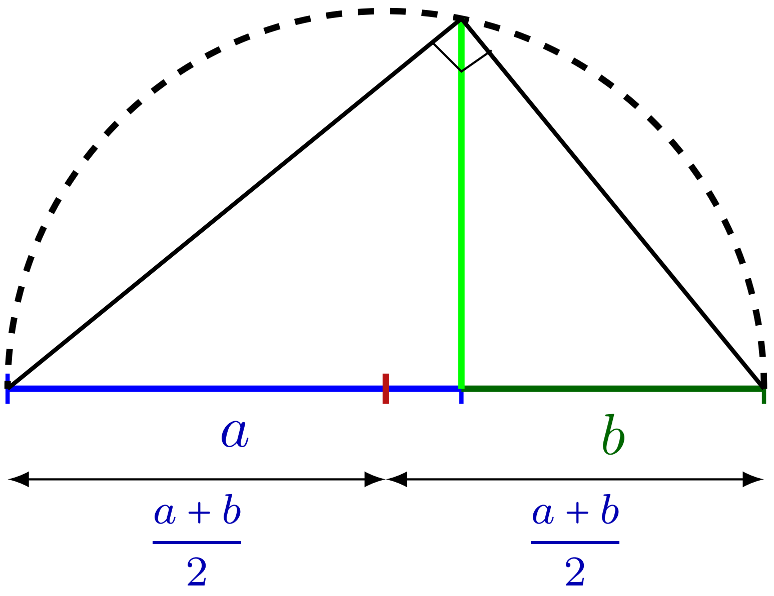

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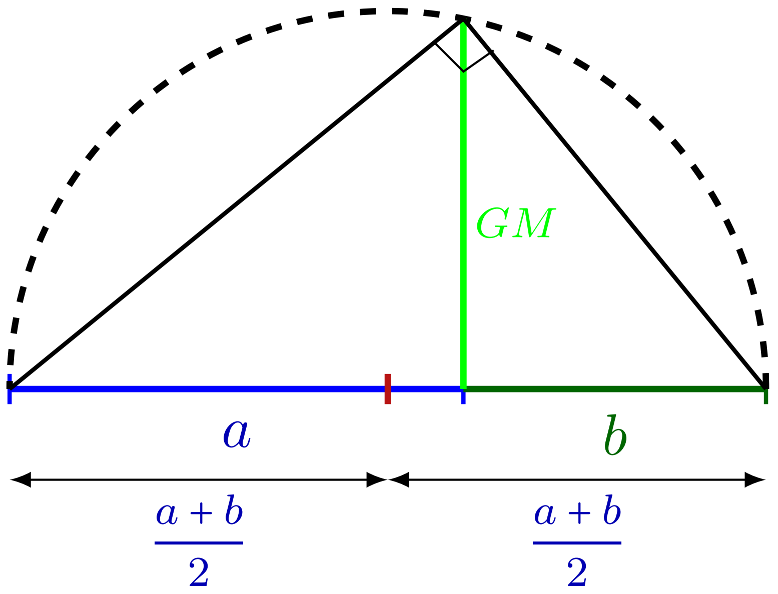

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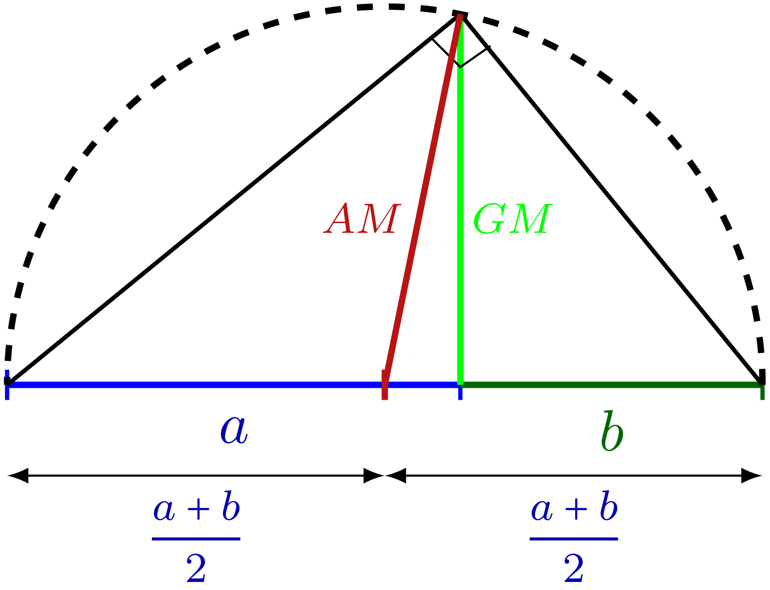

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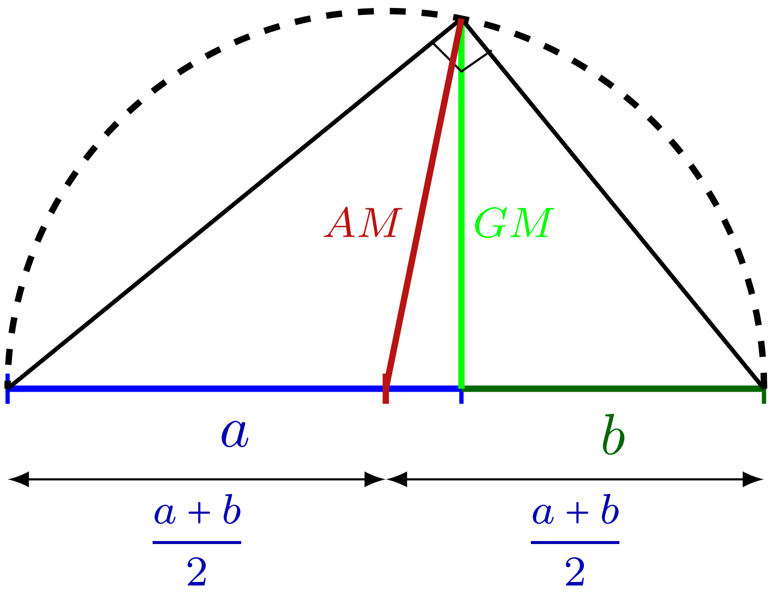

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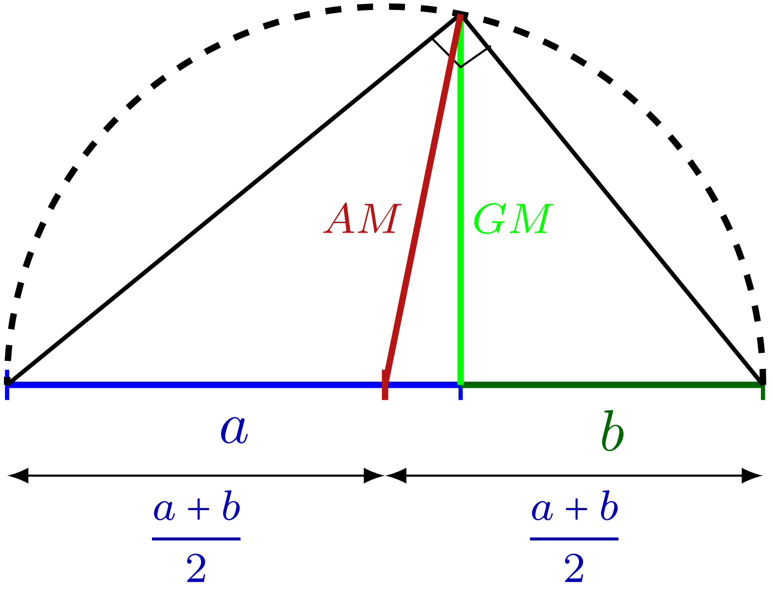

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Why not to prove like this: n^2 is odd $\implies$ $\sqrt{n^2} = n$ is odd?

What to choose: direct proof or proof by contraposition?

For a **direct** proof of

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This suggests a proof by **contraposition**:

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Non-zero integral

Example 3. Let f be integrable on $[0,1]$. Prove that

$$\text{if } \int_0^1 f(x)\,dx \neq 0\text{, then } f(x) \neq 0 \text{ for some } x \in [0,1].$$

Proof. Have to prove:

$$\int_0^1 f(x)\,dx \neq 0 \implies \exists\, x \in [0,1] \;\; f(x) \neq 0\,.$$

Assume that $f(x) = 0$ for **all** $x \in [0,1]$. Then $\int_0^1 f(x)\,dx = 0$.

Therefore, by contraposition,

if $\int_0^1 f(x)\,dx \neq 0$, then $f(x) \neq 0$ for some $x \in [0,1]$, as required.

Proof by contradiction (indirect proof)

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Method: Assume (let) $\neg P$.

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Example 1. Prove that $\sqrt{2}$ is irrational.

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Assume, to the contrary, that $\boxed{\sqrt{2} \text{ is rational}}$.

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Proof. The statement to prove: $\underbrace{\boxed{\sqrt{2} \text{ is irrational}}}_{P}$.

Assume, to the contrary, that $\underbrace{\boxed{\sqrt{2} \text{ is rational}}}_{\neg P}$.

Then $\sqrt{2} = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z},\ q \neq 0$.

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Since any fraction $\dfrac{p}{q}$ can be reduced to lowest terms,

we may assume, without loss of generality, that $\underset{Q}{\boxed{\gcd(p,q) = 1}}$.

According to our assumption, $\sqrt{2} = \dfrac{p}{q}$.

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we may assume, without loss of generality, that $\underset{Q}{\boxed{\gcd(p,q)=1}}$.

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According to our assumption, $\sqrt{2}=\frac{p}{q}$. By squaring, we get $2=\frac{p^2}{q^2}$, so $2q^2=p^2$.

It means that p^2 is even.

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It means that p^2 is even. Since p has the same parity as p^2

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It means that p^2 is even. Since p has the same parity as p^2

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It means that p^2 is even. Since p has the same parity as p^2
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By this, q^2 is even, and, therefore, q is even too:

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By this, q^2 is even, and, therefore, q is even too: $2\,|\,q$.

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By this, q^2 is even, and, therefore, q is even too: $2 \mid q$.

But p is also even, that is $2 \mid p$.

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By this, q^2 is even, and, therefore, q is even too: $2\,|\,q$.

But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

Therefore, $\gcd(p,q)\neq 1$

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Therefore $\underbrace{\boxed{\gcd(p,q)\neq 1}}_{\neg Q}$,

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By this, q^2 is even, and, therefore, q is even too: $2\,|\,q$.

But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

Therefore $\underbrace{\boxed{\gcd(p,q)\neq 1}}_{\neg Q}$, which contradicts to the fact that $\gcd(p,q)=1$.

$\sqrt{2}$ is irrational

Since any fraction $\frac{p}{q}$ can be reduced to lowest terms,

we may assume, without loss of generality, that $\underbrace{\boxed{\gcd(p,q)=1}}_{Q}$.

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By this, q^2 is even, and, therefore, q is even too: $2\,|\,q$.

But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

Therefore $\underbrace{\boxed{\gcd(p,q)\neq 1}}_{\neg Q}$, which contradicts to the fact that $\gcd(p,q)=1$.

This contradiction shows that the original assumption ($\sqrt{2}$ is rational)

$\sqrt{2}$ is irrational

Since any fraction $\frac{p}{q}$ can be reduced to lowest terms,

we may assume, without loss of generality, that $\underbrace{\boxed{\gcd(p,q)=1}}_{Q}$.

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By this, q^2 is even, and, therefore, q is even too: $2\,|\,q$.

But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

Therefore $\underbrace{\boxed{\gcd(p,q)\neq 1}}_{\neg Q}$, which contradicts to the fact that $\gcd(p,q)=1$.

This contradiction shows that the original assumption ($\sqrt{2}$ is rational) was erroneous,

$\sqrt{2}$ is irrational

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In this case, the identity $2q^2=p^2$ is equivalent to $2q^2=(2k)^2$, or $q^2=2k^2$.

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But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

Therefore $\underbrace{\boxed{\gcd(p,q)\neq 1}}_{\neg Q}$, which contradicts to the fact that $\gcd(p,q)=1$.

This contradiction shows that the original assumption ($\sqrt{2}$ is rational) was erroneous, and $\sqrt{2}$ is actually irrational,

$\sqrt{2}$ is irrational

Since any fraction $\frac{p}{q}$ can be reduced to lowest terms,

we may assume, without loss of generality, that $\underbrace{\boxed{\gcd(p,q)=1}}_{Q}$.

According to our assumption, $\sqrt{2}=\frac{p}{q}$. By squaring, we get $2=\frac{p^2}{q^2}$, so $2q^2=p^2$.

It means that p^2 is even. Since p has the same parity as p^2
(see Theorem about the same parity of an integer and its square),
we conclude that p should be even, that is, $p=2k$ for some integer k.

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Theorem (Euclid). *There are infinitely many prime numbers.*

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Proof.

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For source and comments see

Euclid's Elements, Book IX, Proposition 20.

http://aleph0.clarku.edu/ djoyce/java/elements/bookIX/propIX20.html

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Listen to the proof and try to write it down...

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Therefore, $|a+b| \le |a| + |b|$ for all real numbers a and b ,

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- Case 4. $a < 0$ and $b < 0$ .Then $|a| = -a$, $|b| = -b$, $|a+b| = -a-b$, so $|a+b| = -a-b = |a| + |b|$, and, by this $|a+b| \leq |a| + |b|$.

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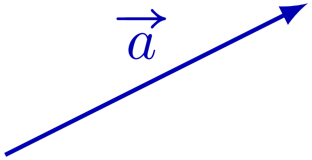

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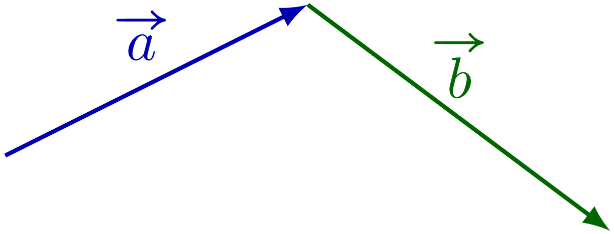

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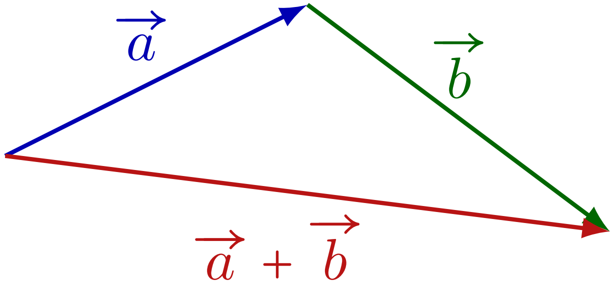

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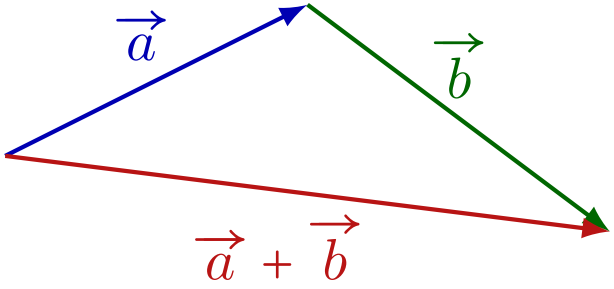

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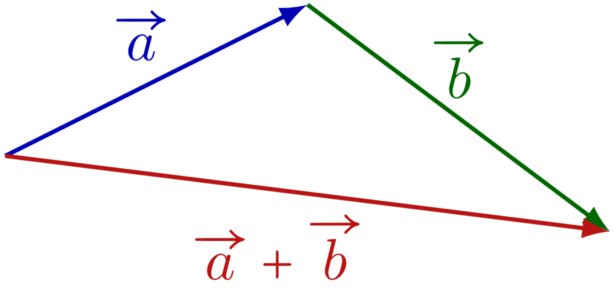

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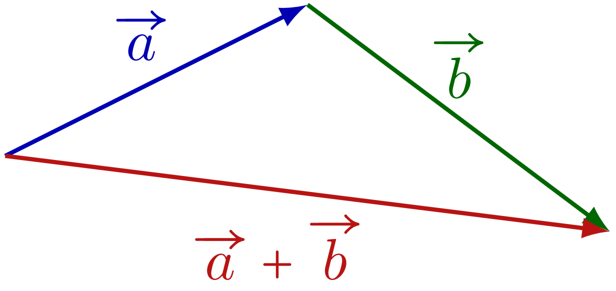

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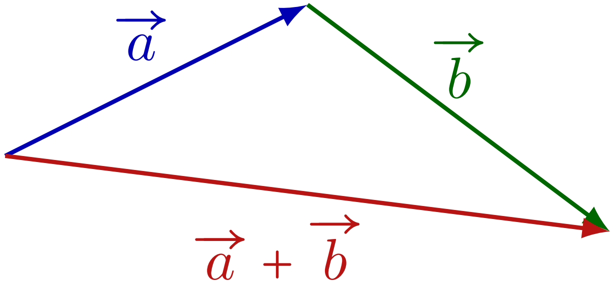

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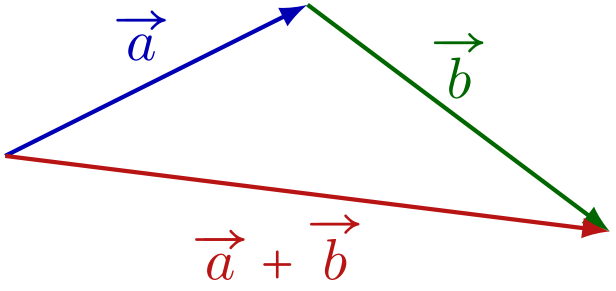

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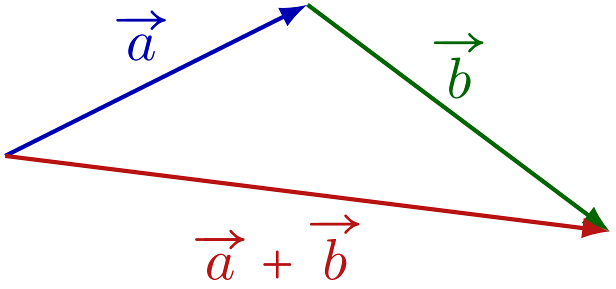

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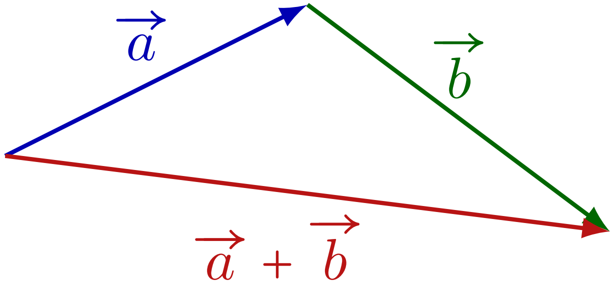

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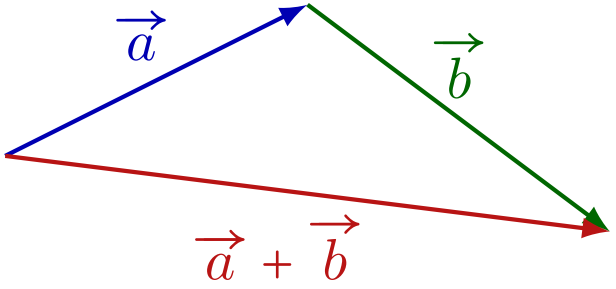

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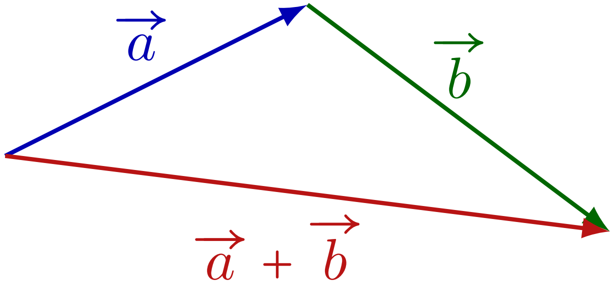

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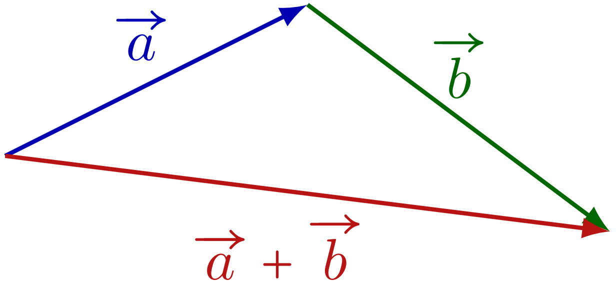

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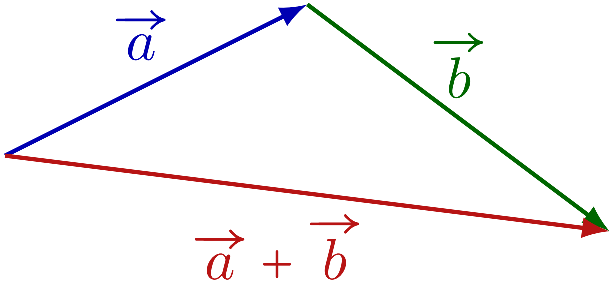

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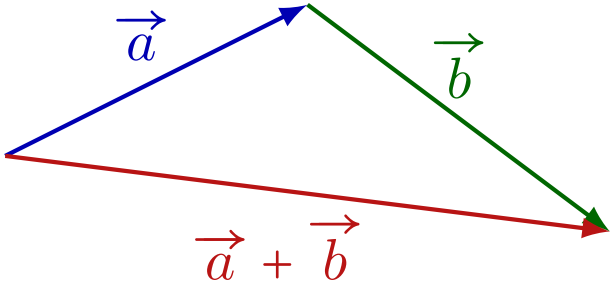

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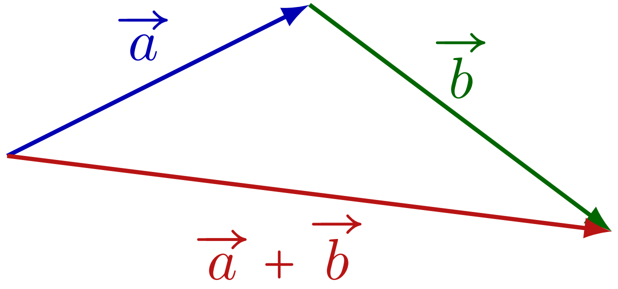

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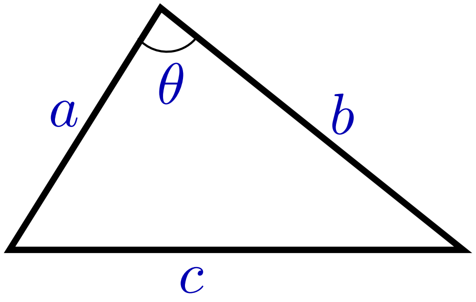

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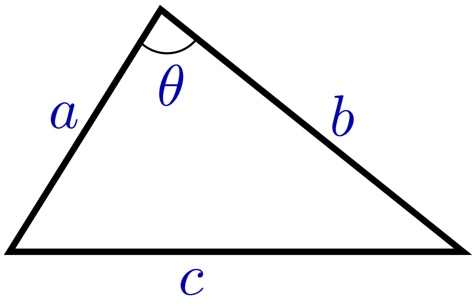

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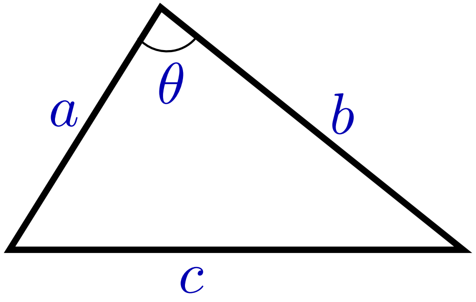

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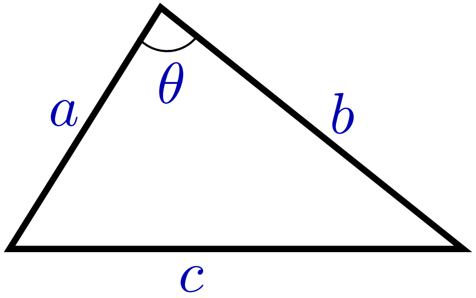

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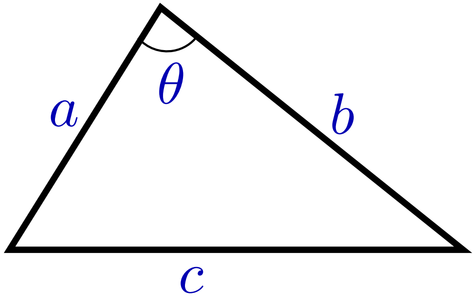

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A triangle with the sides a, b, c is right $\underset{?}{\iff}$ $\theta = 90^\circ$ $\underset{?}{\iff}$ $\cos\theta = 0$

How to prove an equivalence

To prove a statement of type $P \iff Q$, we may use one of two alternatives:

Alternative 1: $P \iff R \iff S \iff \cdots \iff Q$

Alternative 2: $P \implies Q$ and $Q \implies P$.

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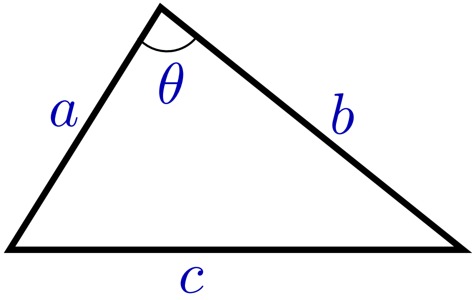

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