Lecture 4

Definitions in Mathematics

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This is a **descriptive** (or implicit) definition.
There are also **constructive** (or explicit) definitions.

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(4) if the name is an adjective,
then instead of **is called** one may use **is said to be**.

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Let X, Y and Z be sets, and let $f : X \to Y$, $g : Y \to Z$ be maps.

Then the map $g \circ f : X \to Z$ defined by formula $g \circ f(x) = g(f(x))$ is called the **composition** of f and g.

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Remarks.

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Since $k + l$ is an integer, a is a factor of $b + c$.

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Remarks. 1. The definition of divisibility is made in terms of multiplication, not division. Why? Is there a division?How would it be with division?

2. Why $d \neq 0$? Why we can't divide by 0?

Let us see **how** this **definition is used in** the **proof of a theorem**.

Theorem. Let a, b and c be integers, and $a \neq 0$.
If a divides both b and c, then a divides $b + c$.

Proof. Since $a|b$, then, by definition of divisibility, $b = a \cdot k$ for some integer k. Since $a|c$, then $c = a \cdot l$ for some integer l. Therefore,

$$b + c = ak + al = a(k + l).$$

Since $k + l$ is an integer, a is a factor of $b + c$. Therefore, a divides $b + c$.

Example of a definition: divisibility

Definition. Let d and n be **integers** and $d \neq 0$. One says that
d **divides** n (or, equivalently, n is **divisible** by d)
if $n = d \cdot k$ for some integer k.

Notation: $d \mid n$

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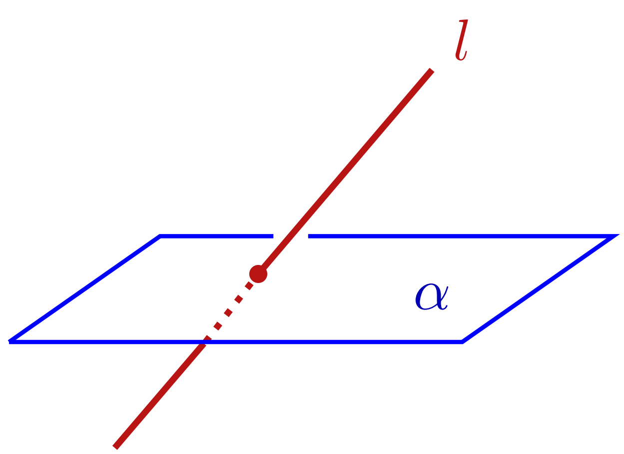

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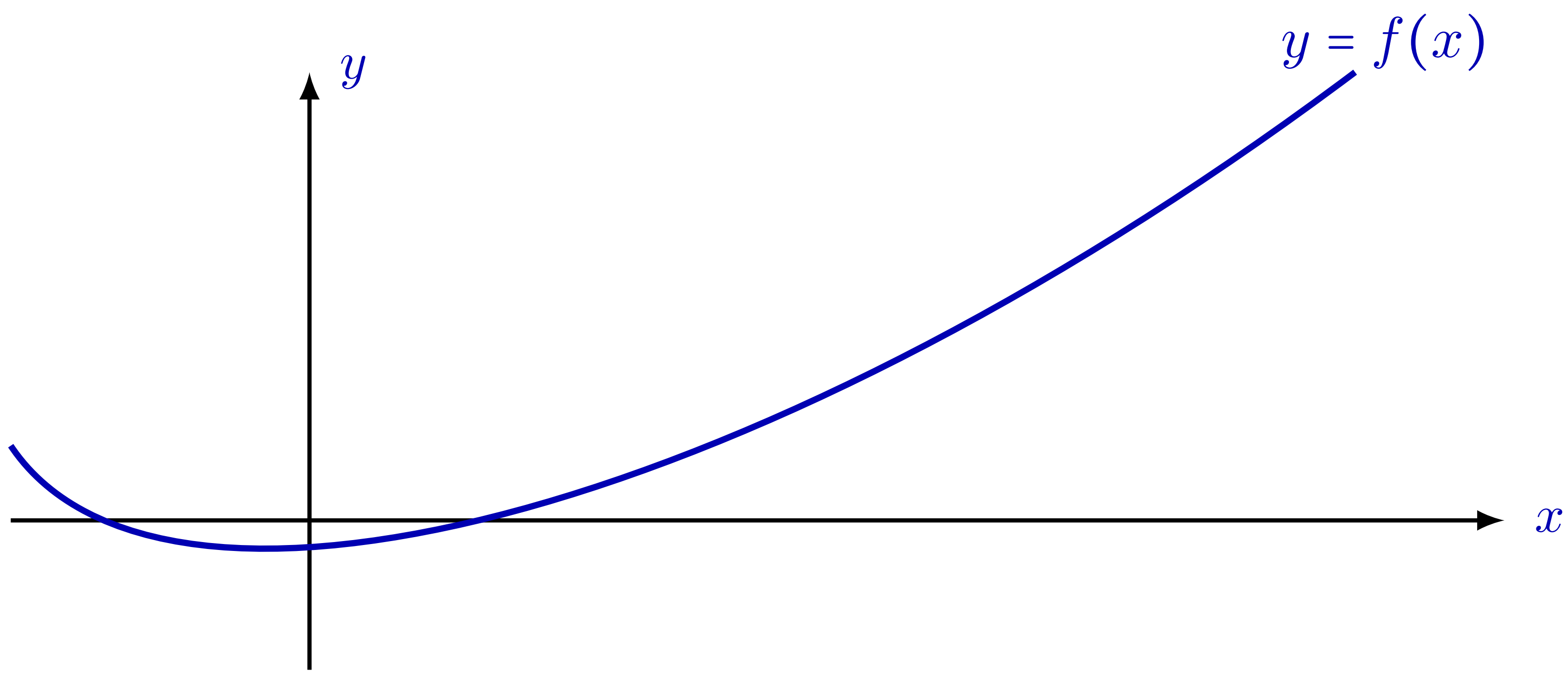

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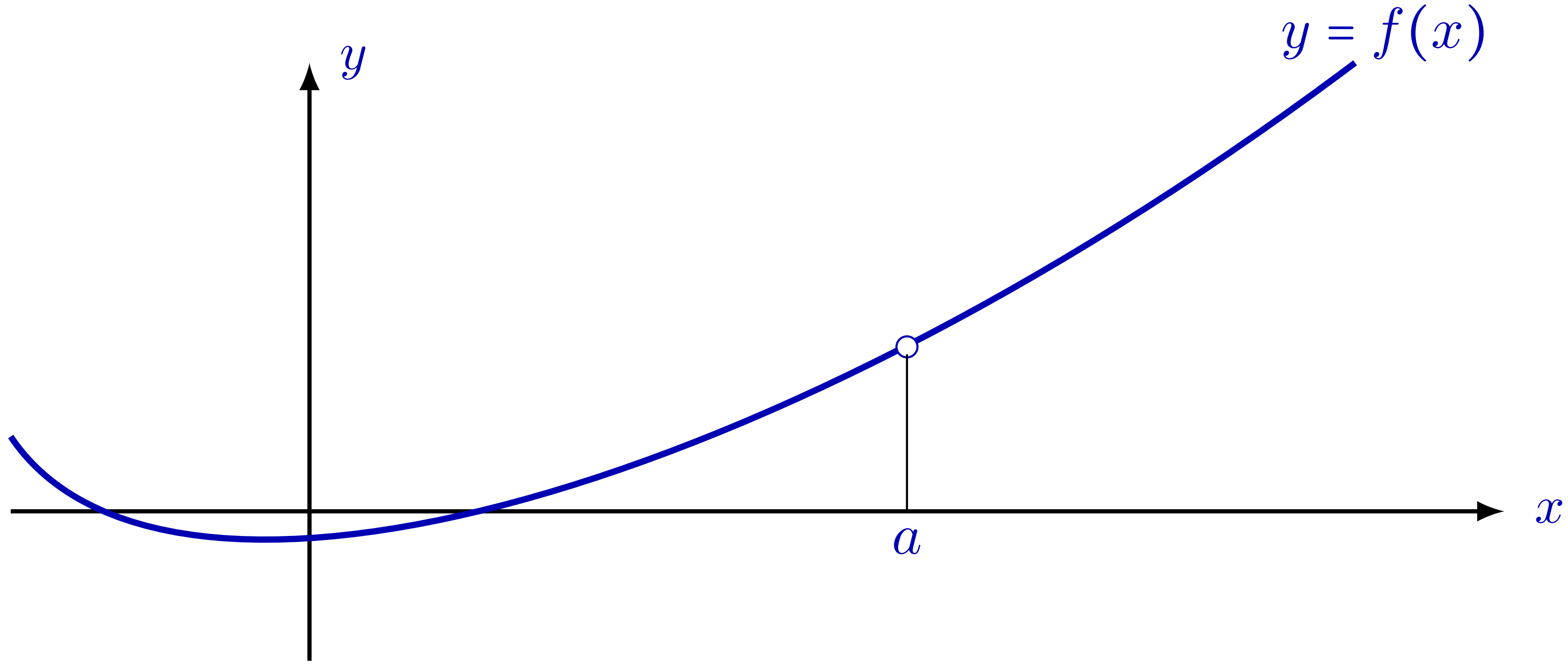

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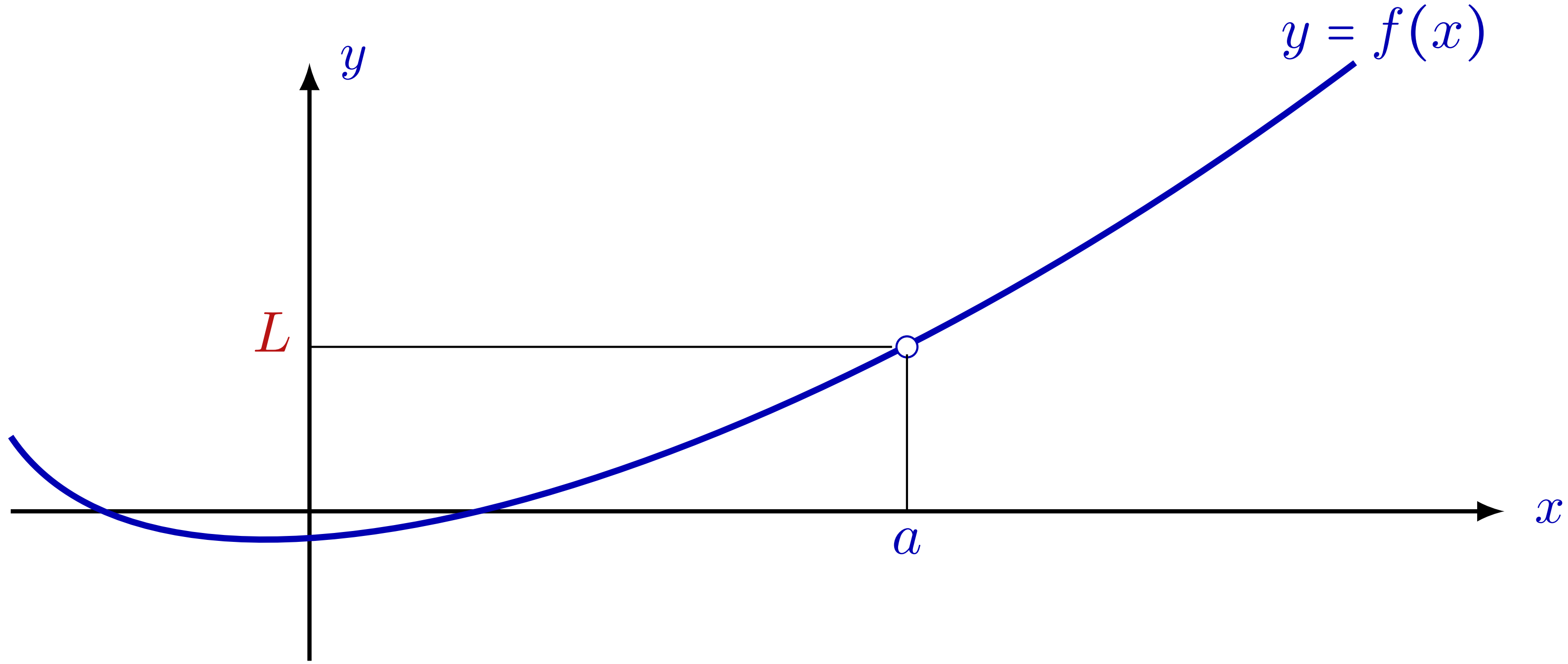

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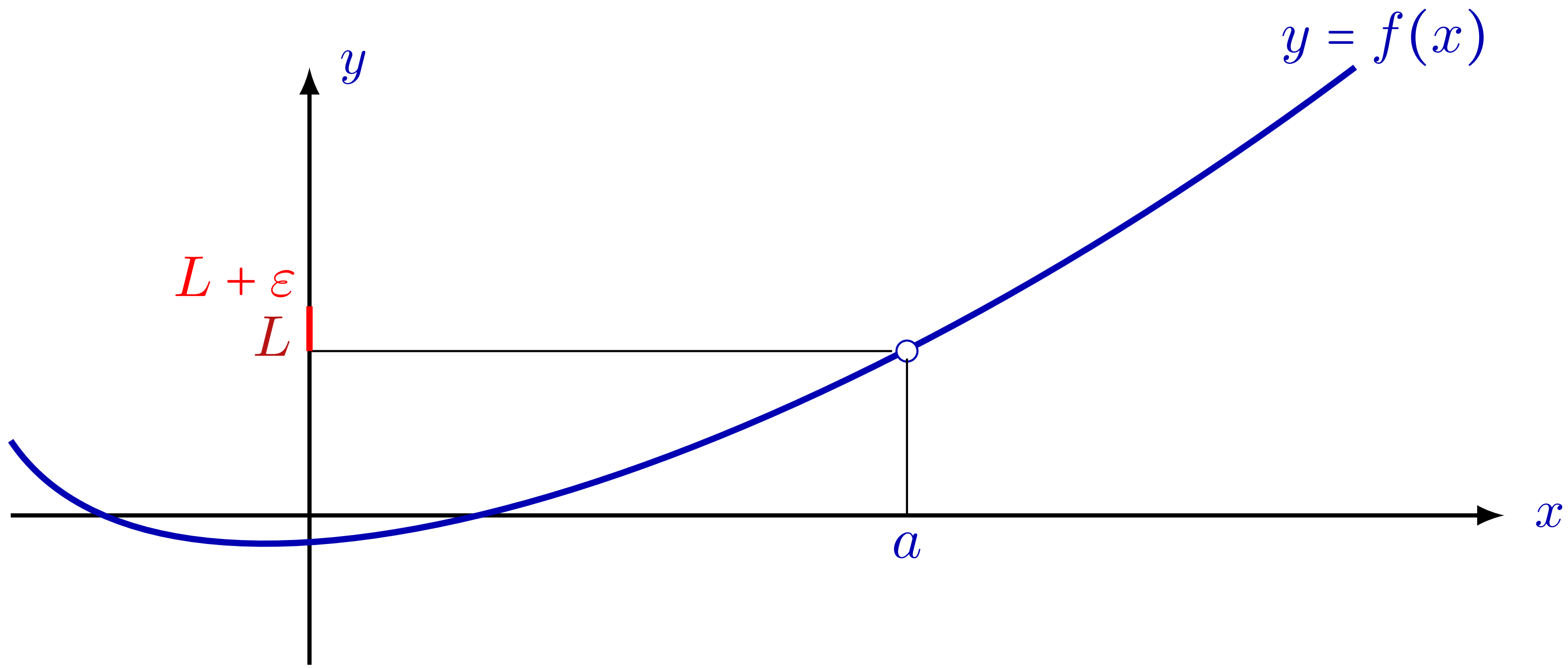

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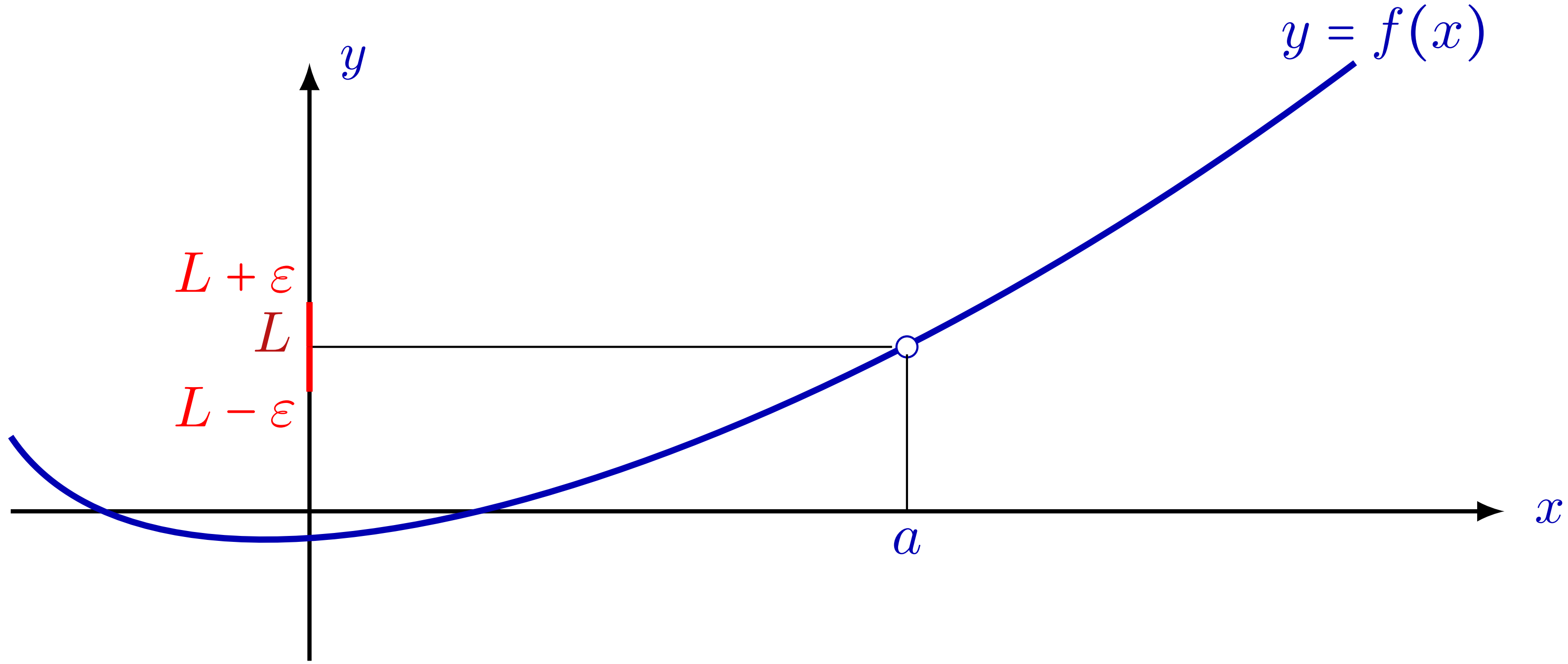

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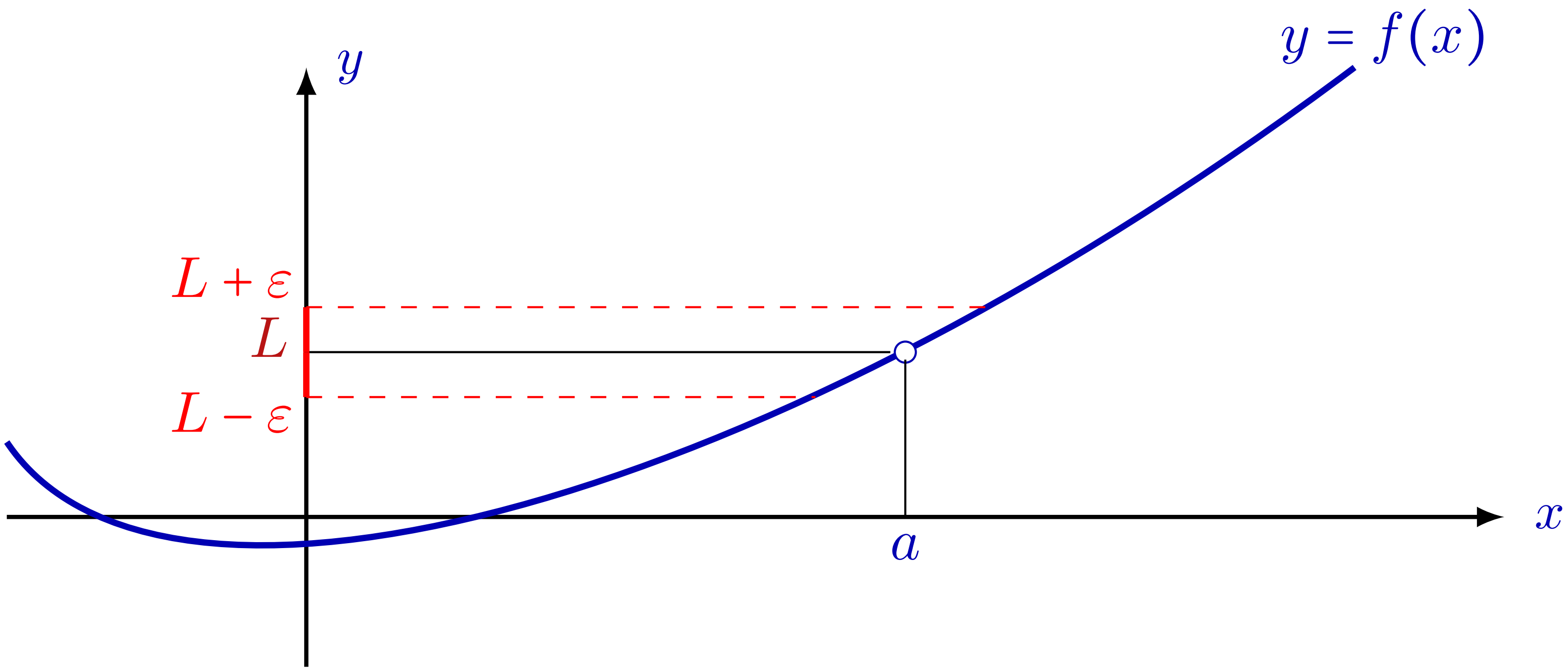

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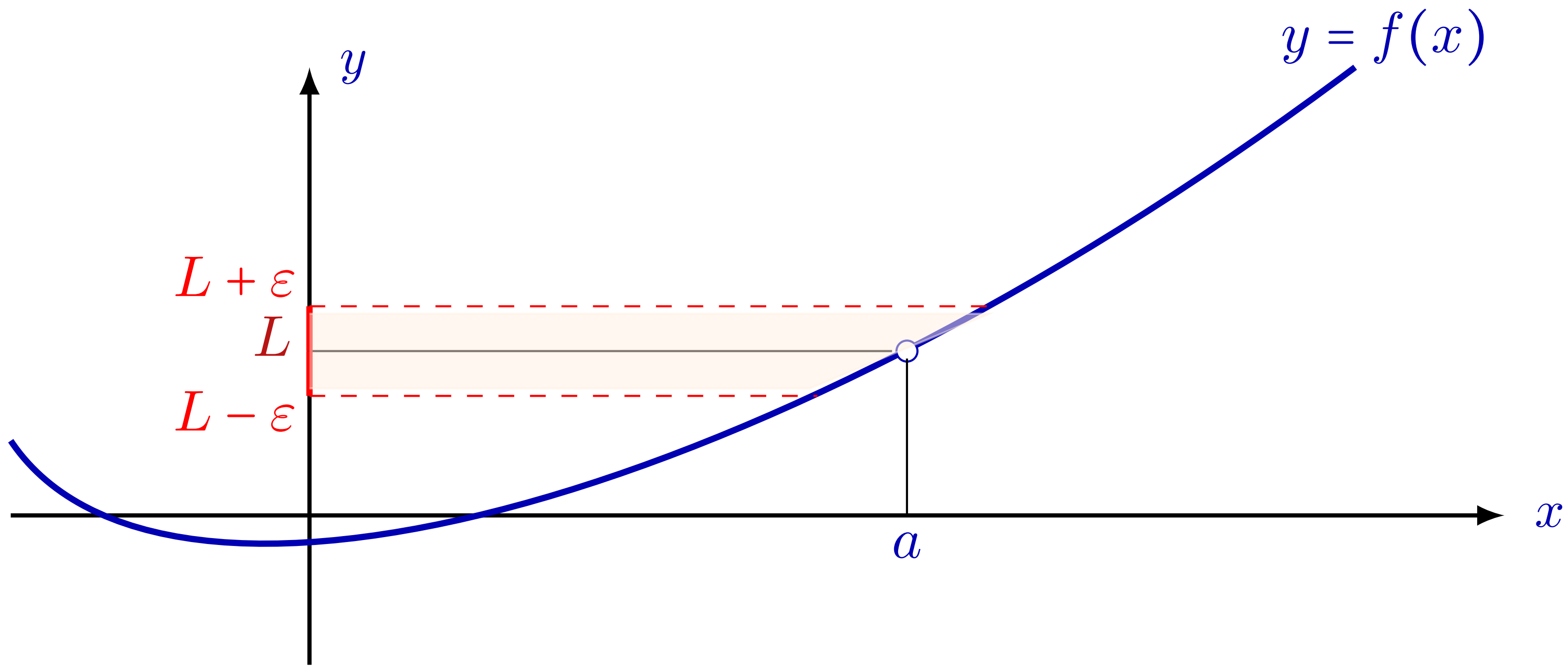

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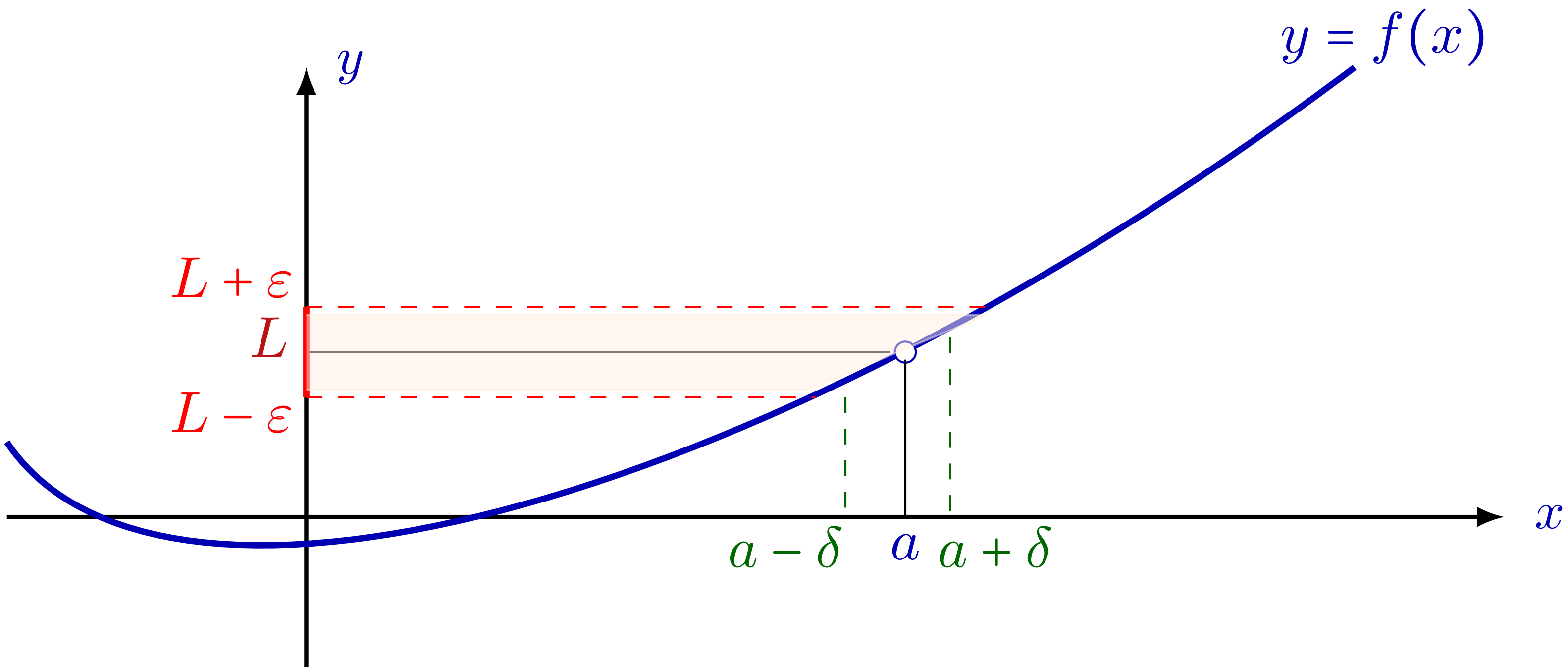

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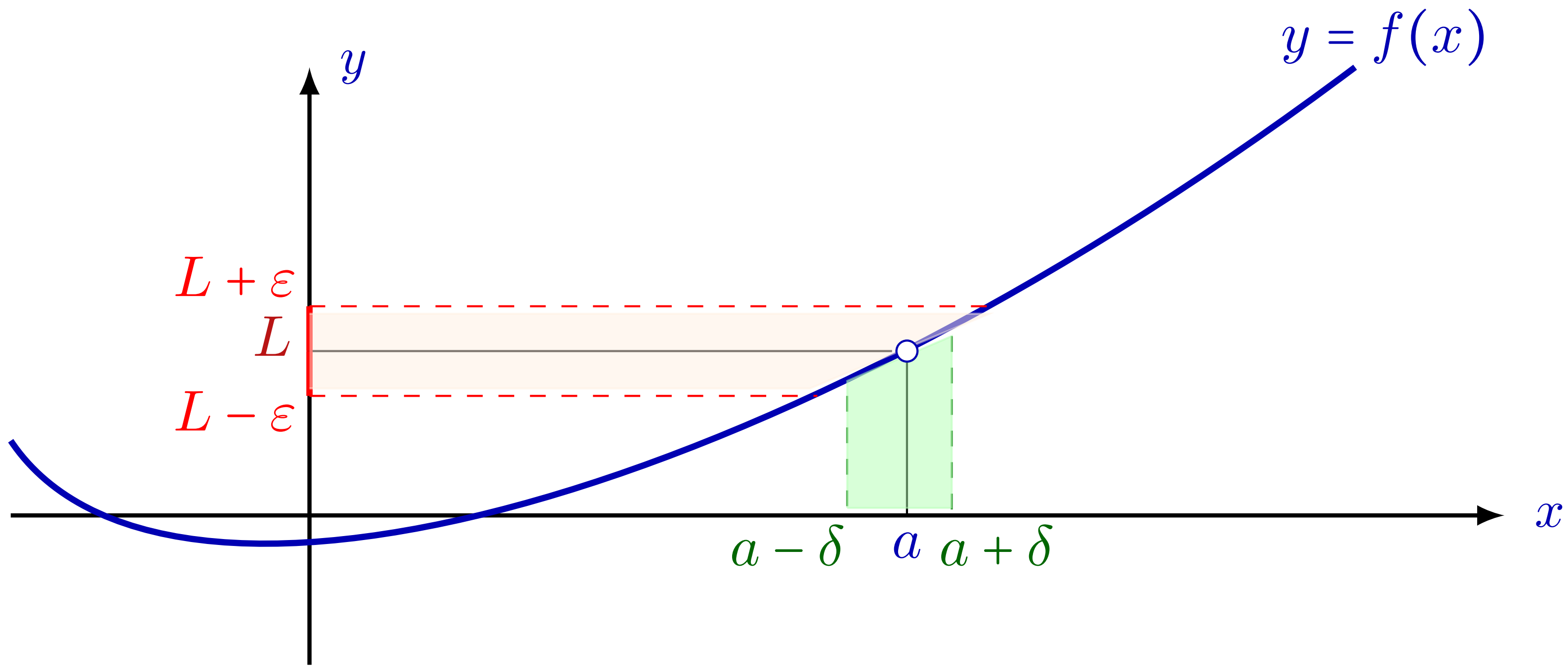

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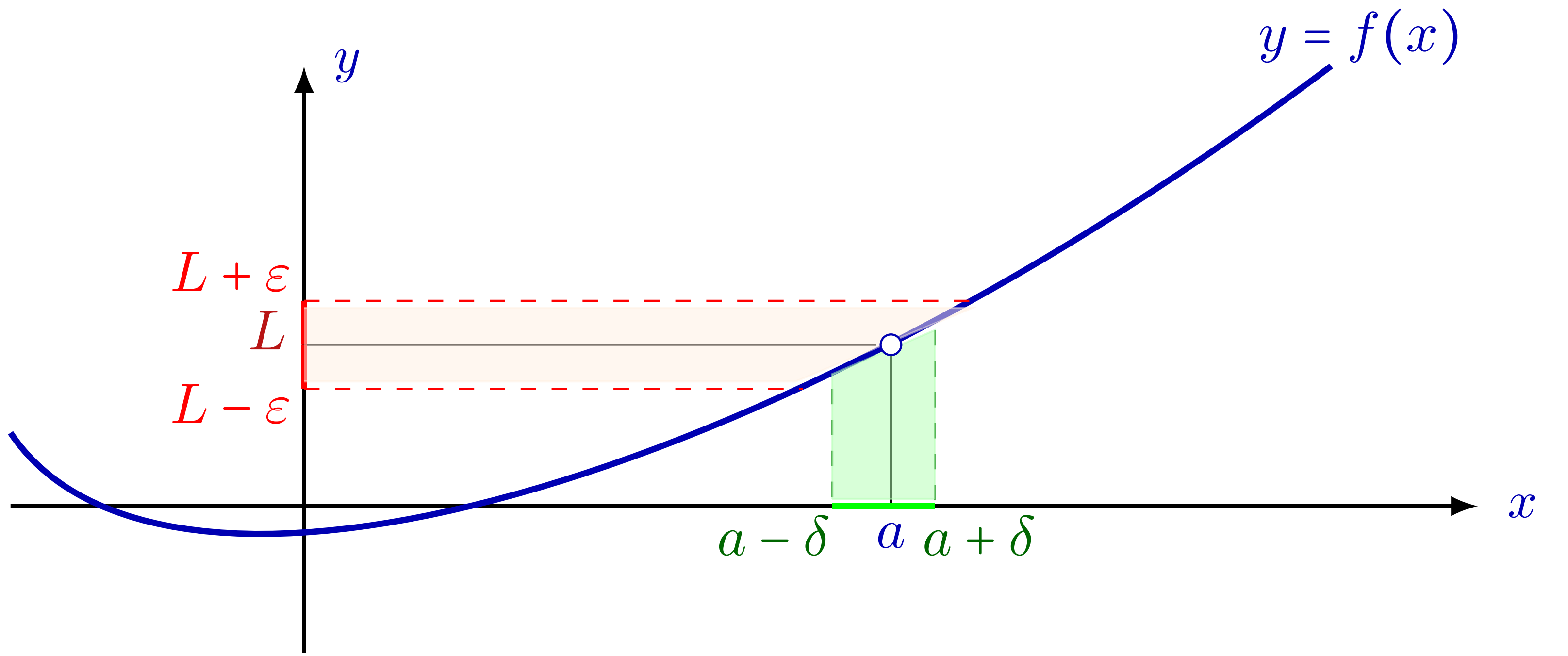

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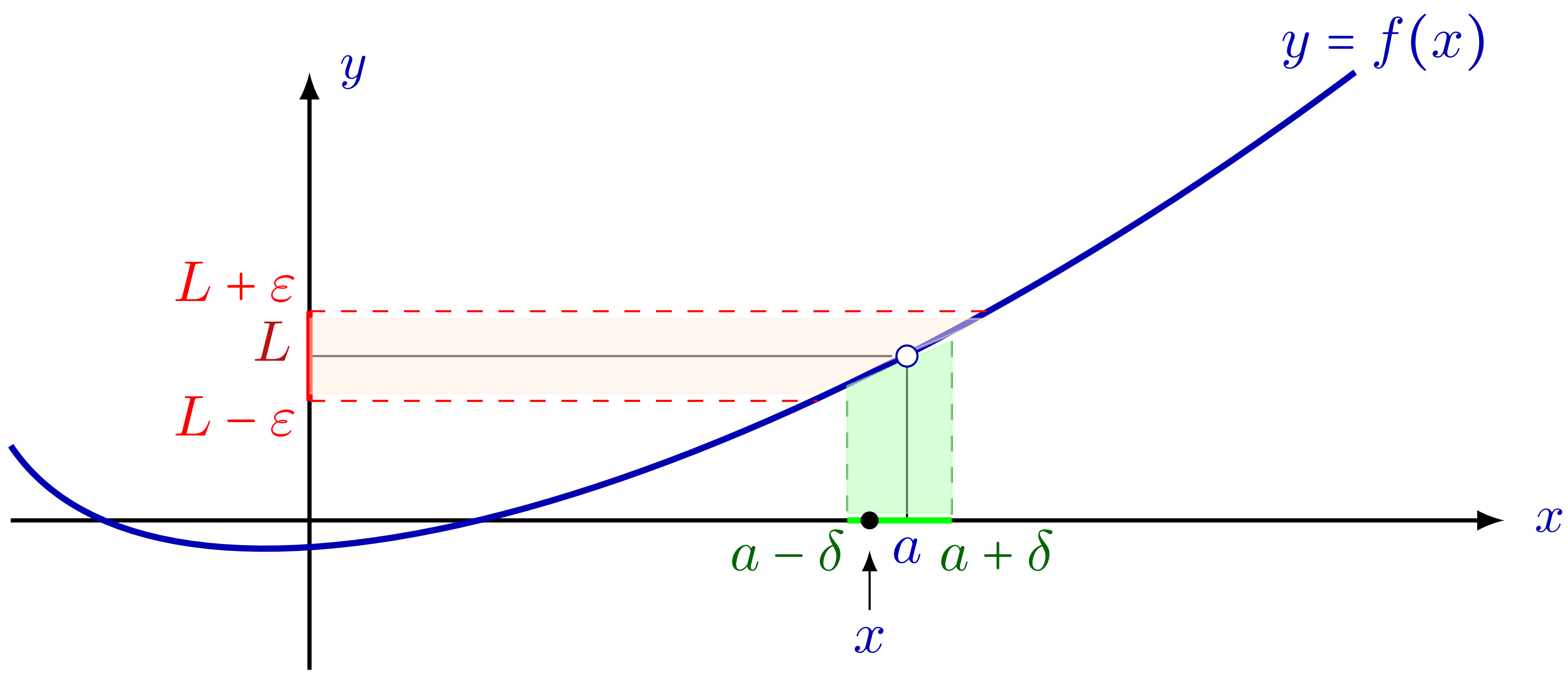

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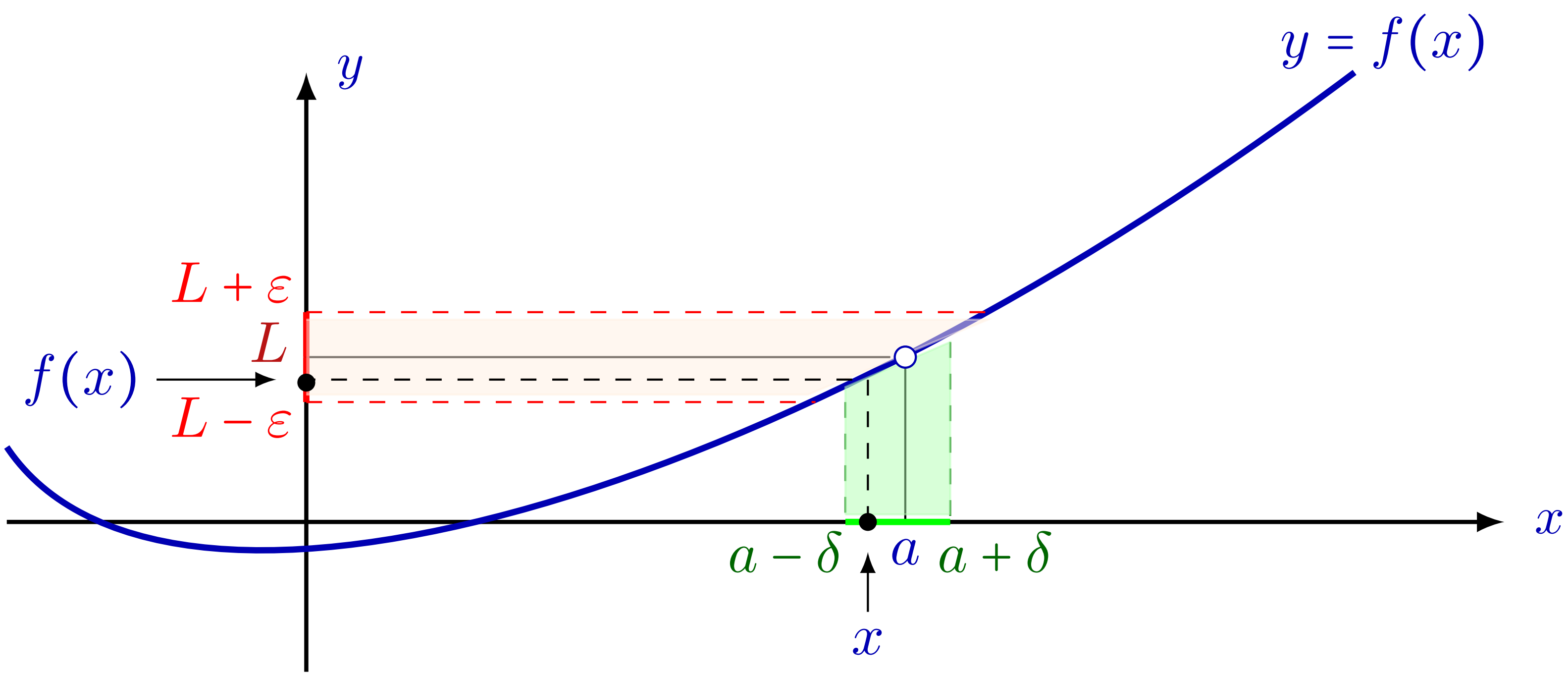

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The properties are called the **axioms** of a ring.

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Important: To prove that each of the listed above objects is a ring, we have to verify all ring axioms.

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Next come examples and **detailed** reading of proofs.

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As the first step towards classifying the lengths which can be constructed by straightedge and compass, this chapter introduces the concept of an algebraic number. Each such number will satisfy many polynomial equations and our immediate goal is to choose the simplest one.

A number $\alpha \in \mathbb{C}$ is said to be *algebraic over a field* $\mathbb{F} \subseteq \mathbb{C}$ if there exists a nonzero polynomial $f(x) \in \mathbb{F}[x]$ such that α is a zero of $f(x)$.

For each field $\mathbb{F}$, every number α in $\mathbb{F}$ is algebraic over $\mathbb{F}$ because α is a zero of the polynomial $f(x) = x - \alpha \in \mathbb{F}[x]$. This implies that e and π are algebraic over $\mathbb{R}$, though they are not algebraic over $\mathbb{Q}$ as we will prove later.

Let us read!

The number $\sqrt{2}$ is algebraic over $\mathbb{Q}$ because it is zero of the polynomial $f(x) = x^2 - 2$, which is nonzero and has coefficients in $\mathbb{Q}$.

In order to show that a number is algebraic, we look for a suitable polynomial having that number as zero. Try to prove that $1 + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

It is useful to be able to recognize the definition of "algebraic over a field $\mathbb{F}$" when it appears in different guises: a number $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ if and only if there is a positive integer n such that $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n\}$ are linearly dependent over $\mathbb{F}$.

Indeed, if $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ then there exists a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients $a_0, a_1, \ldots, a_n$ all belong to $\mathbb{F}$, at least one of these coefficients is nonzero, and $f(\alpha) = 0$, that is

$$a_0 + a_1\alpha + a_2\alpha^2 \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n = 0. \qquad (*)$$

Let us read!

Since $\mathbb{F}$ is a subfield of $\mathbb{C}$, we can regard $\mathbb{C}$ as a vector space over $\mathbb{F}$. The numbers $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n$ are all elements in $\mathbb{C}$, and hence can be regarded as vectors in the vector space $\mathbb{C}$ over $\mathbb{F}$.

The coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$, on the other hand, are all in $\mathbb{F}$ so we can regard them as scalars. Thus, the equality $(*)$ can be interpreted as a linear dependence of vectors $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n$ in $\mathbb{C}$.

You will often meet the terms "algebraic number" and "transcendental number" where no field is specified. In such cases the field is taken to be $\mathbb{Q}$. We formalize this as follows.

A complex number is said to be an *algebraic number* if it is algebraic over $\mathbb{Q}$; a *transcendental number* if it is not algebraic over $\mathbb{Q}$.