Lecture 6

Maps

The set of all maps $X \to Y$

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That's why $\mathcal{M}ap(X, Y)$ is often denoted by Y^X.

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$1 \in X$, $1 \notin \mathcal{P}(X)$, $1 \not\subset \mathcal{P}(X)$

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How large is the power set?

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Theorem.

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Another notation for $\mathcal{P}(X)$ is 2^X.

Characteristic function of a set

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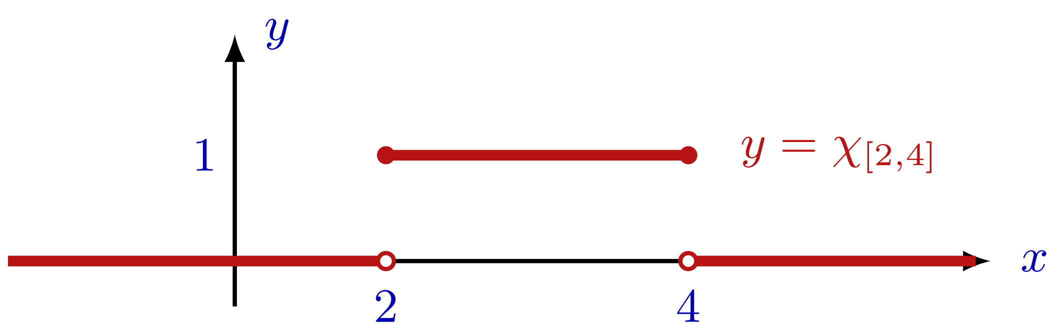

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Corollary. $|\mathcal{P}(X)| = |\mathcal{M}ap\,(X,\{0,1\})| = 2^{|X|}$, as we already know.

Working with power set

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And the half of the proof is done!

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Proof. Let $A \subset B$. We have to prove that $\mathcal{P}(A) \subset \mathcal{P}(B)$.

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Therefore, by the transitivity of inclusion, $X \subset B$. So $X \in \mathcal{P}(B)$.

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Overall, $A \subset B \iff \mathcal{P}(A) \subset \mathcal{P}(B)$ □

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Exercise 2. Formulate and prove a similar identity for $(g \circ f)^*$.

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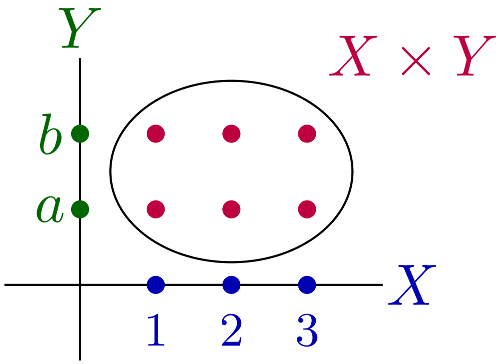

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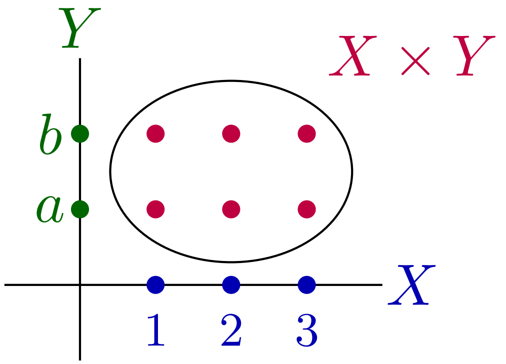

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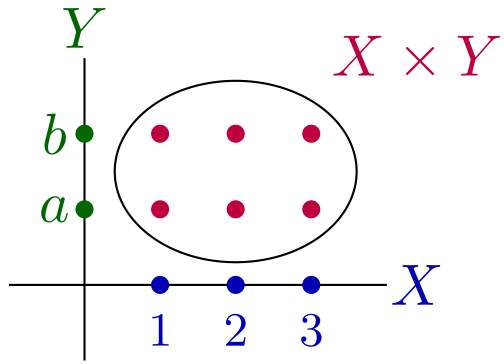

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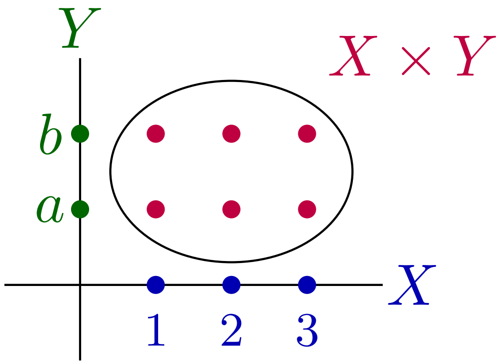

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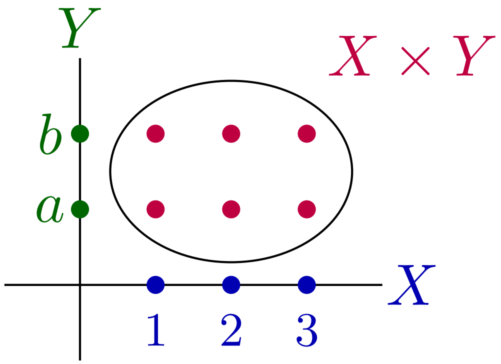

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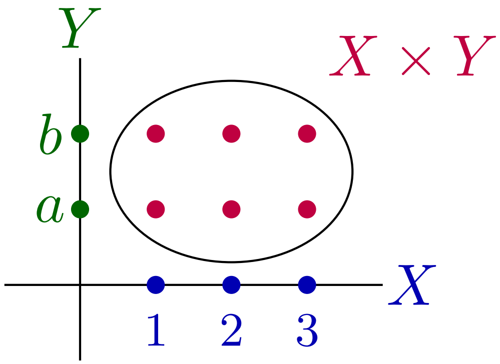

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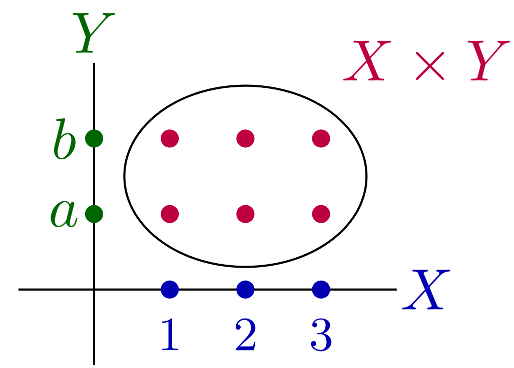
Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Then
$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.



However, there is a **natural bijection** $X \times Y \to Y \times X : (x, y) \mapsto (y, x)$.
In this sense, the Cartesian product is **commutative**.

Theorem. If X has p elements, and Y has q elements, then $X \times Y$ has pq elements.

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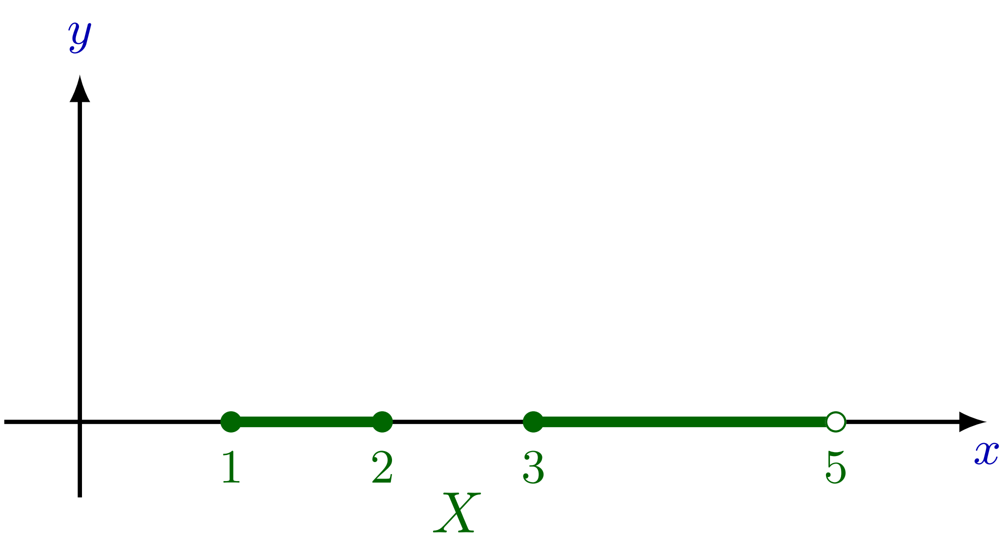

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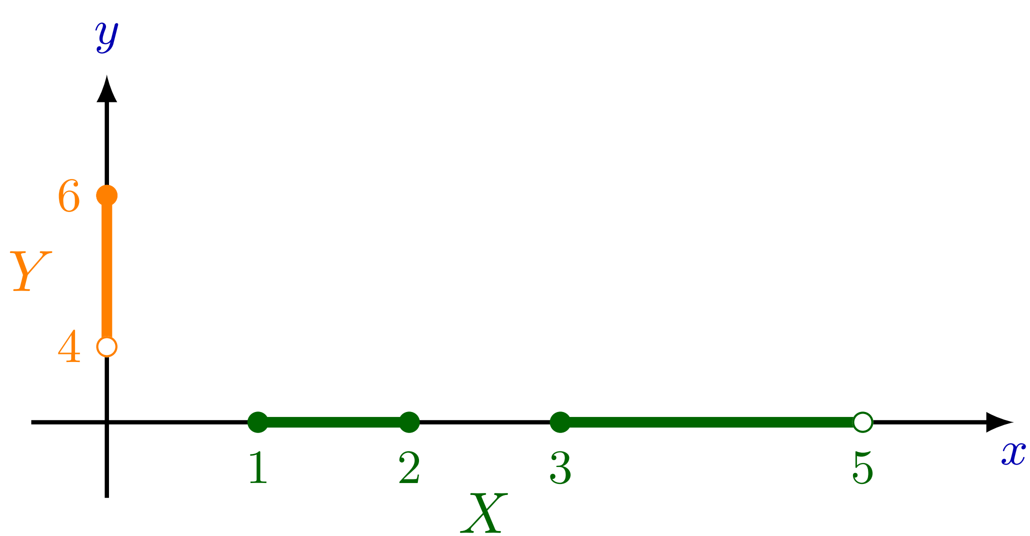

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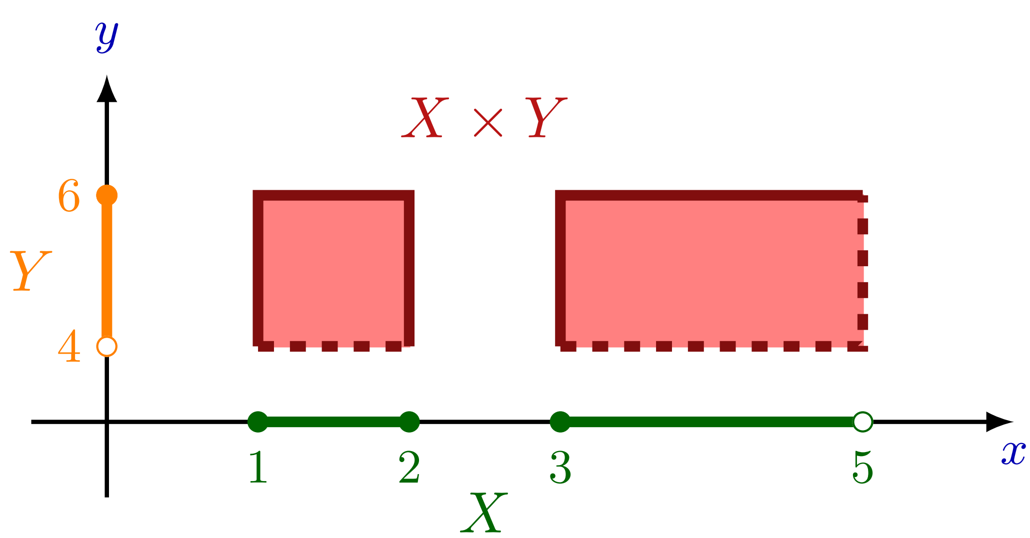

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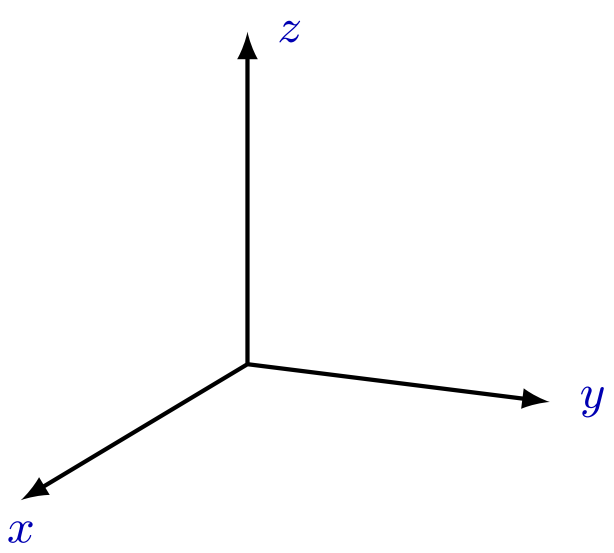

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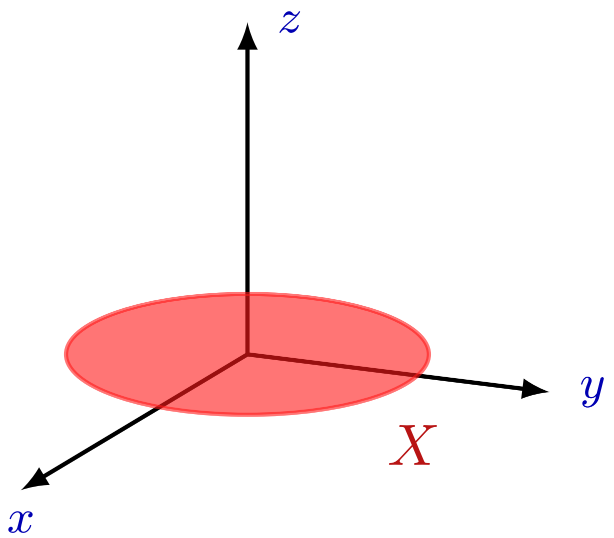

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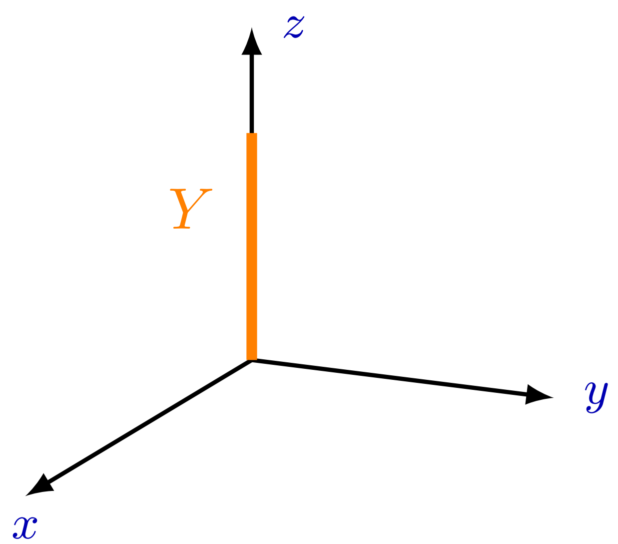

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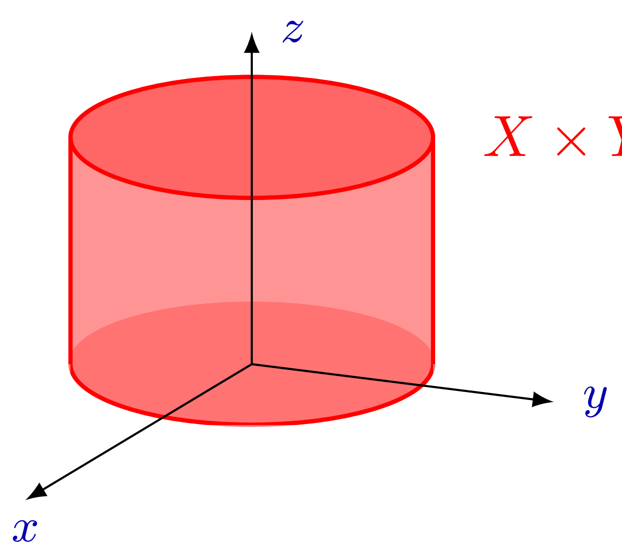

$X \times Y$ is a solid cylinder in $\mathbb{R}^3$

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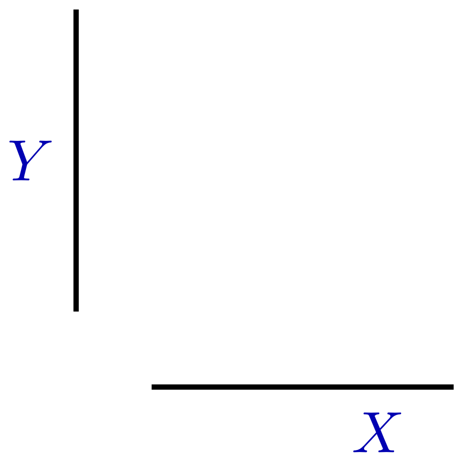

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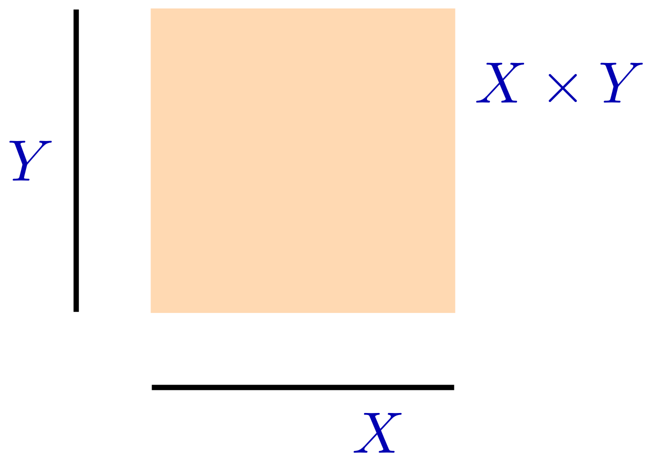

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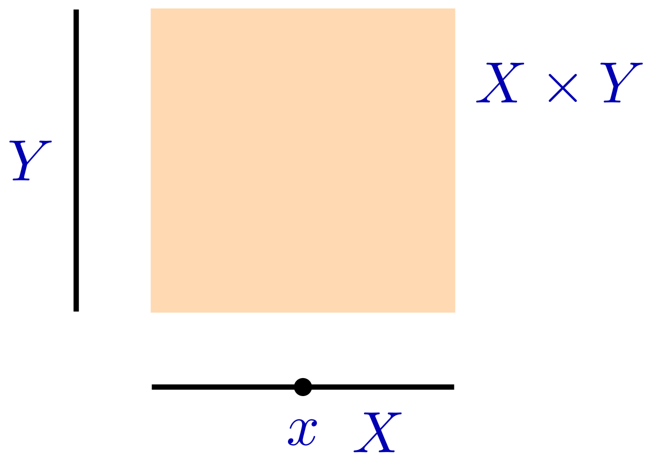

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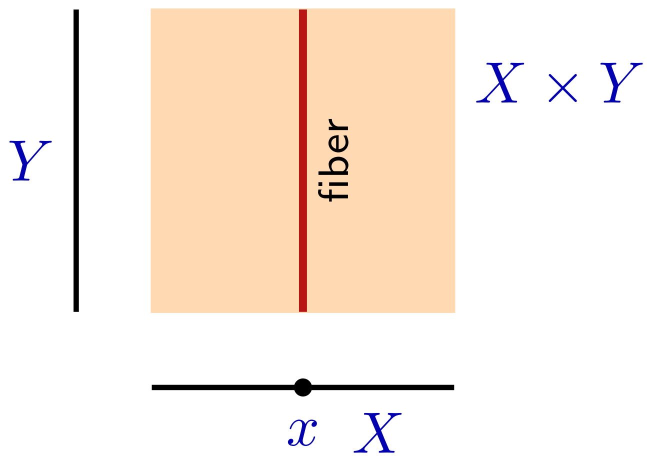

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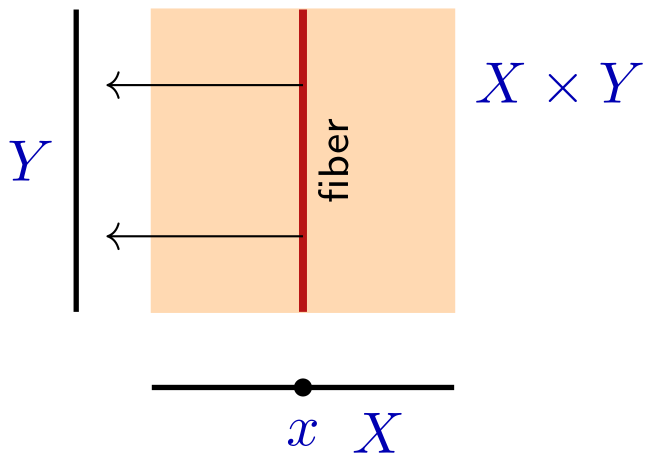

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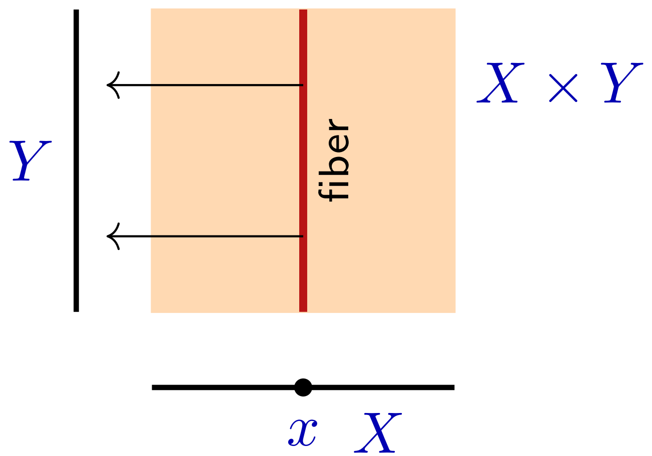

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Similarly, $\text{proj}_X\big|_{X\times\{y\}} : X \times \{y\} \to X$ is a bijection.

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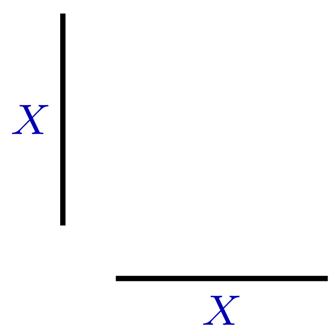

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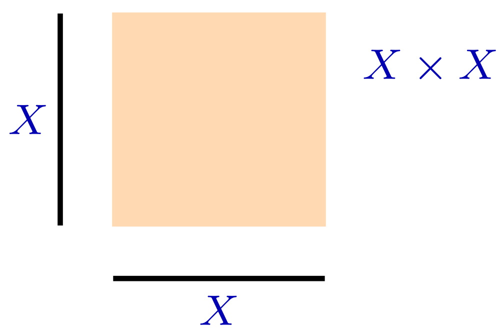

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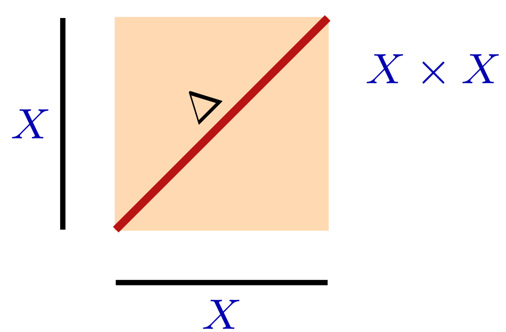

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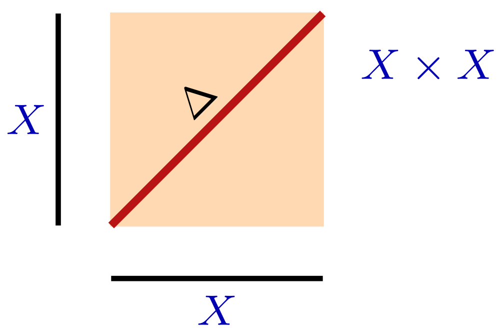

The diagonal is the image of $\text{id}_X \odot \text{id}_X$.

Graph of a map

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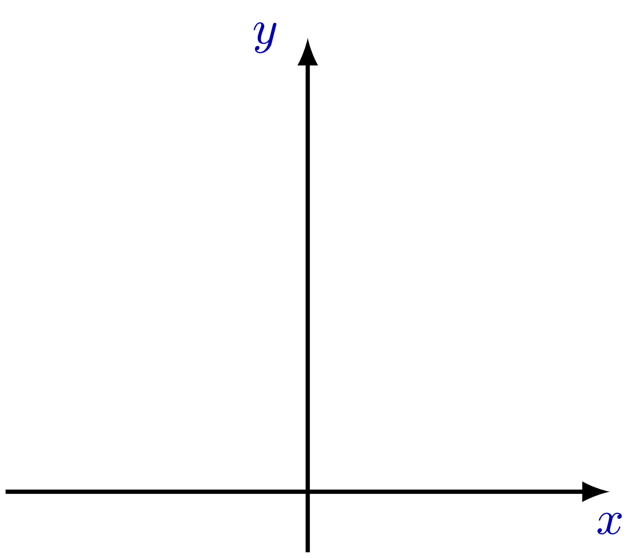

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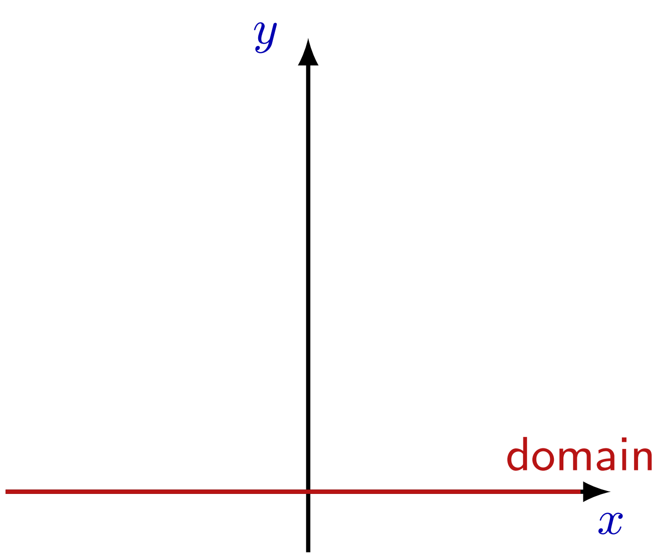

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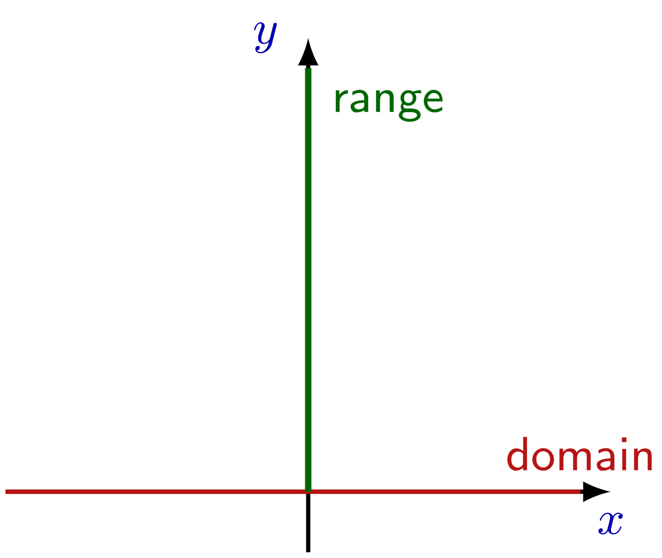

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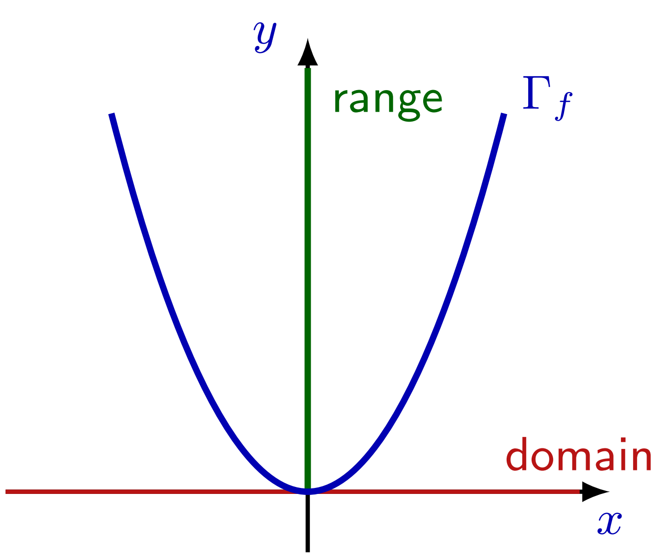

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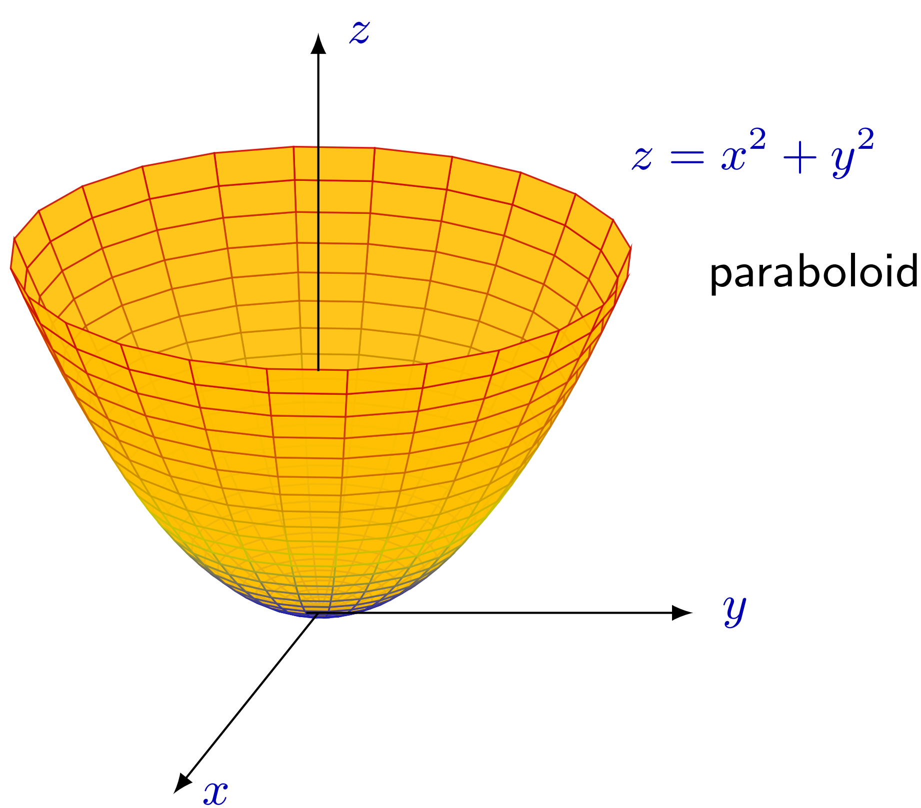

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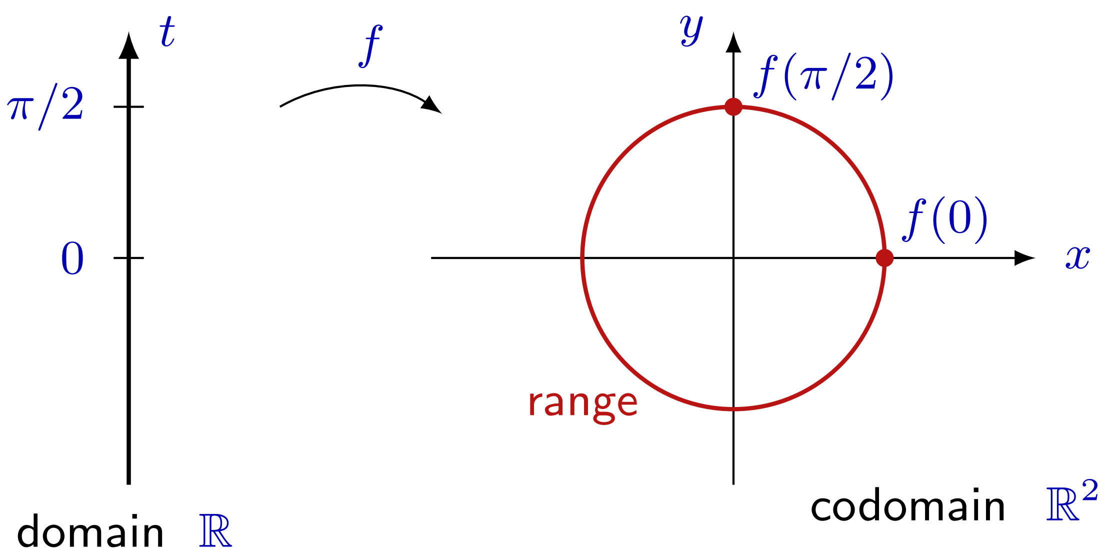

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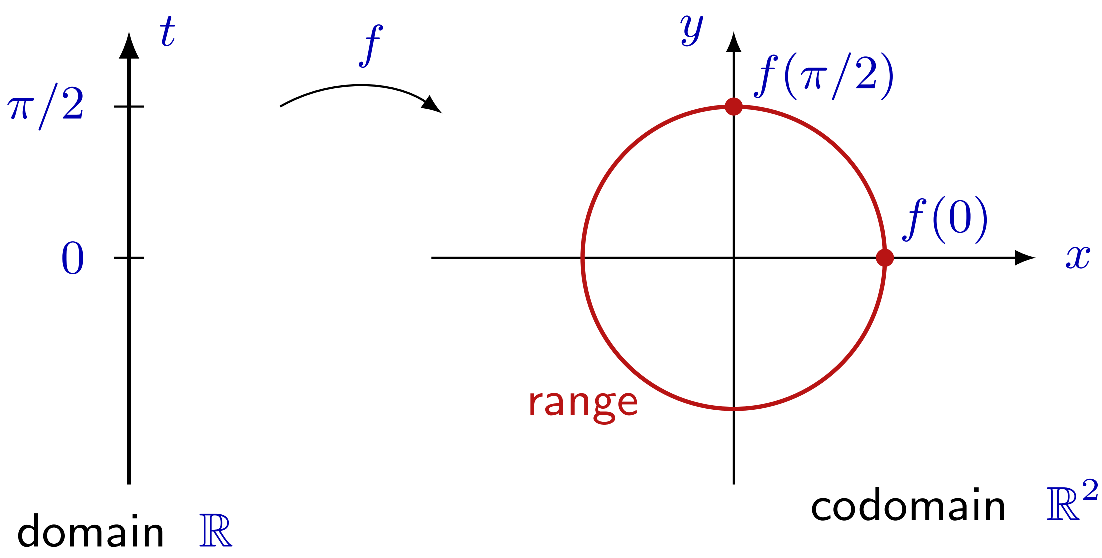

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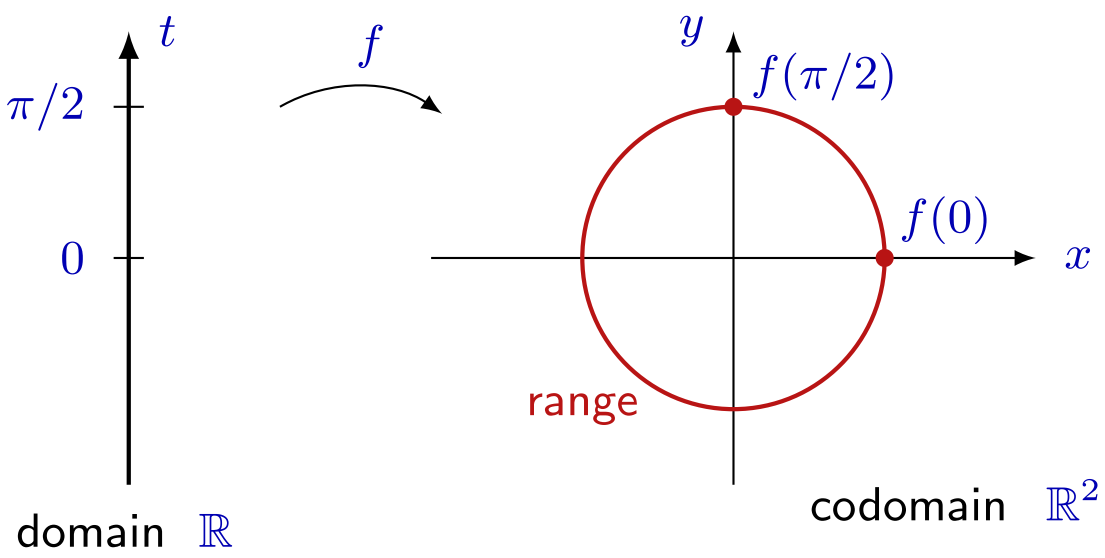

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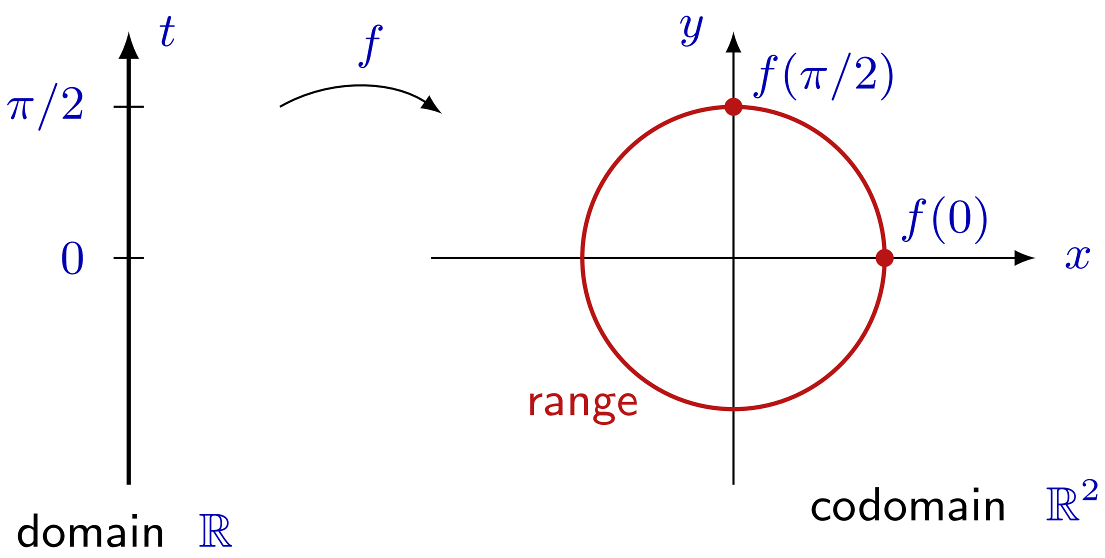

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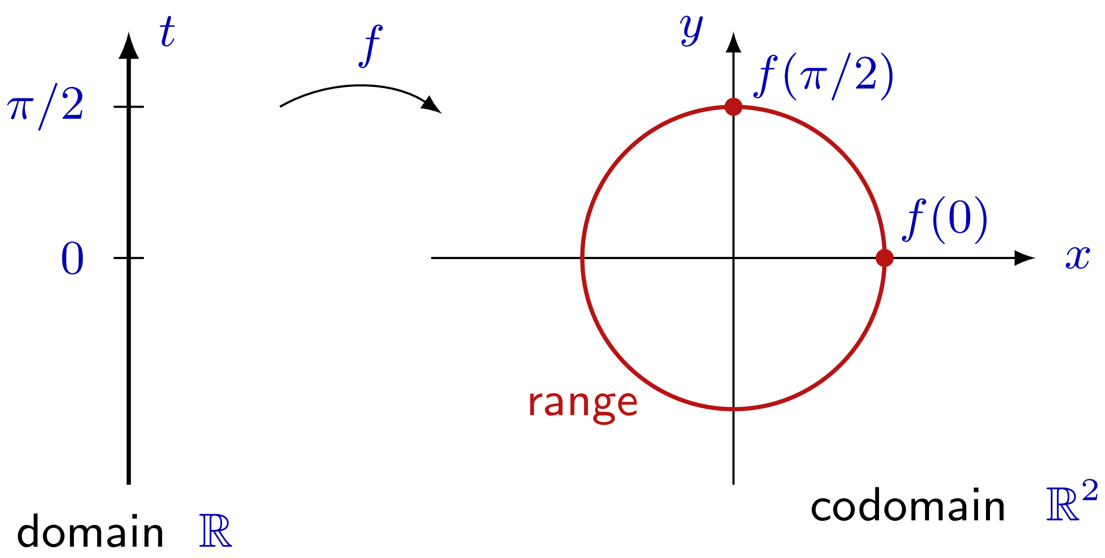

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Γ_f is a curve in $\mathbb{R}^3$. It is called helix.

Helix

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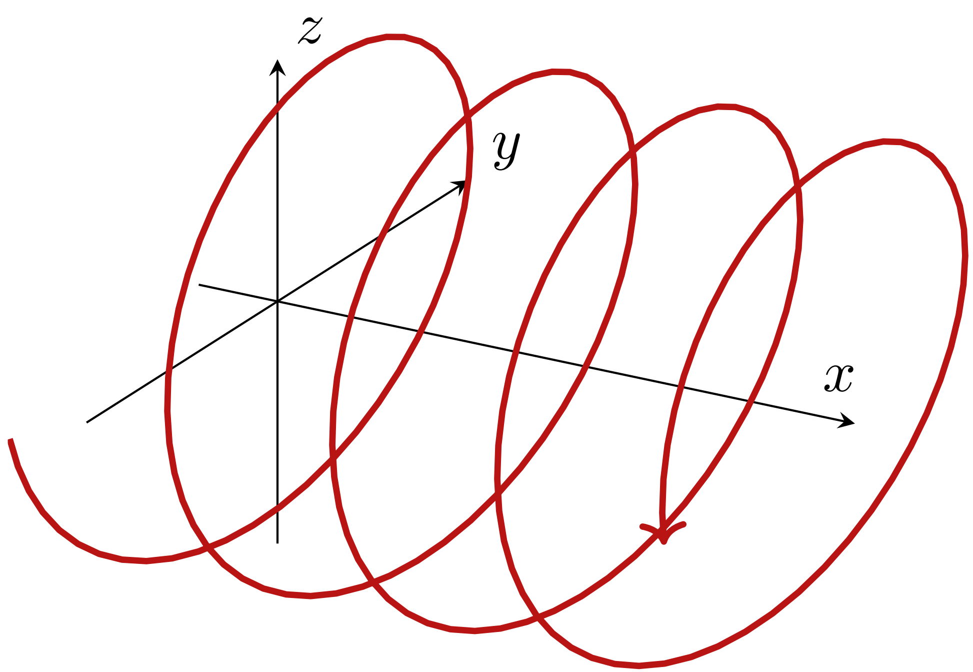

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The conditions $\mathbf{1, 2, 3}$ are called the **axioms** of metric space.

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2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

A pair (X, d) is called a **metric space**.

The conditions $\mathbf{1, 2, 3}$ are called the **axioms** of metric space.

Metric

Definition. A **metric** (or distance function) on a set X is a map

$$d : X \times X \to \mathbb{R}_{\geq 0} = [0, \infty)$$

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Euclidean metric on a line

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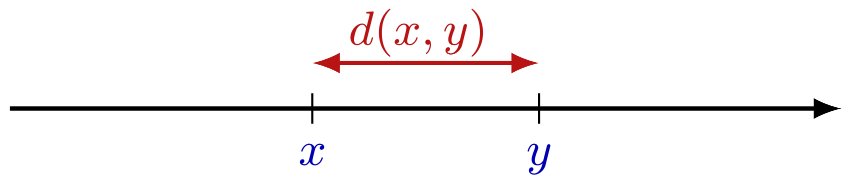

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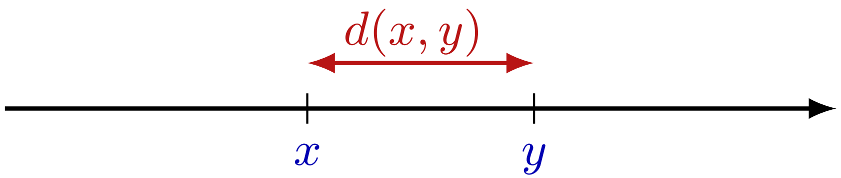

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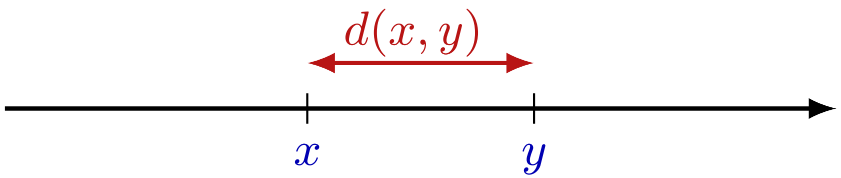

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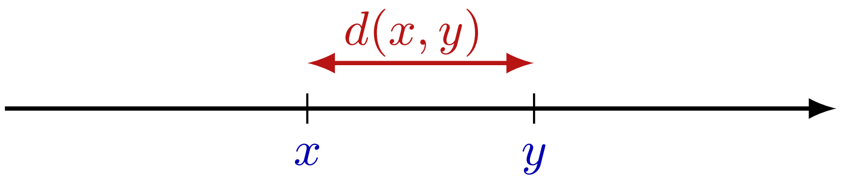

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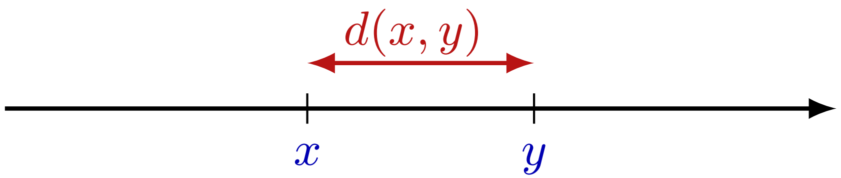

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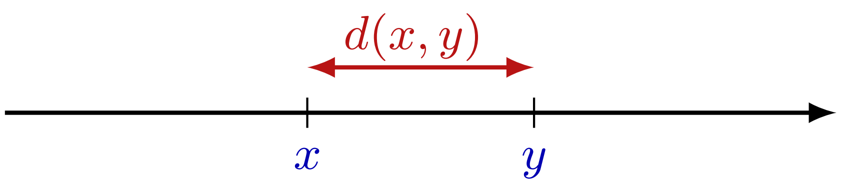

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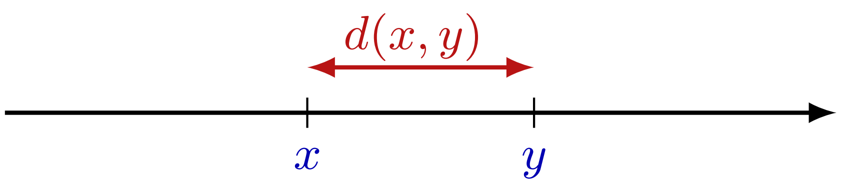

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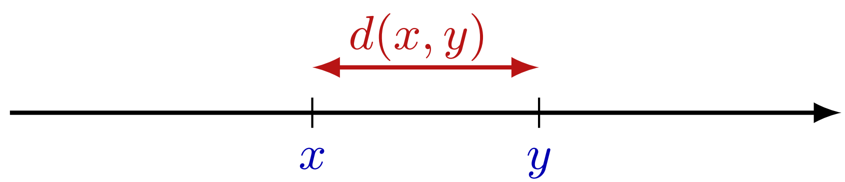

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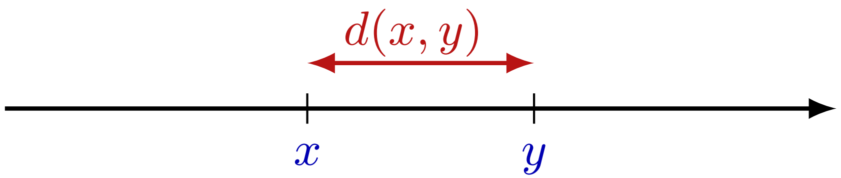

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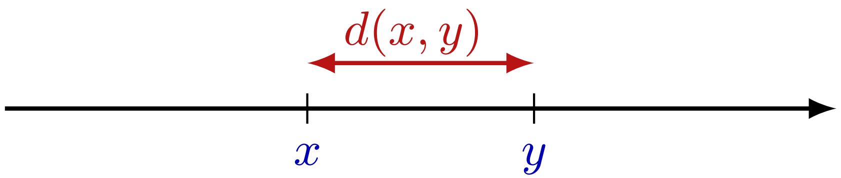

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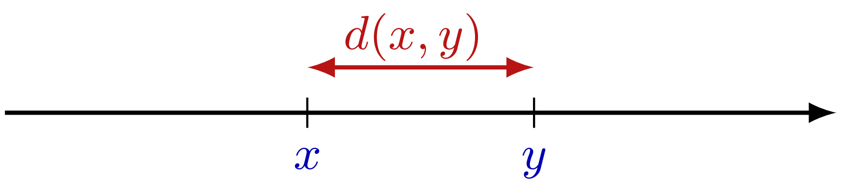

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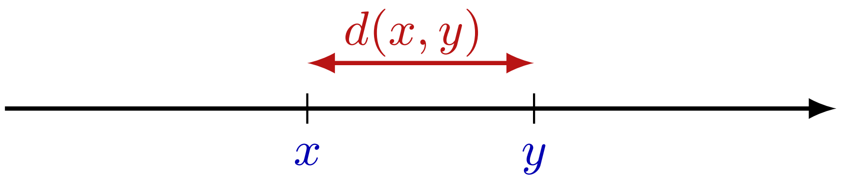

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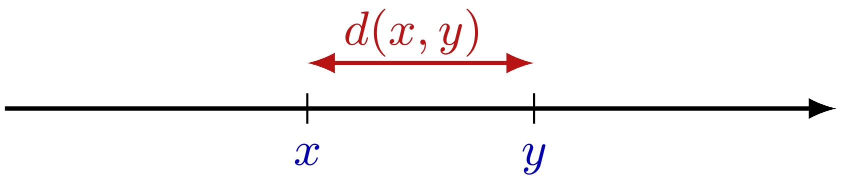

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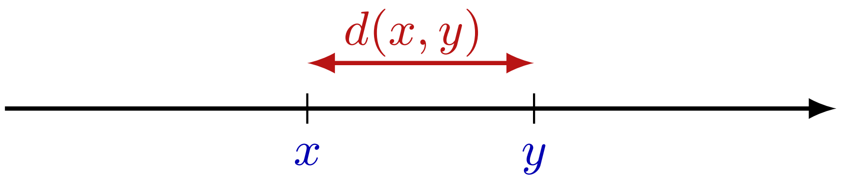

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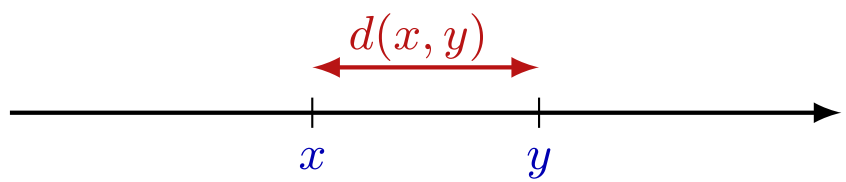

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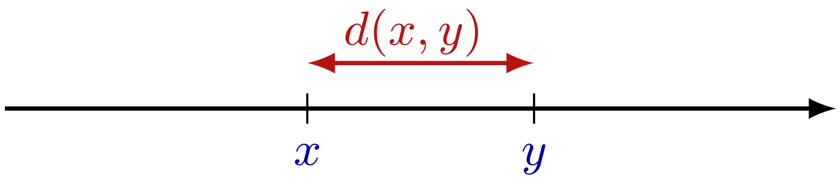

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Therefore, all axioms are satisfied and the map d is a metric.

Euclidean metric on a plane

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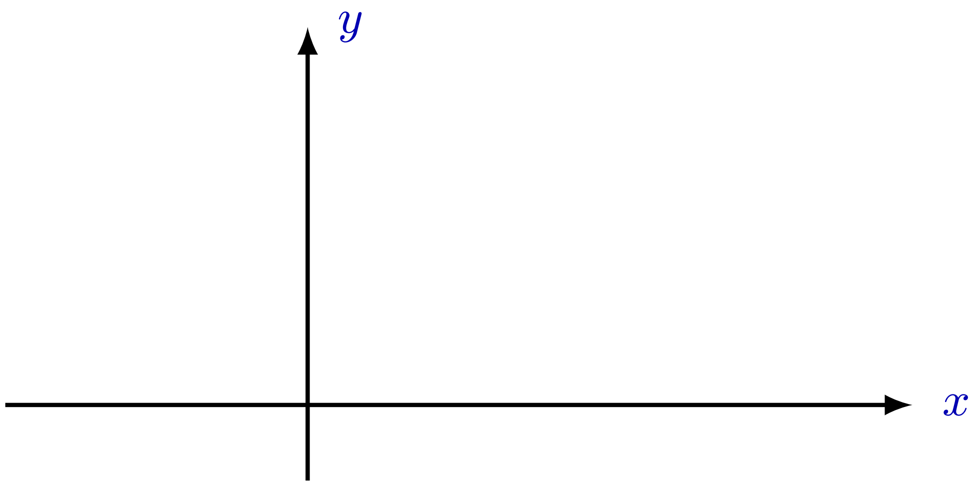

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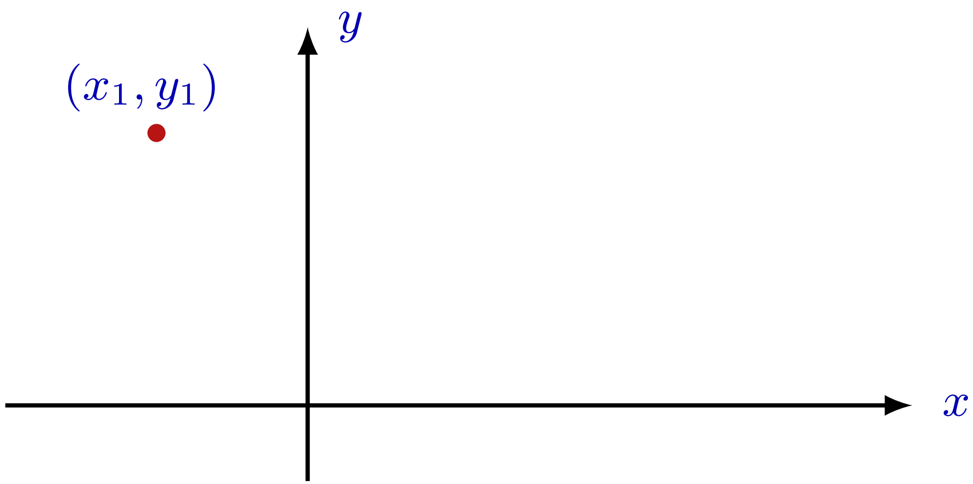

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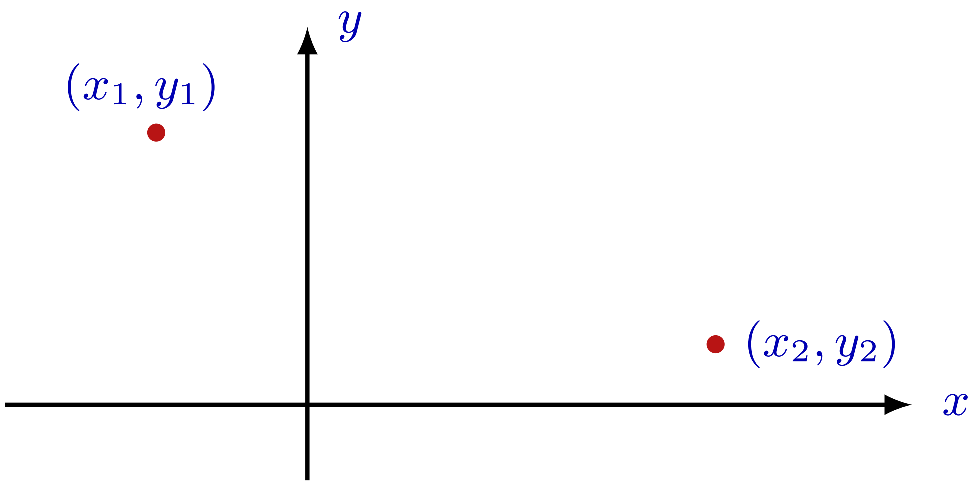

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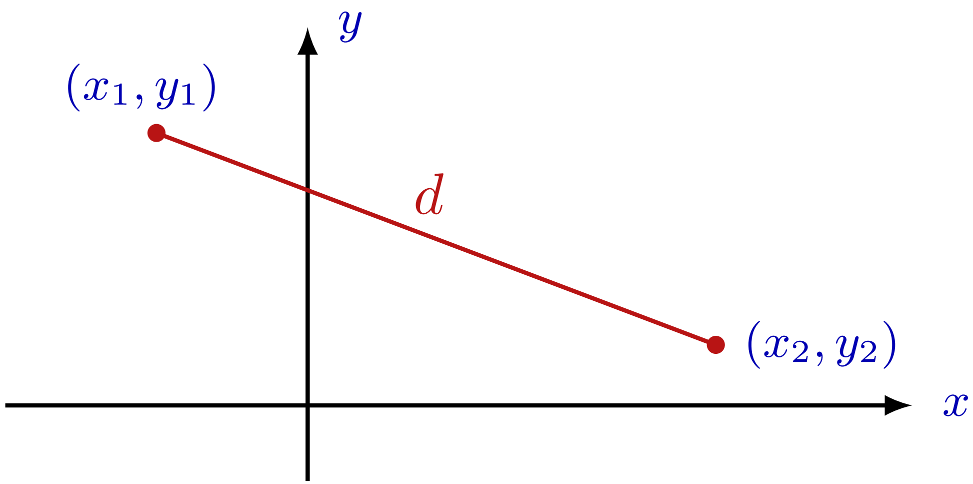

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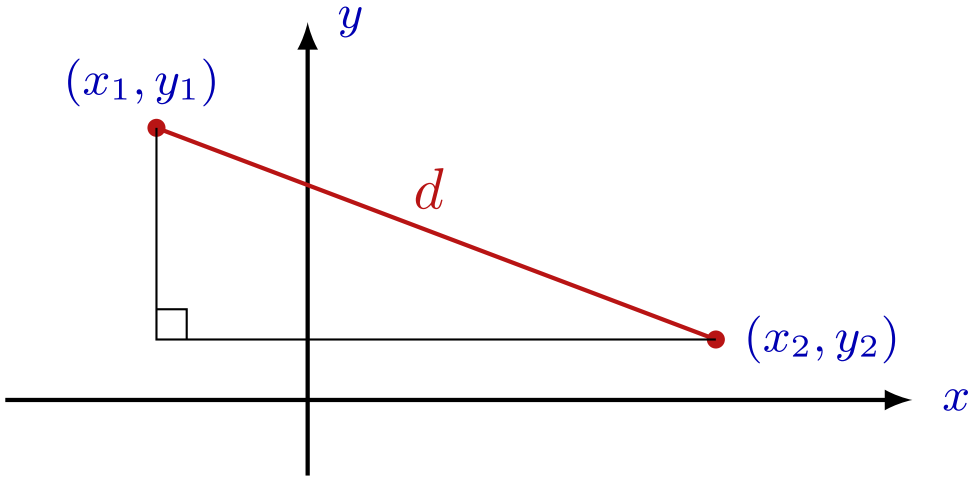

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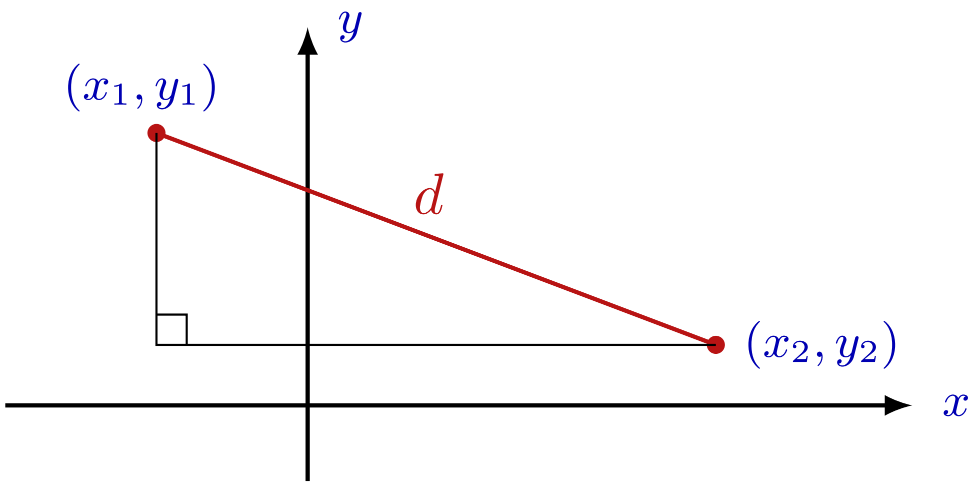

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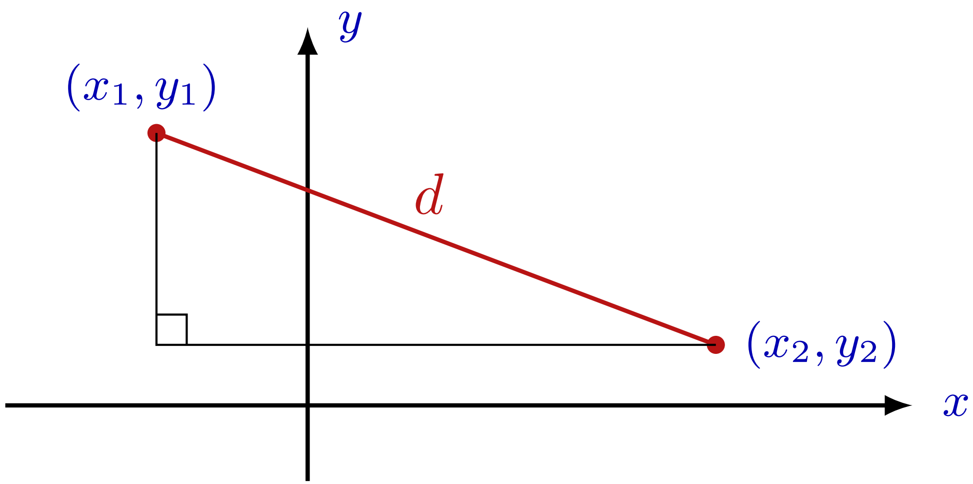

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Proof will be given in a course of Linear Algebra.

Taxi driver metric on a plane

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$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

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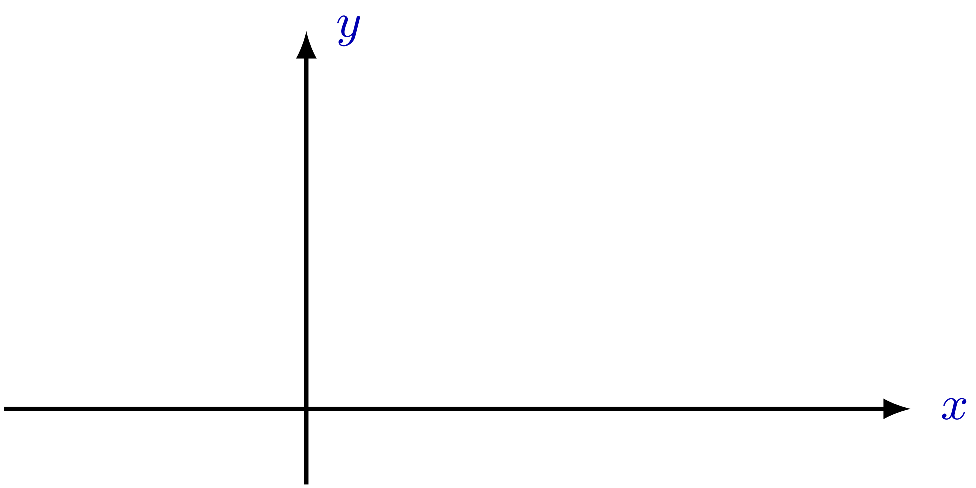

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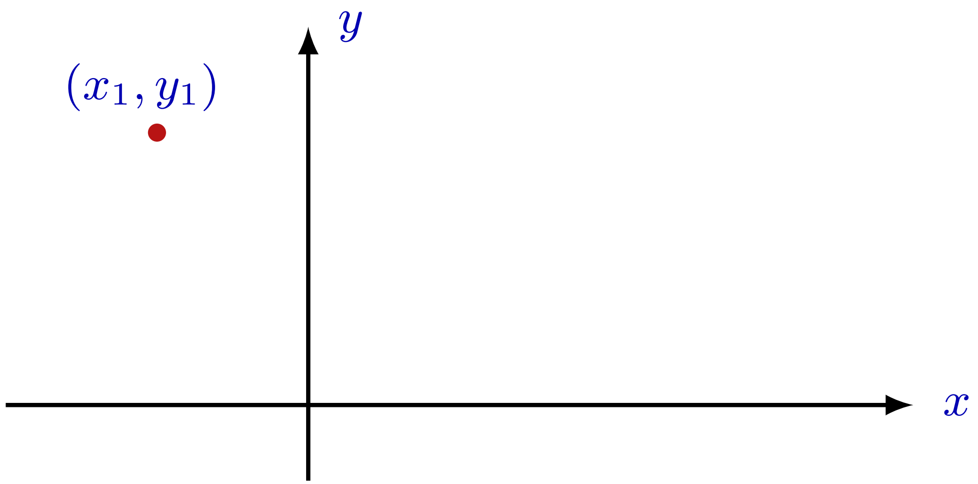

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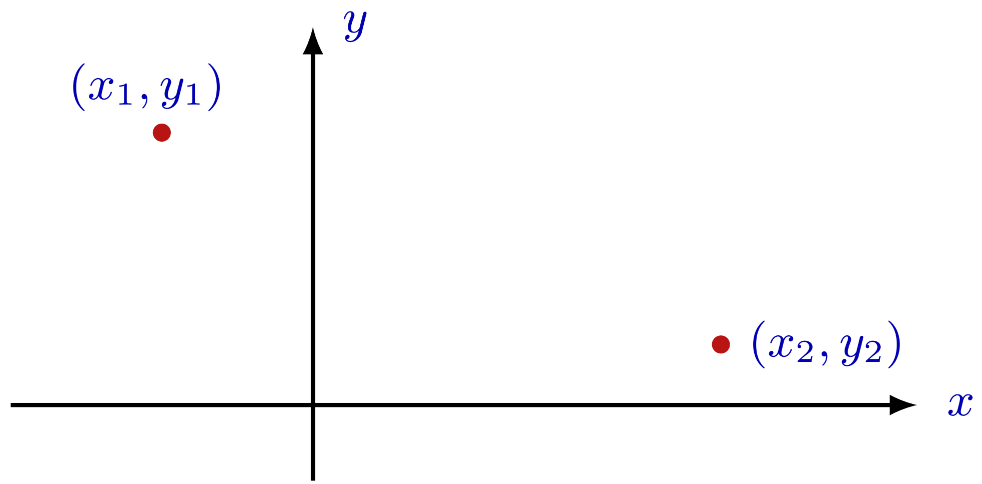

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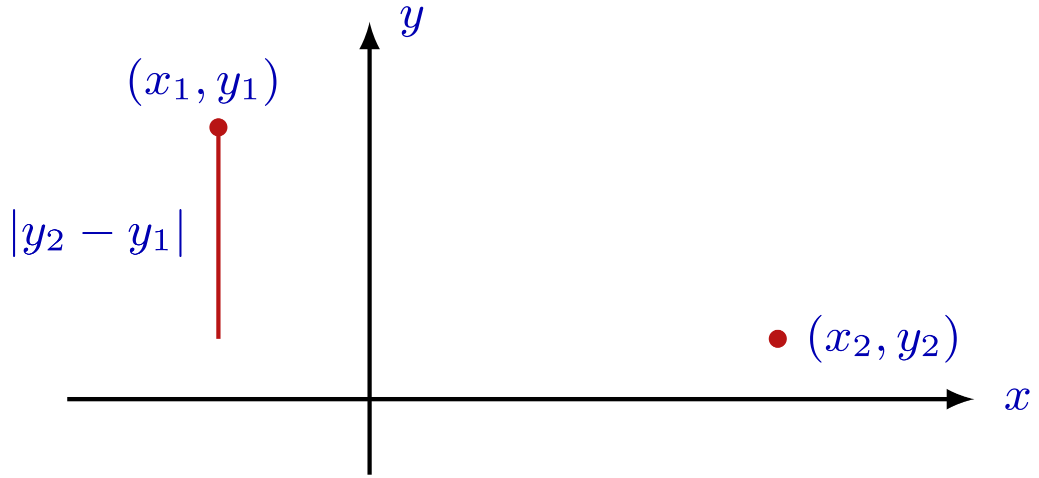

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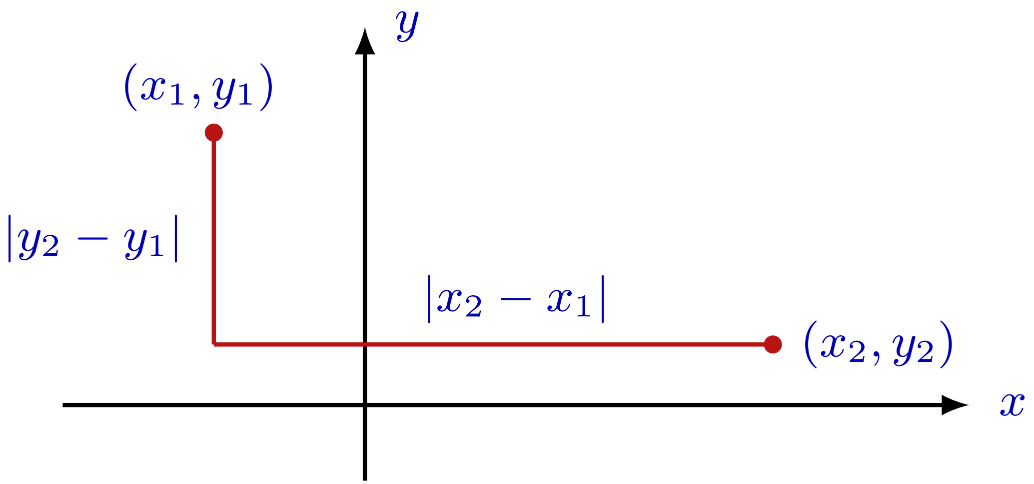

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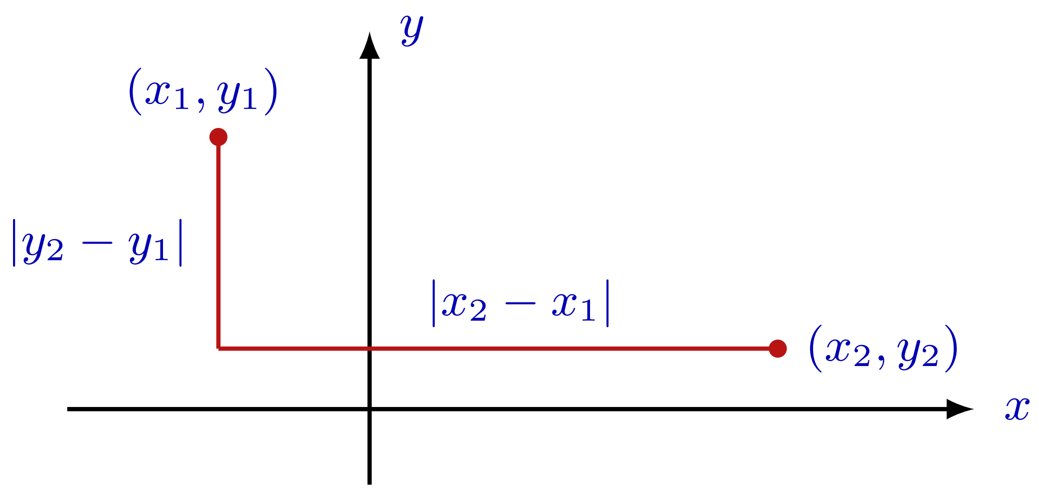
for any (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$



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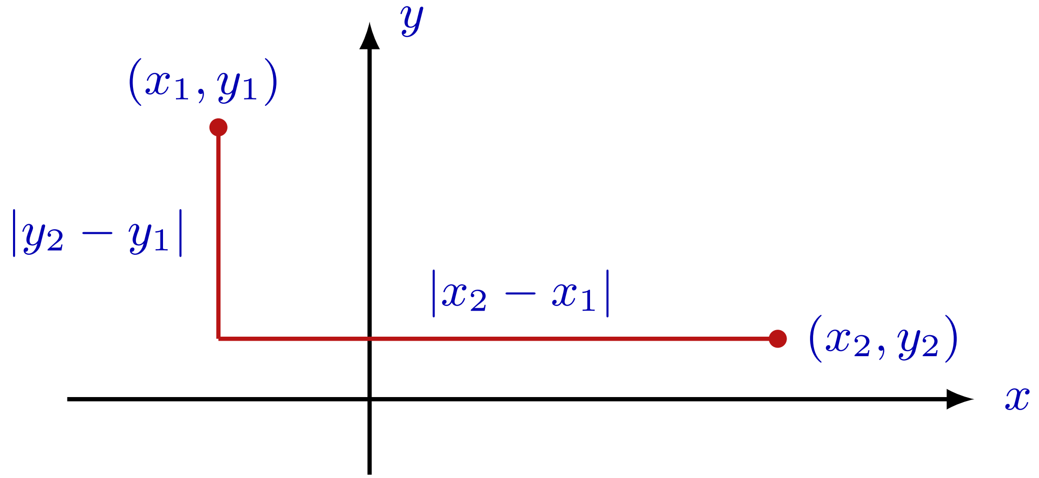

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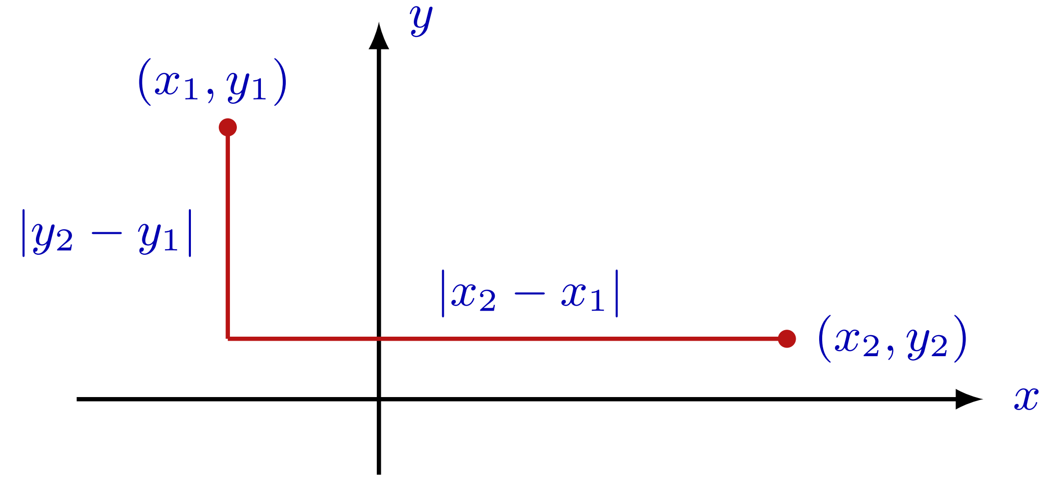

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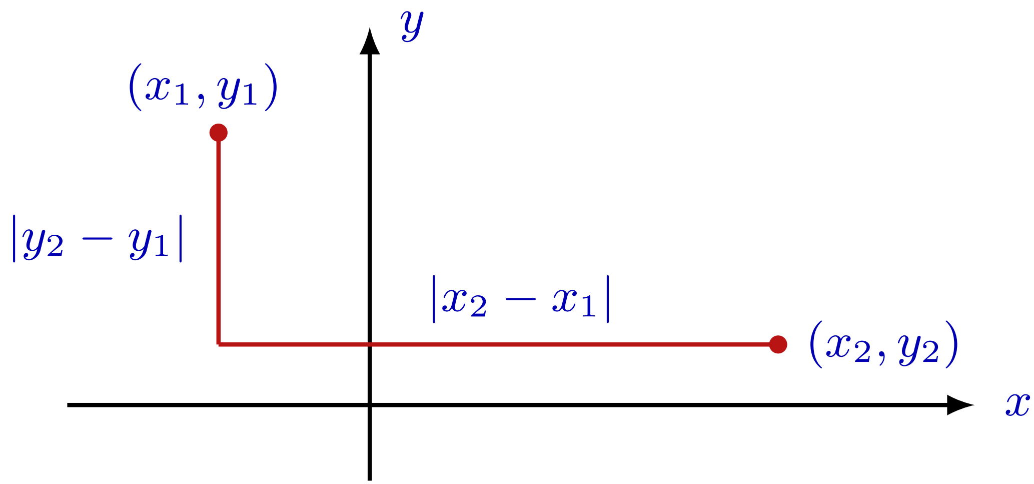

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The plane with Euclidean metric

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Relations

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We will deal mostly with **binary** relations on a **single** set.

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Let a set X have 3 elements.

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$\mathcal{P}(X \times X)$ is a huge set!

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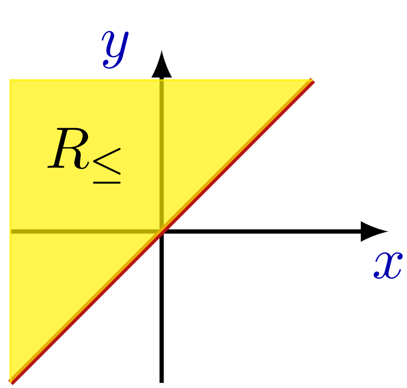

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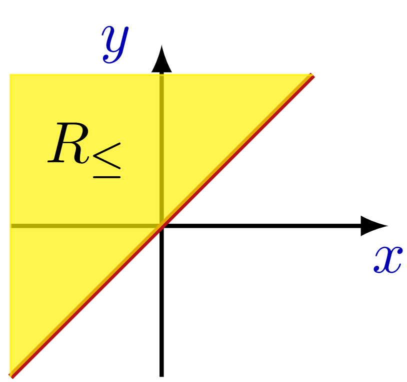

$$\forall x, y \in \mathbb{R} \quad \underbrace{(x,y) \in R_{\leq}}_{x \leq y} \text{ or } \underbrace{(y,x) \in R_{\leq}}_{y \leq x}\,.$$

Relation of inclusion

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Example 2.

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Inclusion $\subset$

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Relation of inclusion

Example 2. Let X be a set, and $\mathcal{P}(X)$ be its power set.

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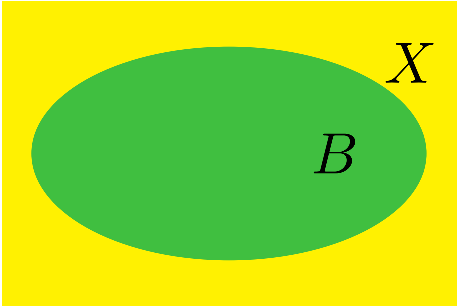

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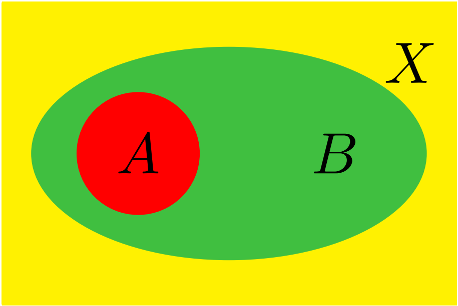

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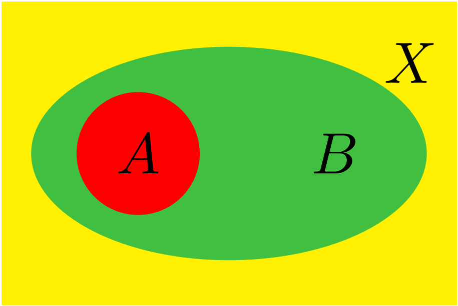

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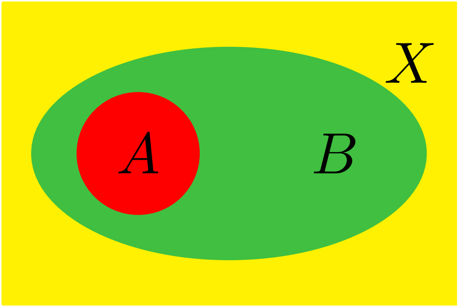

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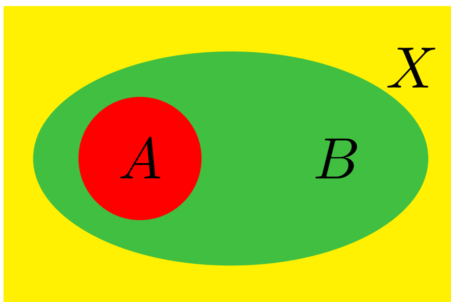

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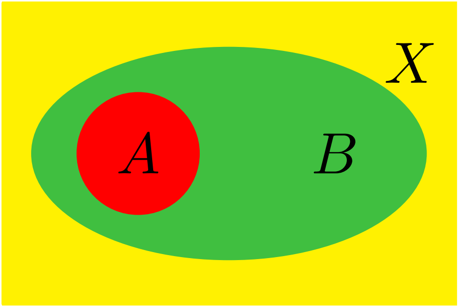

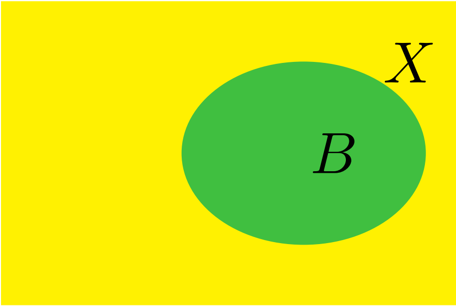

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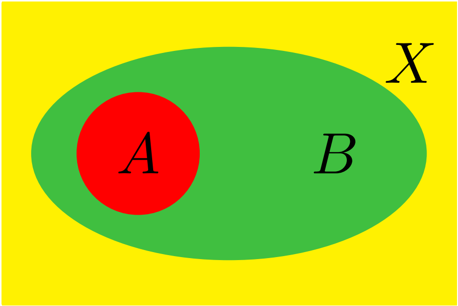

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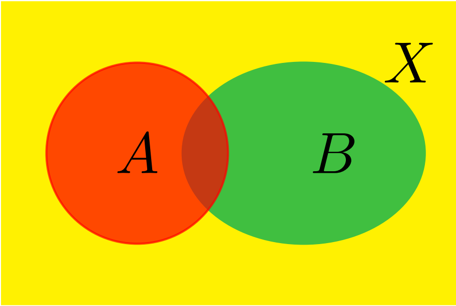

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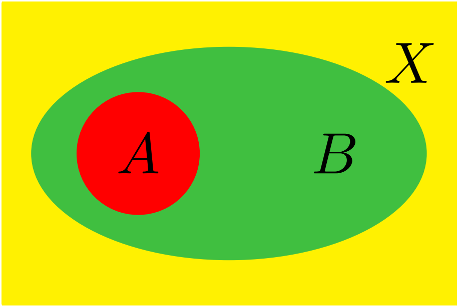

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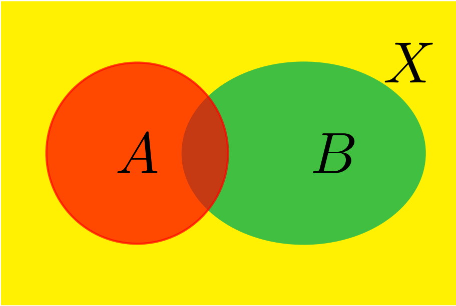

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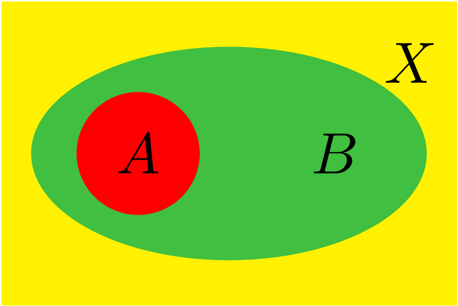

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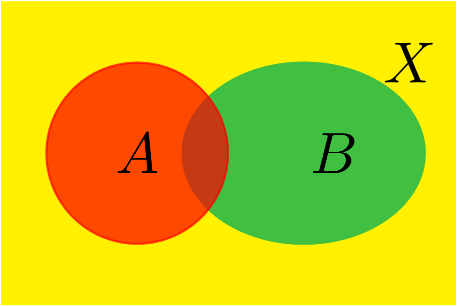

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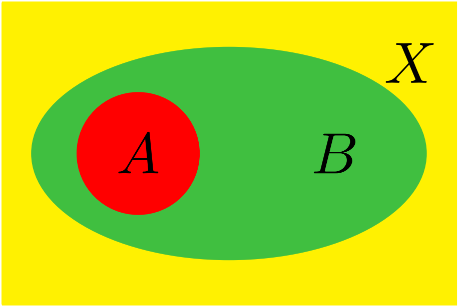

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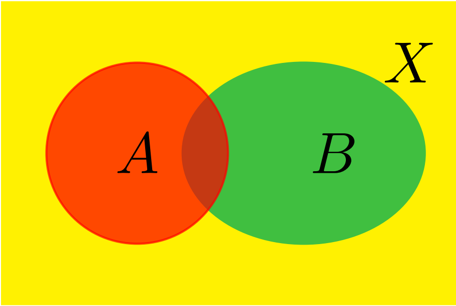

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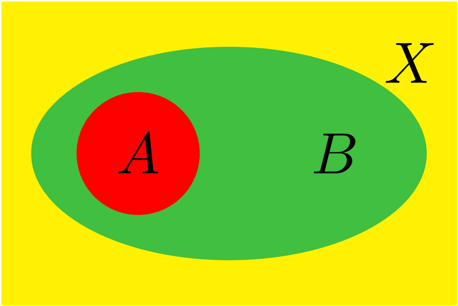

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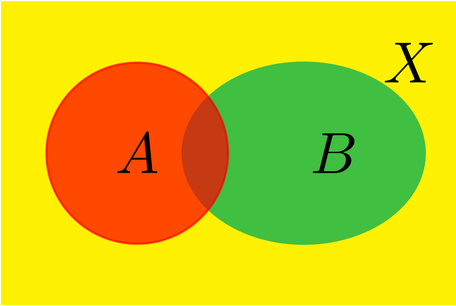

$(A, B) \notin R_\subset$ since $A \not\subset B$

Is it true that $\forall A, B \in \mathcal{P}(X) \quad \underbrace{(A, B) \in R_\subset}_{A \subset B}$ or $\underbrace{(B, A) \in R_\subset}_{B \subset A}$?

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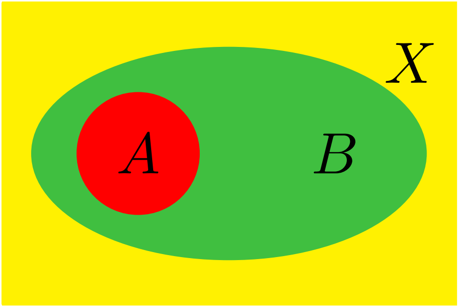

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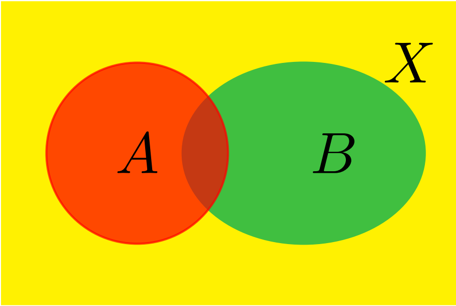

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Is it true that $\forall A, B \in \mathcal{P}(X)$ $\underbrace{(A, B) \in R_\subset}_{A \subset B}$ or $\underbrace{(B, A) \in R_\subset}_{B \subset A}$? No!

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$2019 \equiv 0 \mod 3$ since $3 \mid (2019-0)$

Criteria for divisibility by 3 and 9

Lemma.

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$$= a_n \cdot (\underbrace{99\dots9}_{n} + 1) + a_{n-1} \cdot (\underbrace{99\dots9}_{n-1} + 1) + \dots + a_2 \cdot (99 + 1) + a_1(9 + 1) + a_0$$

$$= \underbrace{(a_n \cdot 99\dots9 + a_{n-1} \cdot 99\dots9 + \dots + a_2 \cdot 99 + a_1 \cdot 9)}_{\text{divisible by } 3}$$

$$+(a_n + a_{n-1} + \dots + a_2 + a_1 + a_0).$$

Therefore, N is divisible by 3 iff
the sum $a_n + a_{n-1} + \dots + a_2 + a_1 + a_0$ of its digits is divisible by 3.

Remark. The same proof proves that,
a number is divisible by 9 iff the sum of its digits is divisible by 9.

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total if $\forall x, y \in X \quad x\,R\,y \vee y\,R\,x$ for example, $\leq$

Properties of relations

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$\leq$ on $\mathbb{R}$	$\equiv \mod 3$ on $\mathbb{Z}$	$\subset$ on $\mathcal{P}(X)$	divisibility on $\mathbb{N}$
reflexive $x \leq x$	reflexive $a \equiv a \mod 3$	reflexive $A \subset A$	reflexive $a \mid a$
antisymmetric $x \leq y \wedge y \leq x$ $\implies x = y$	symmetric $a \equiv b \mod 3$ $\implies b \equiv a \mod 3$	antisymmetric $A \subset B \wedge B \subset A$ $\implies A = B$	antisymmetric $a \mid b \wedge b \mid a$ $\implies a = b$
transitive $x \leq y \wedge y \leq z$ $\implies x \leq z$	transitive $a \equiv b \mod 3 \wedge$ $b \equiv c \mod 3$ $\implies a \equiv c \mod 3$	transitive $A \subset B \wedge B \subset C$ $\implies A \subset C$	transitive $a \mid b \wedge b \mid c$ $\implies a \mid c$
total $\forall x, y \in \mathbb{R}$ $x \leq y \vee y \leq x$			

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Non-strict total (linear) order (antisymmetric+transitive+total)

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Non-strict partial order (reflexive+antisymmetric+transitive)

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- **Equivalence relation** (reflexive+symmetric+transitive)

$\equiv \mod 3$ on $\mathbb{Z}$.