Lecture 5

Maps

Maps: domain, codomain, range

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Definition.

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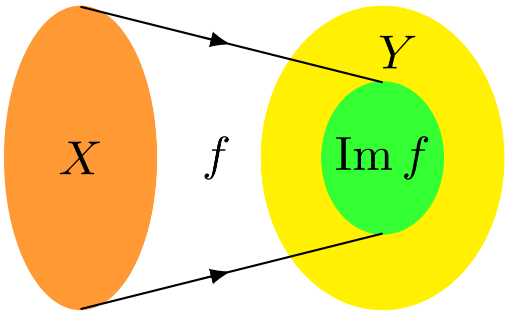

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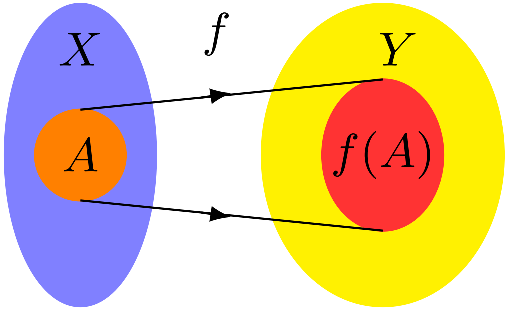

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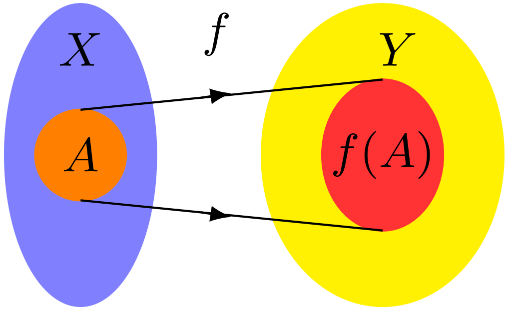

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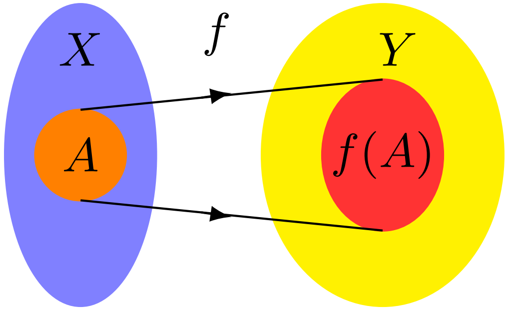

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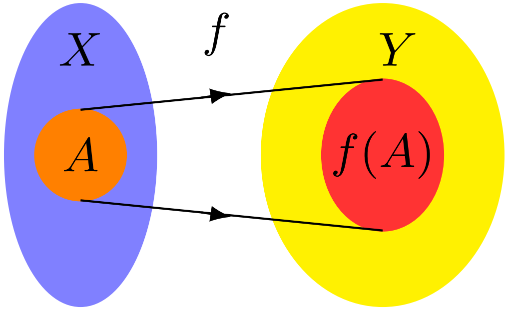

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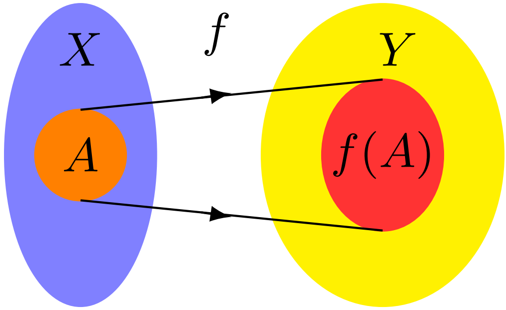

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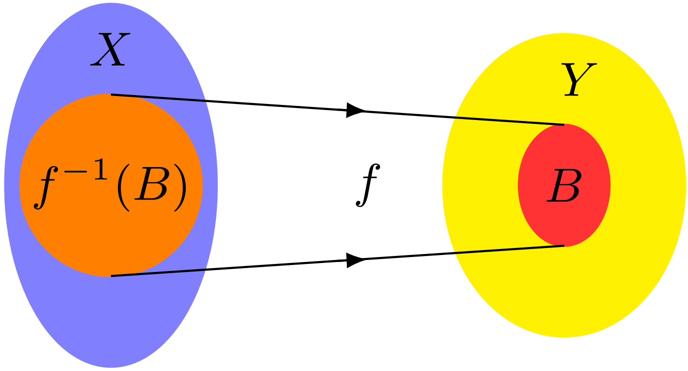

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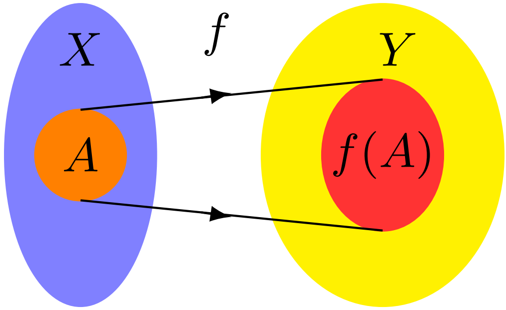

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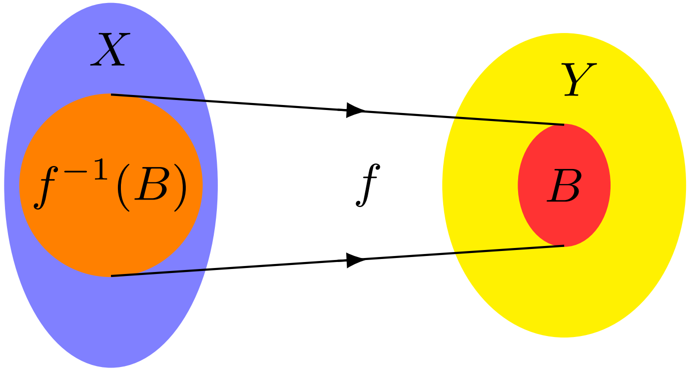

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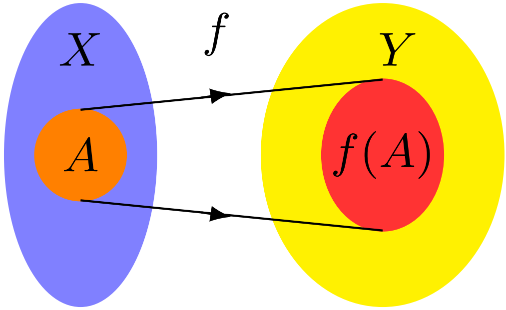

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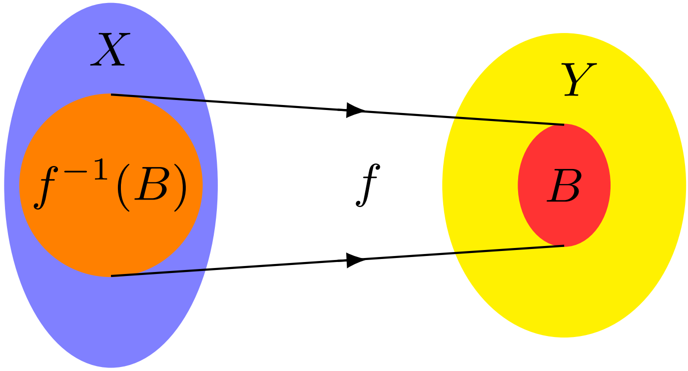

Warning: f^{-1} is **not** the inverse map!

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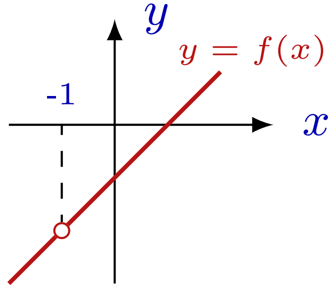

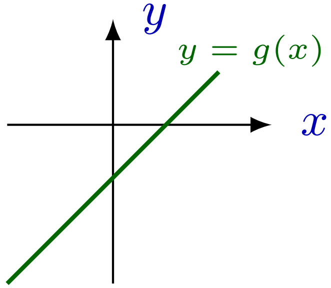

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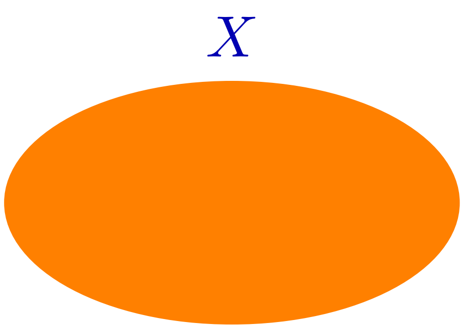

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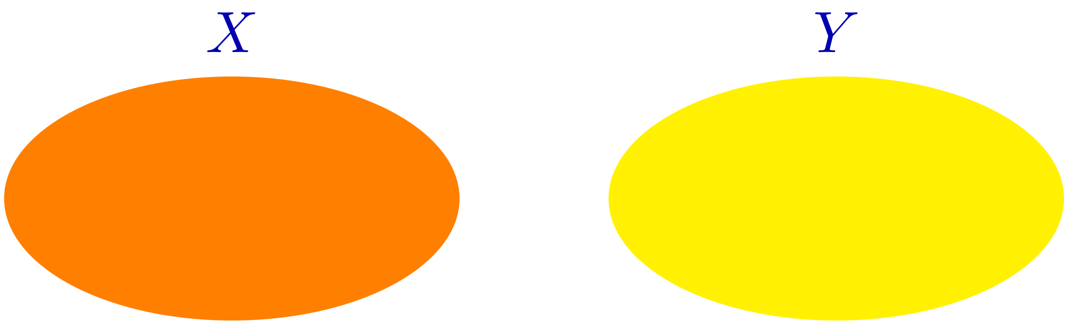

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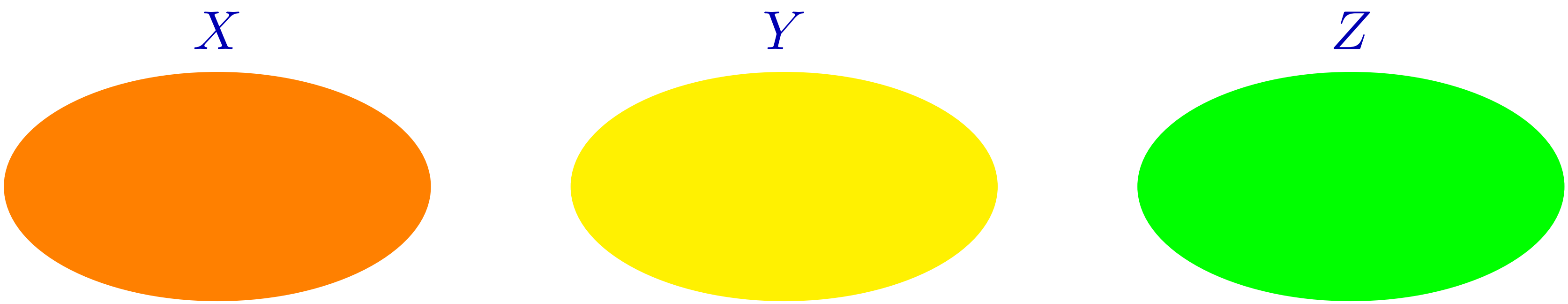

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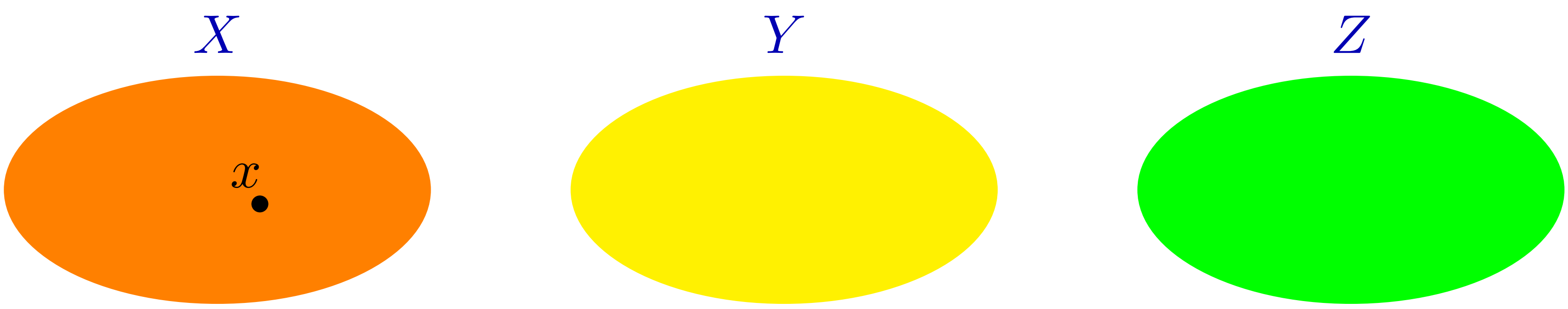

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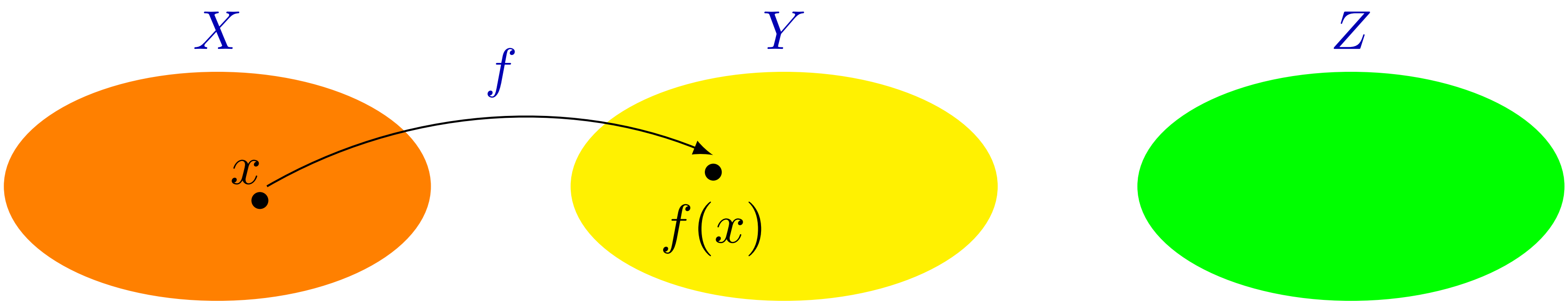

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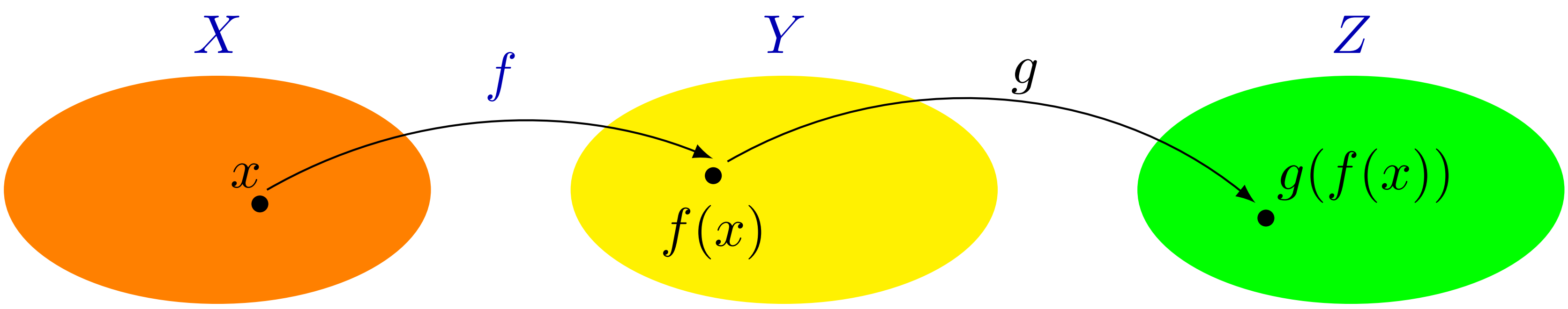

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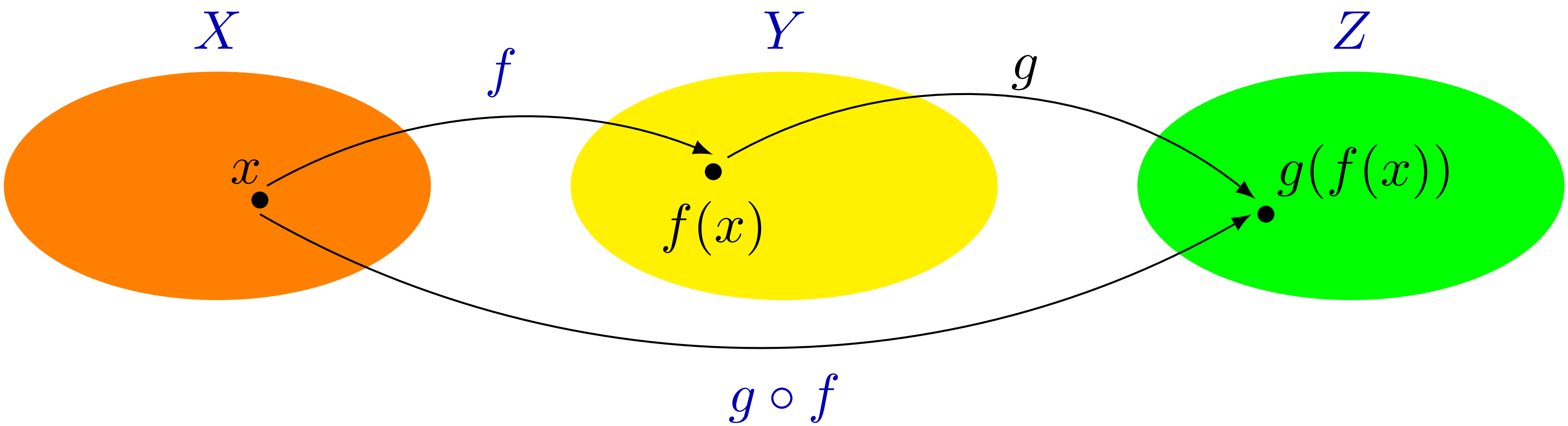

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Let X, Y be sets. Choose any $y_0 \in Y$ and define a map $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$. This map is called a **constant** map.

Give a **descriptive definition** of a constant map.

A map $f : X \to Y$ is said to be **constant** if $\forall a, b \in X \; f(a) = f(b)$.

Examples of maps

- A **function** in one variable $y = f(x)$ is a map $f : D \to \mathbb{R}$,
where $D \subset \mathbb{R}$ is the domain of f.

Domain convention: when a function f is defined without specifying its domain, we assume that the domain is the **maximal** set of x-values for which $f(x)$ is defined.

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Or $\exists c \in Y\ \forall a \in X\ f(a) = c.$

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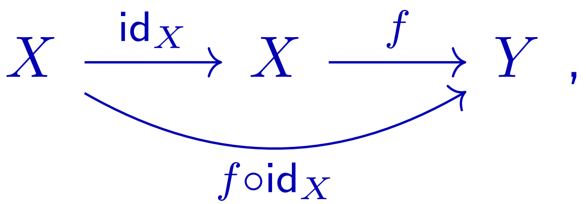

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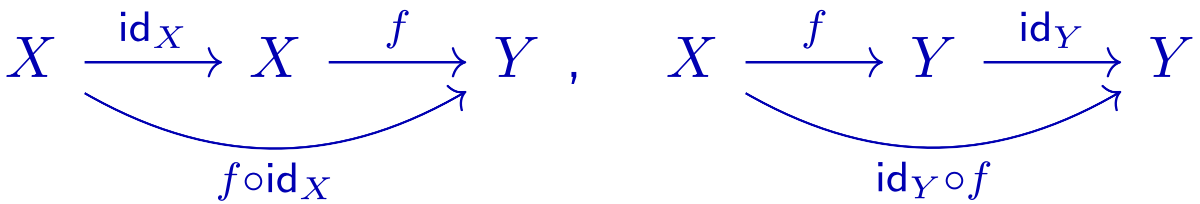

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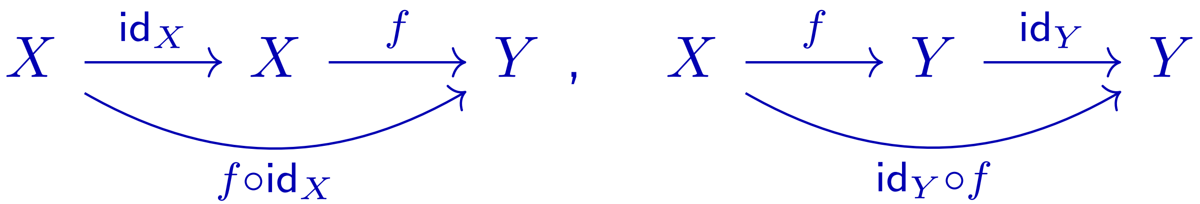

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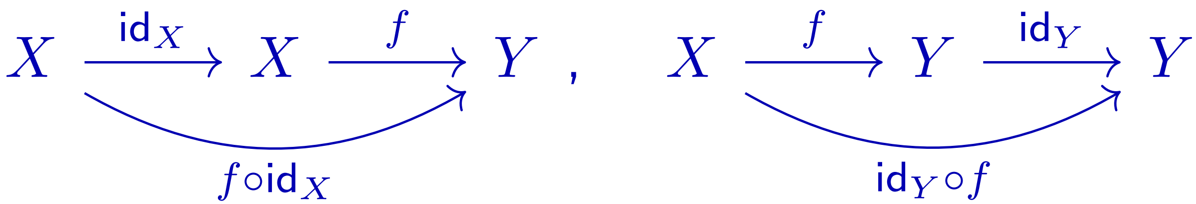

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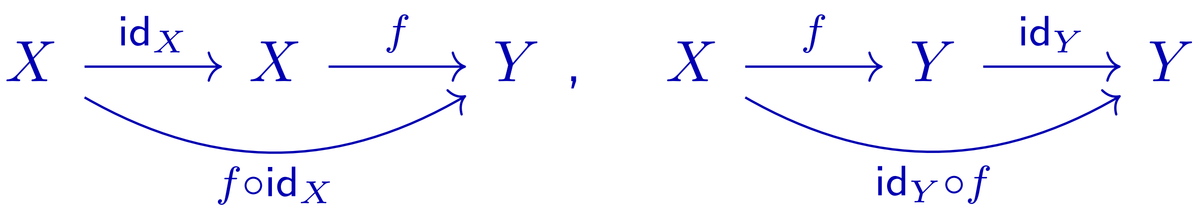

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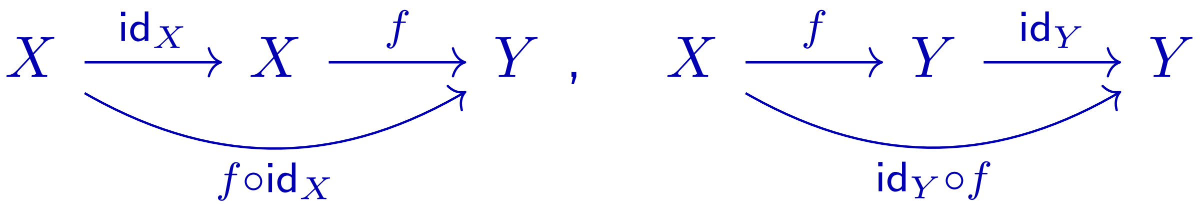

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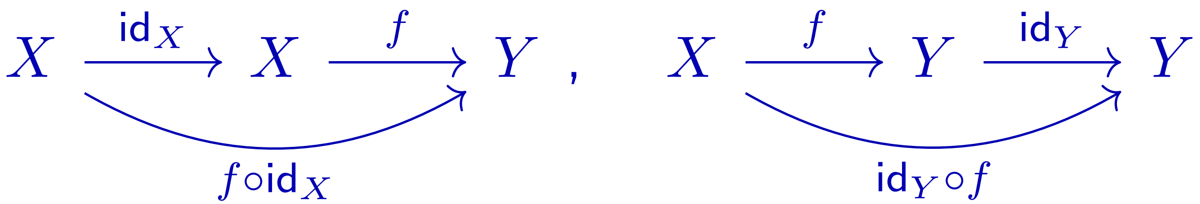

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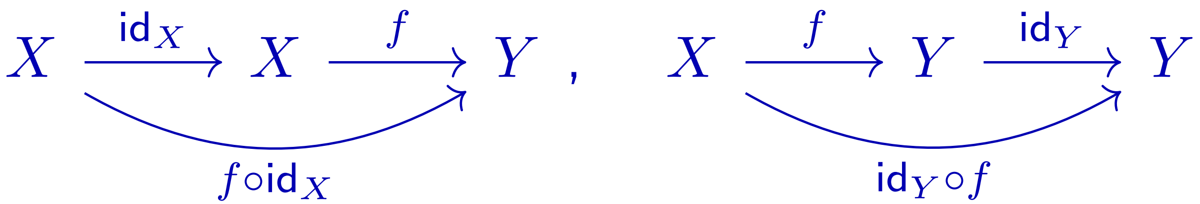

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□

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$\forall\, y \in \operatorname{Im} f$ the equation $y = f(x)$ has **at most one** solution.

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Example.

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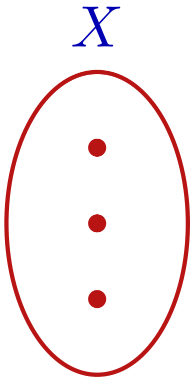

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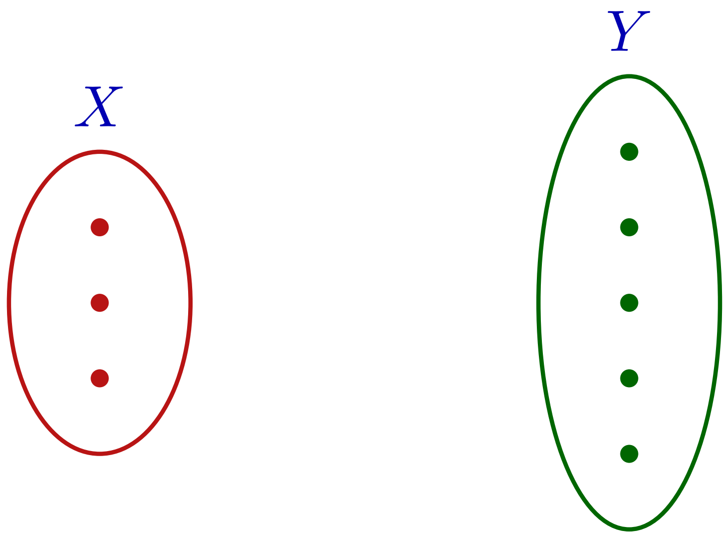

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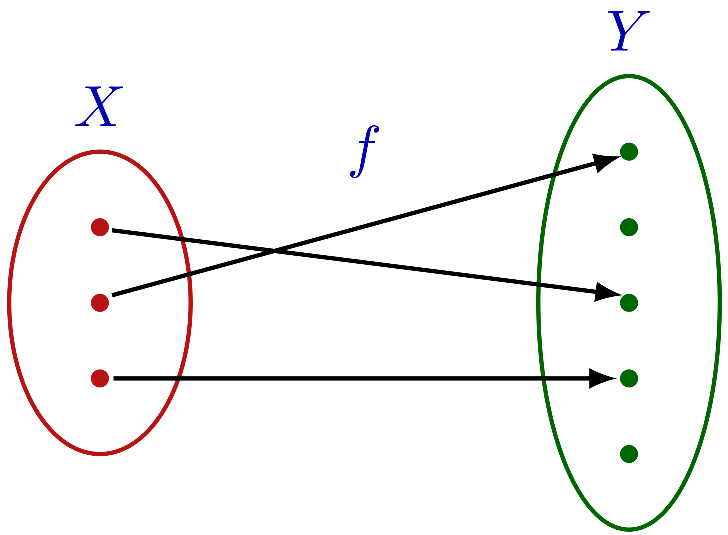

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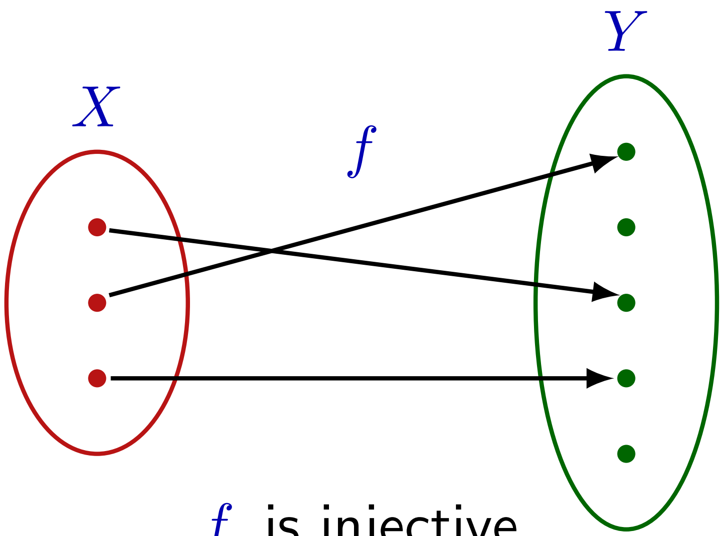

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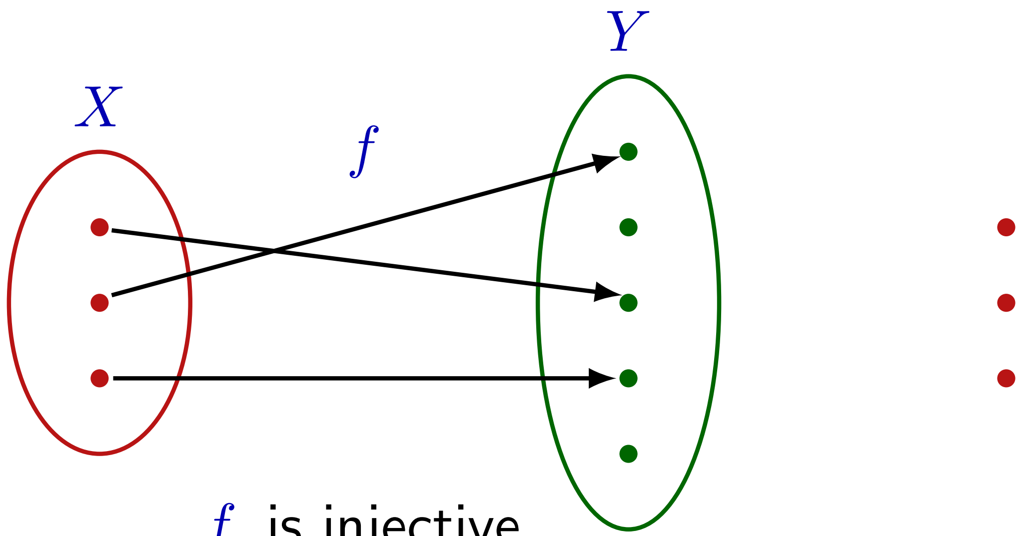

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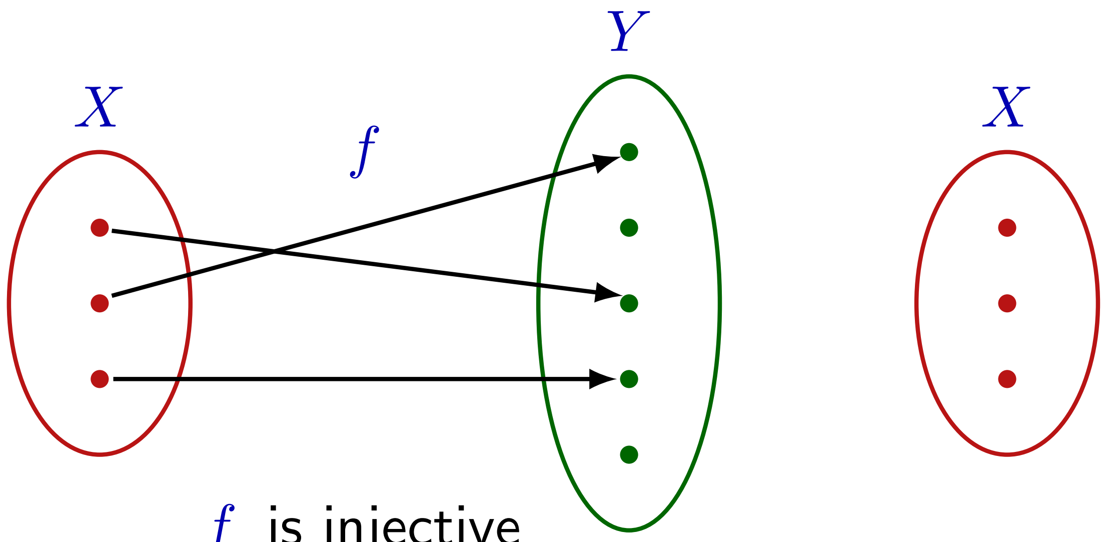

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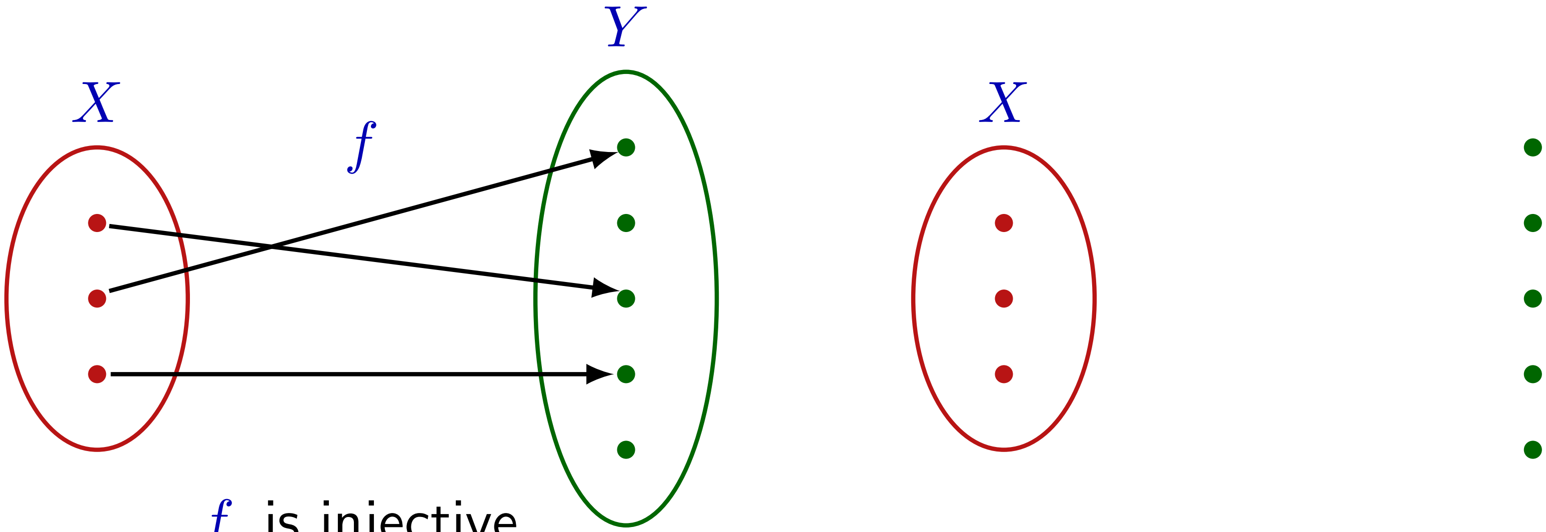

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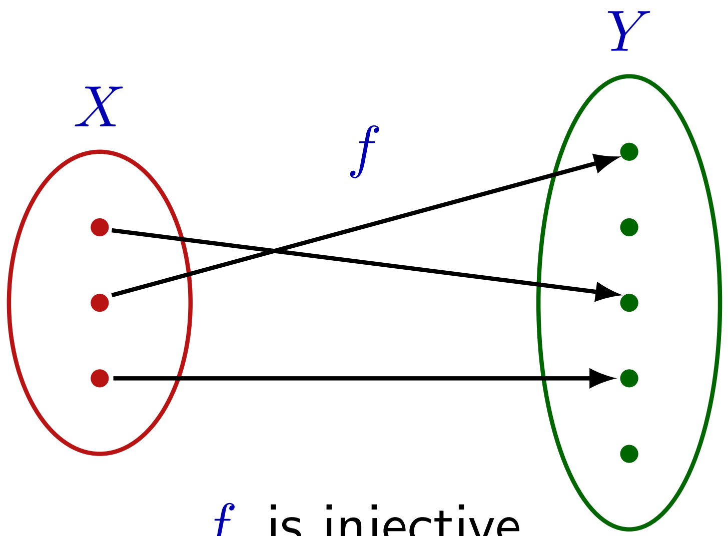

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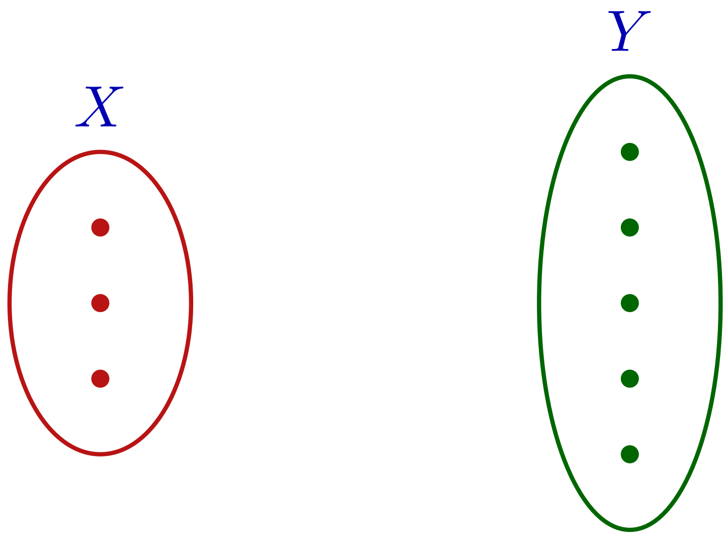

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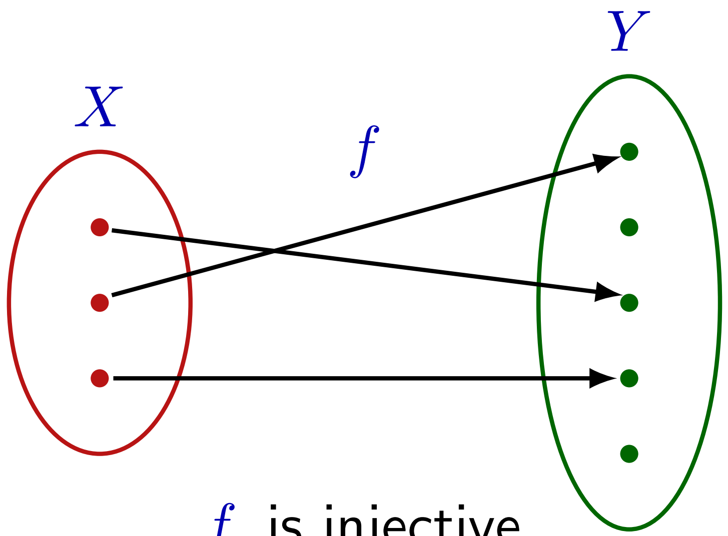

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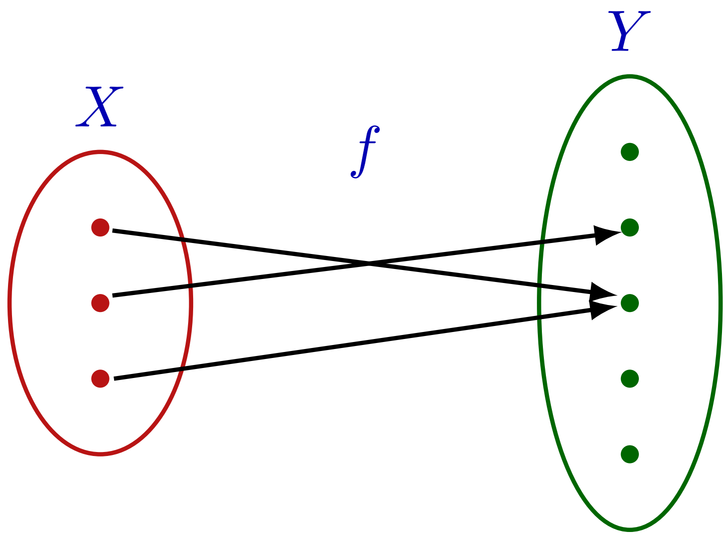

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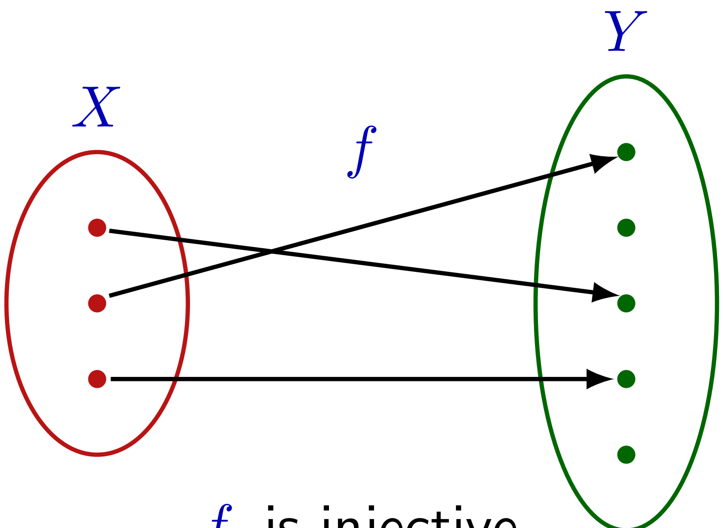

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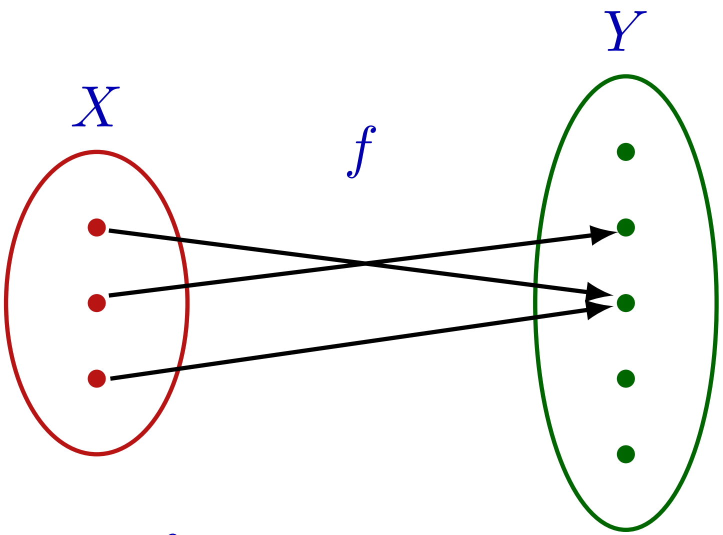

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Non-constant linear function is injective

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Theorem.

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Theorem. A linear map $f : \mathbb{R} \to \mathbb{R}$

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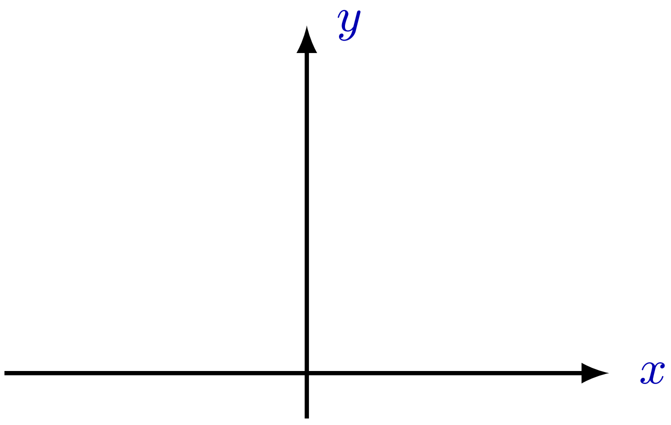

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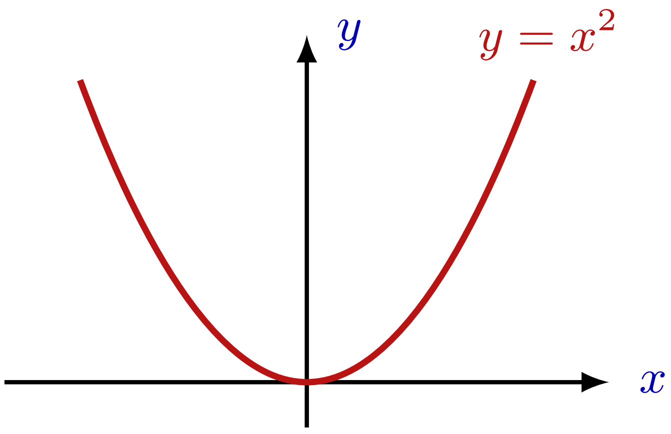

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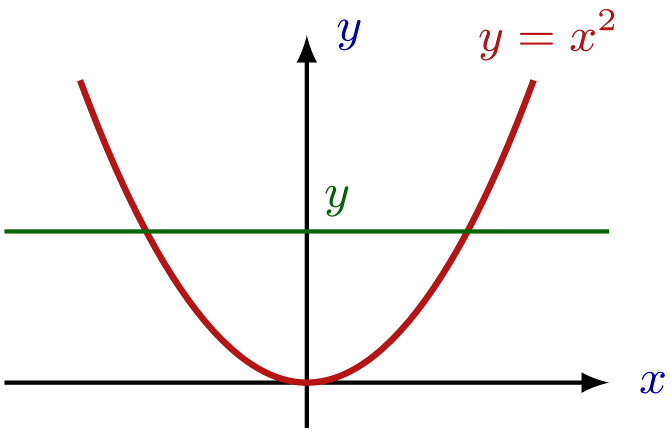

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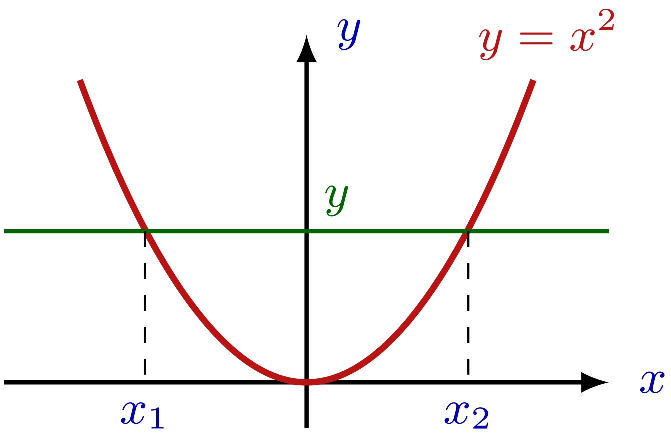

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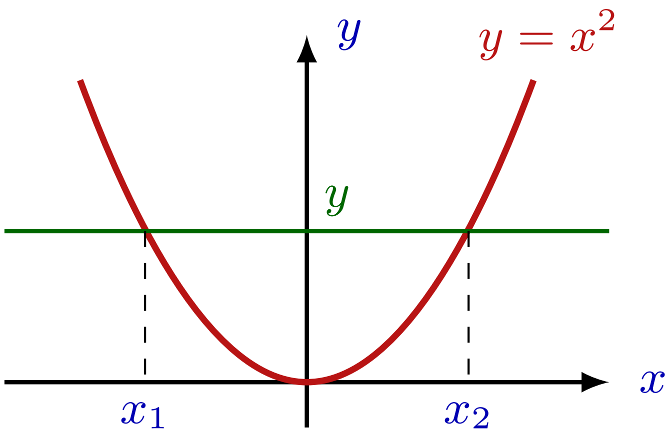

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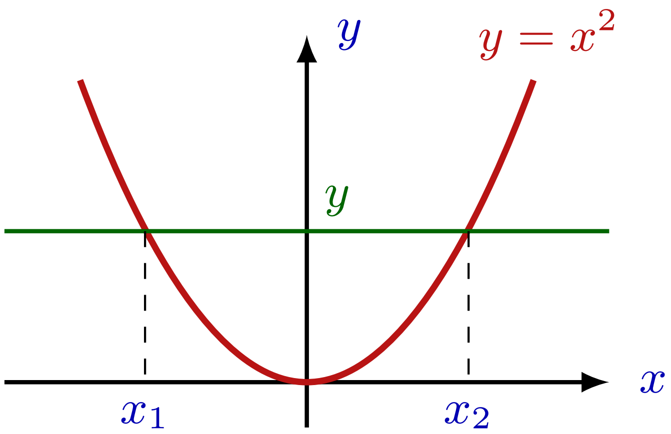

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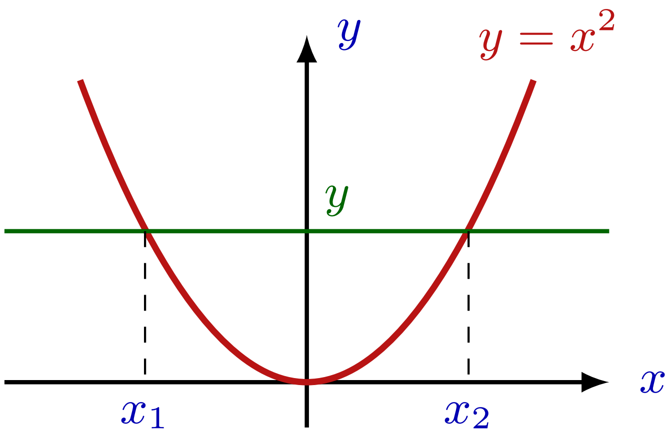

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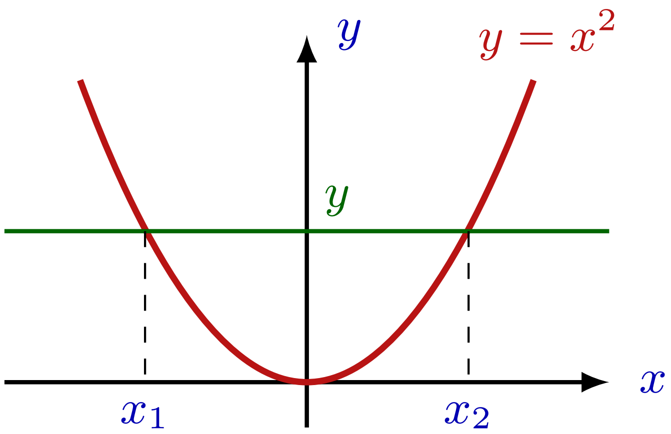

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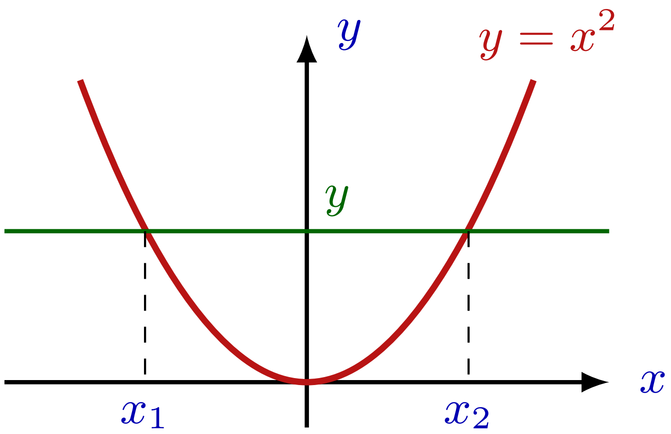

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Remark: The restriction $f\Big|_{\mathbb{R}_+}$ is injective.

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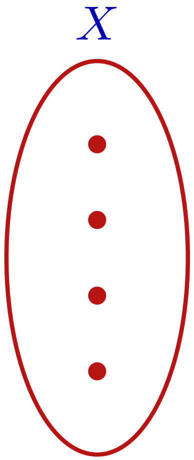

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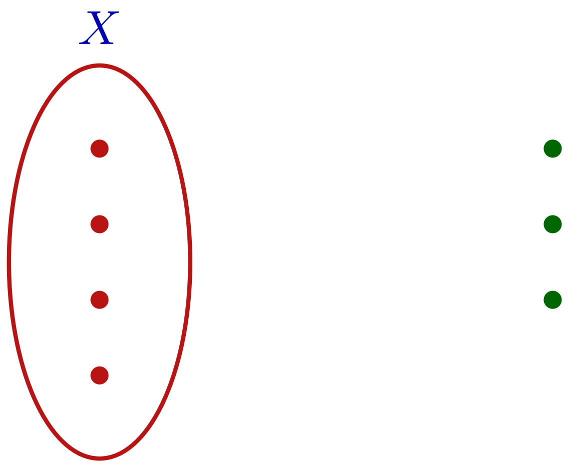

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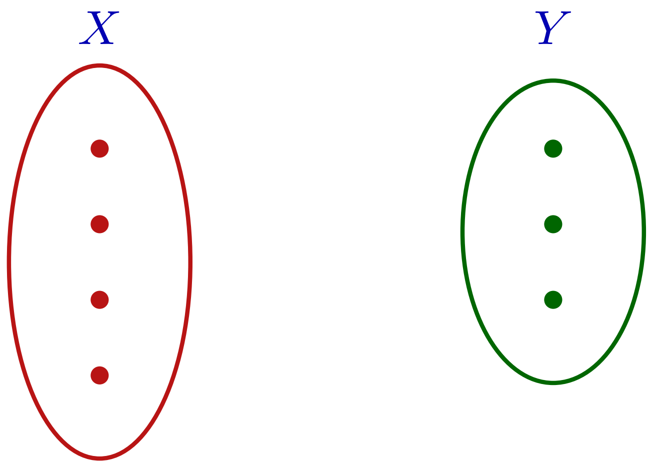

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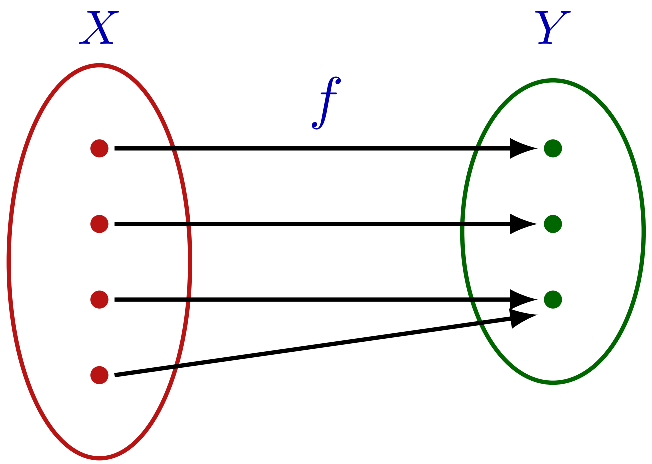

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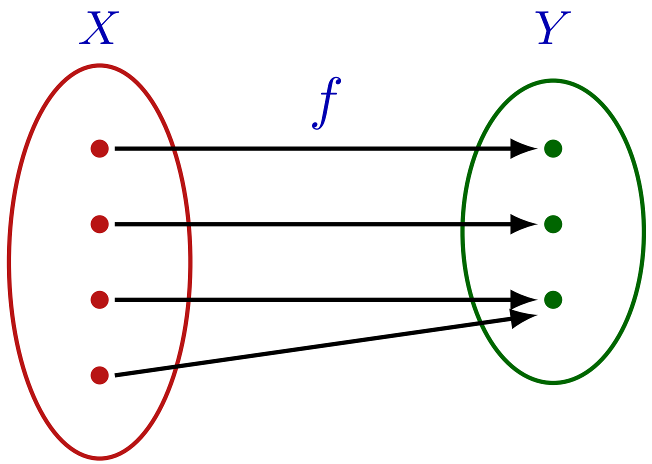

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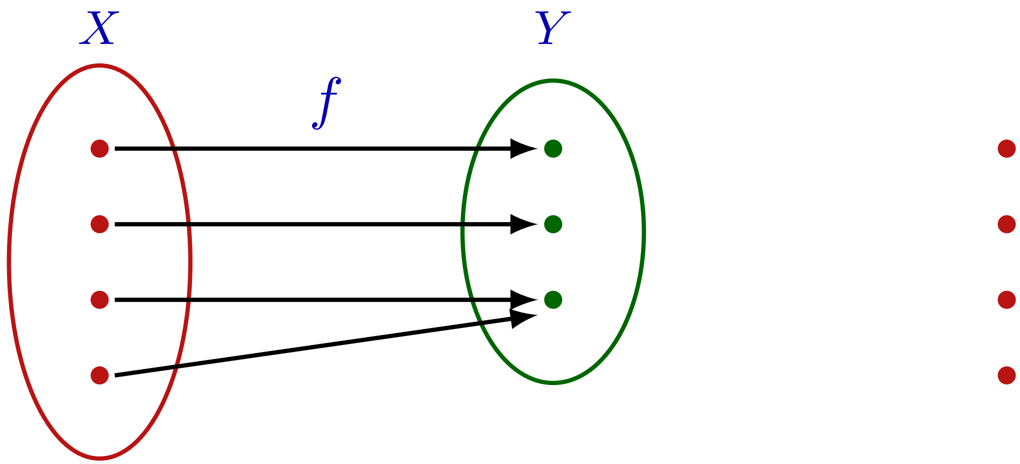

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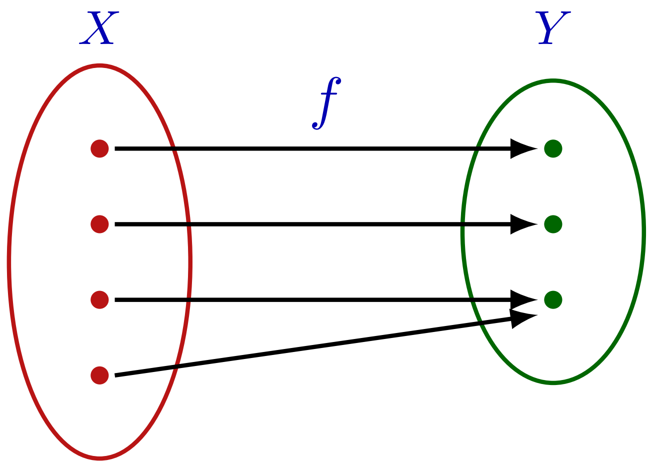

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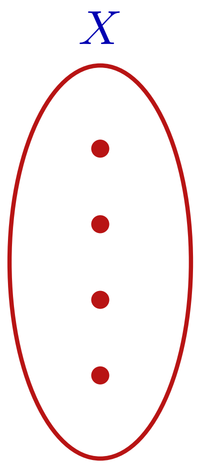

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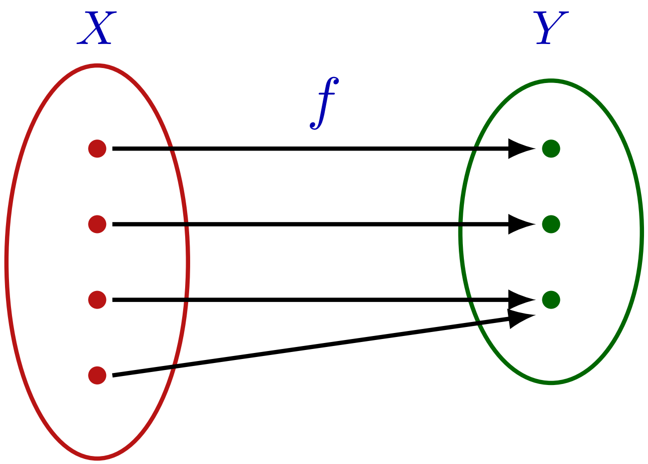

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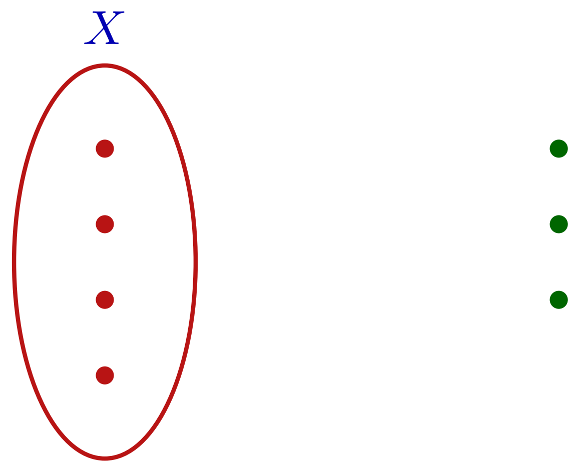

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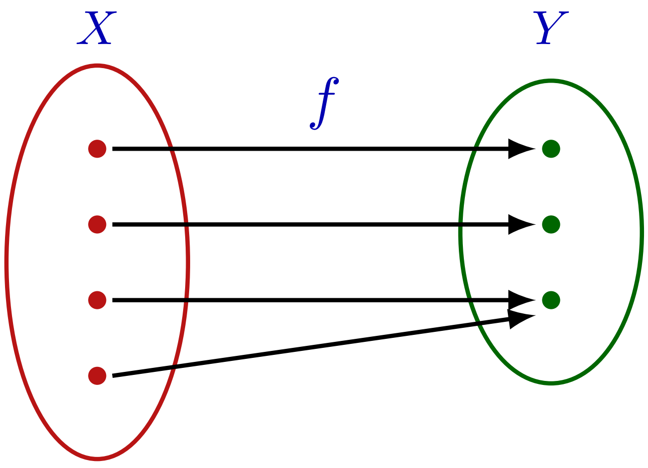

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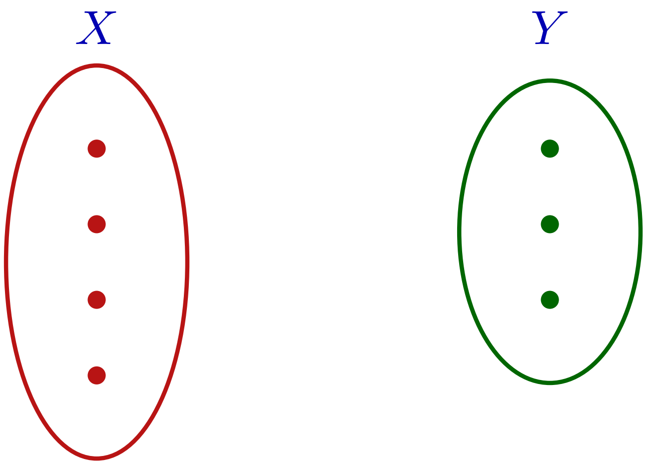

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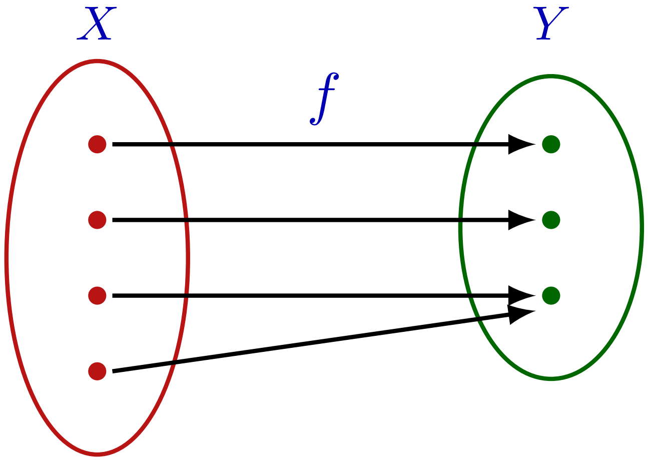

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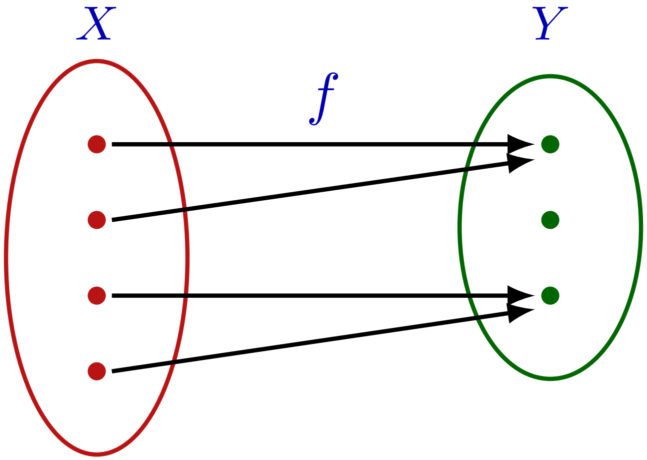

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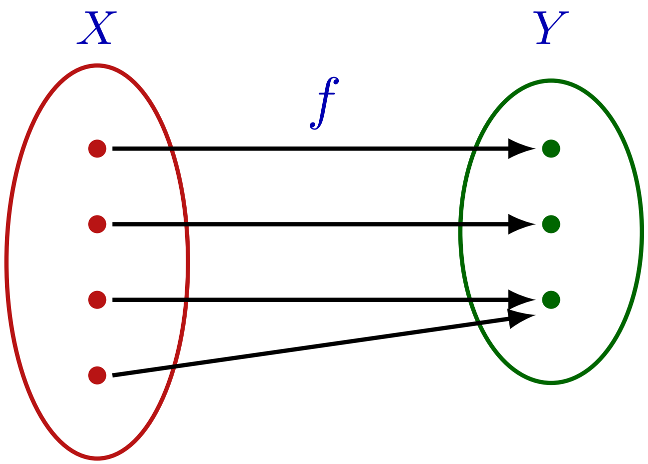
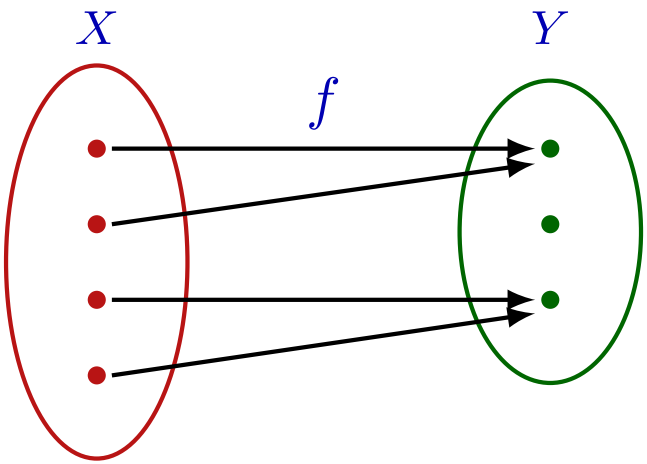


f is surjective



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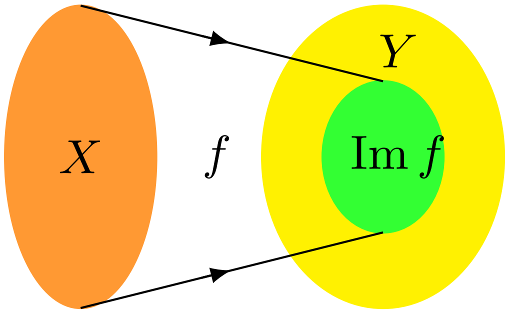

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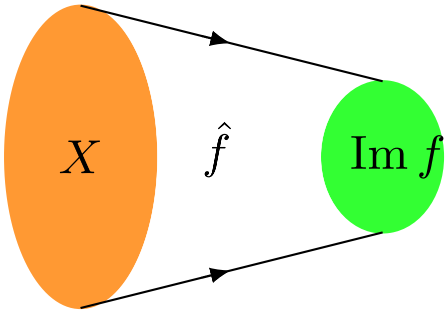

Bijective maps

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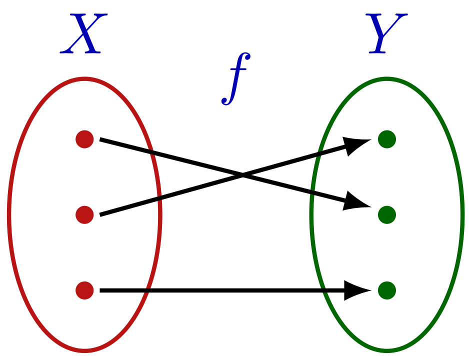

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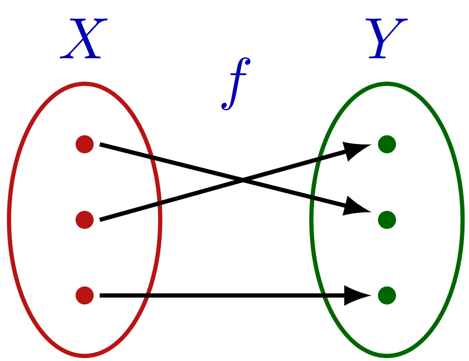

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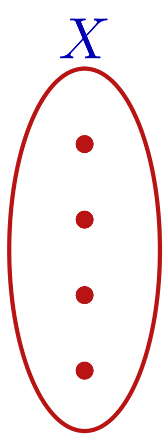

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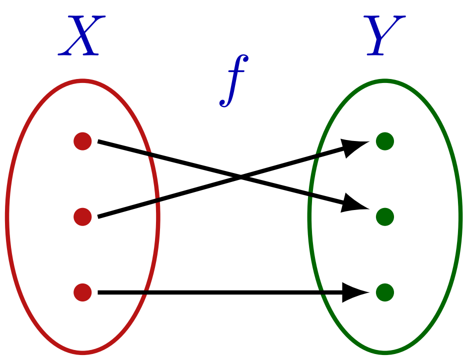

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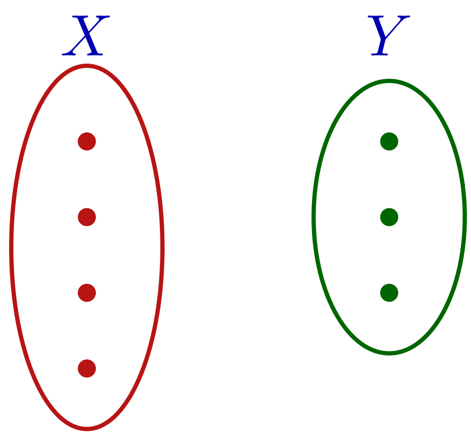

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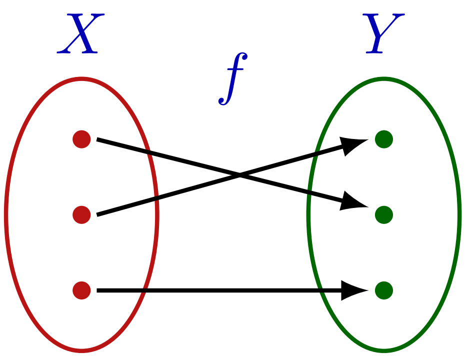

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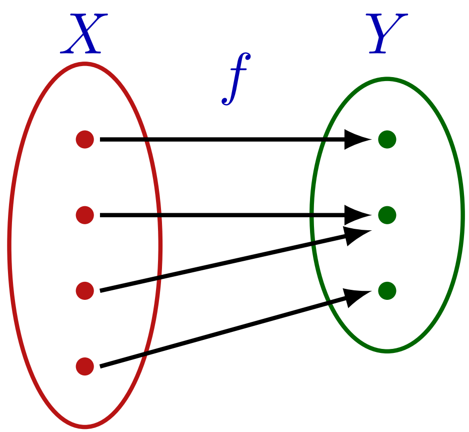

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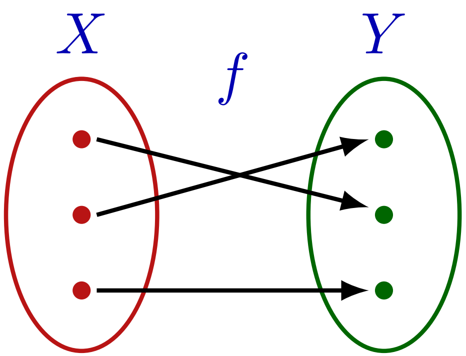

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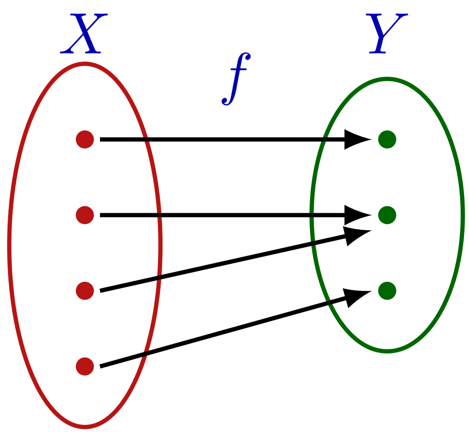

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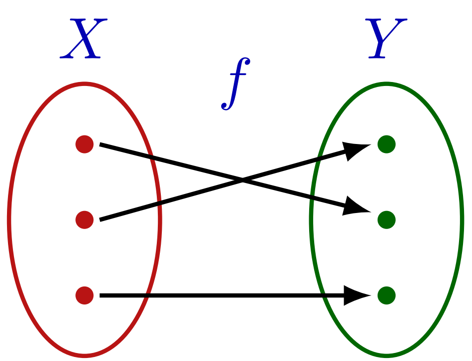

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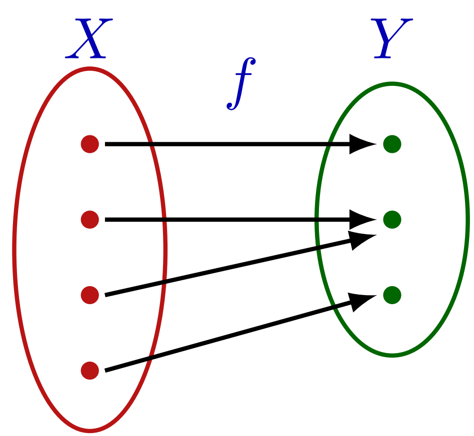

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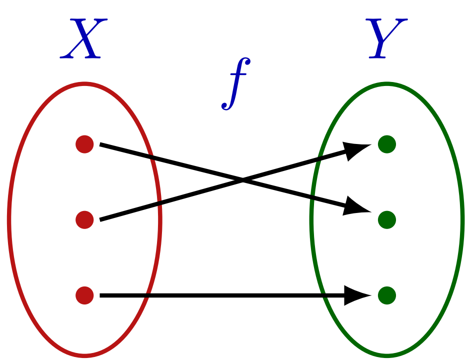

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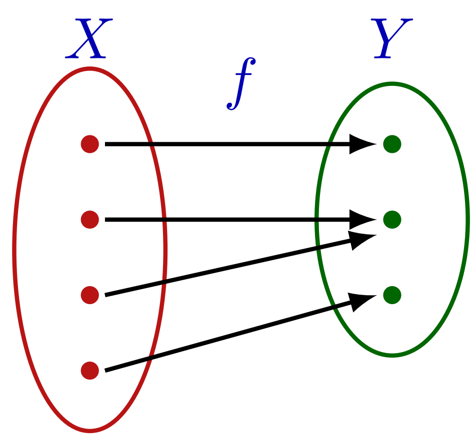

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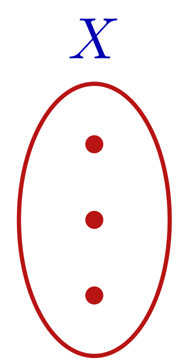

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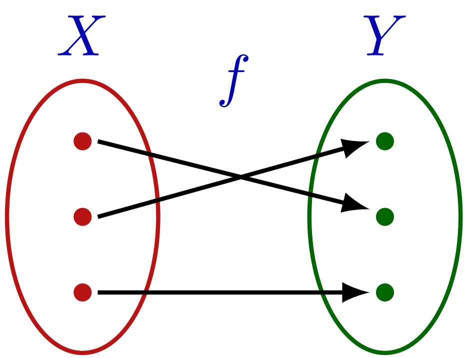

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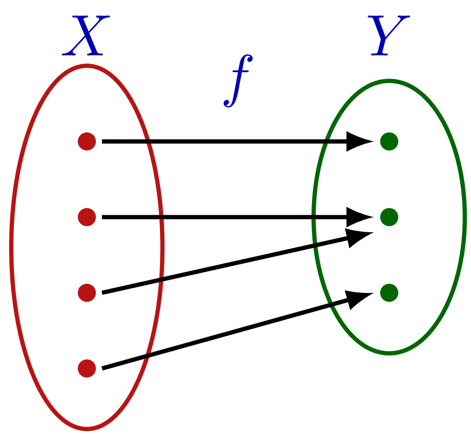

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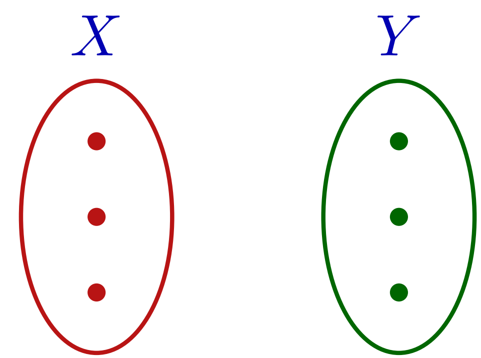

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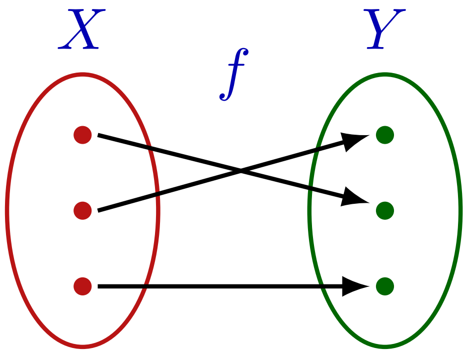

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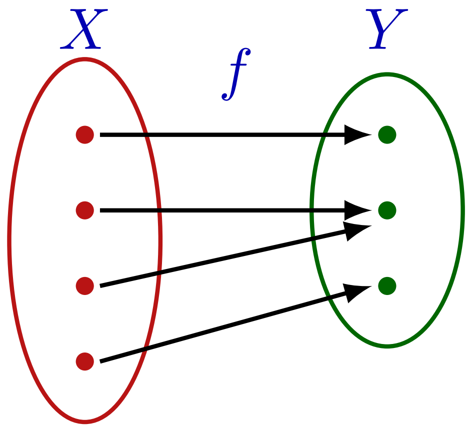

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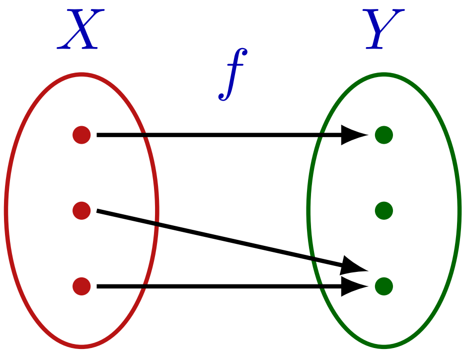

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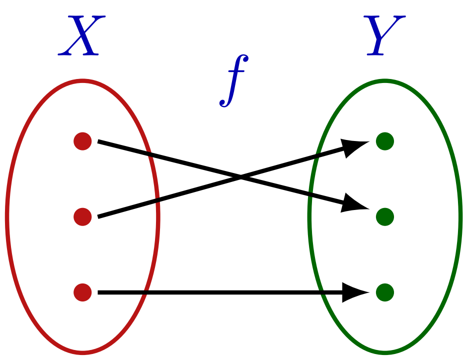

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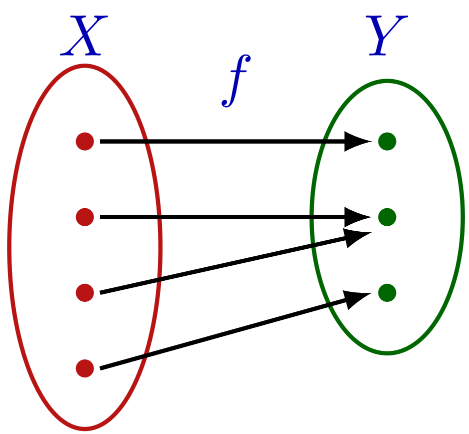

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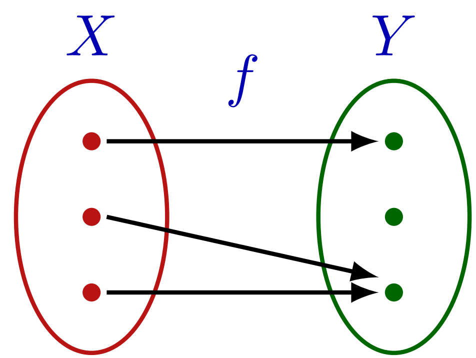

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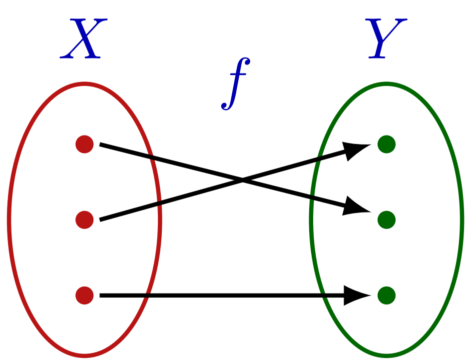
Bijective maps

Definition. Let $f : X \to Y$ be a map.

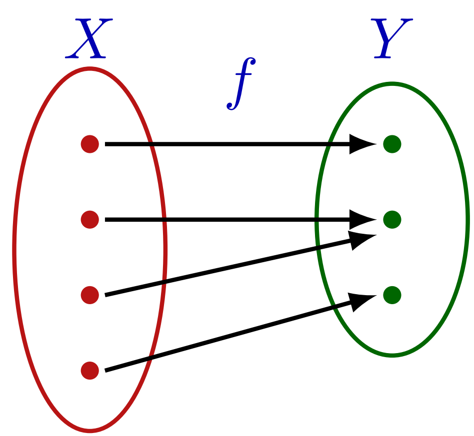
f is called **bijective** (or bijection, or one-to-one correspondence) if

f is injective and surjective: $\forall\, y \in Y \;\; \exists !\, x \in X \;\; y = f(x)$,

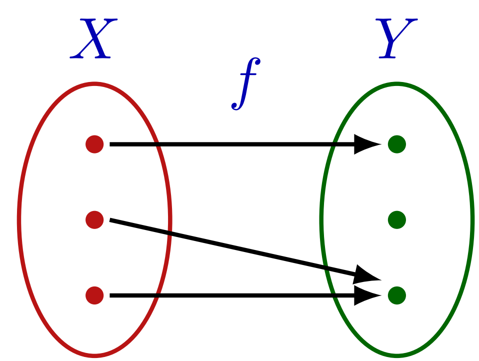
or, equivalently, $\forall\, y \in Y$ the equation $f(x) = y$ has a **unique** solution.



f is bijective



f is not bijective
(f is not injective)



f is not bijective
(f is neither injective
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Linear function is bijective

Linear function is bijective

A linear map $f : \mathbb{R} \to \mathbb{R}$

Linear function is bijective

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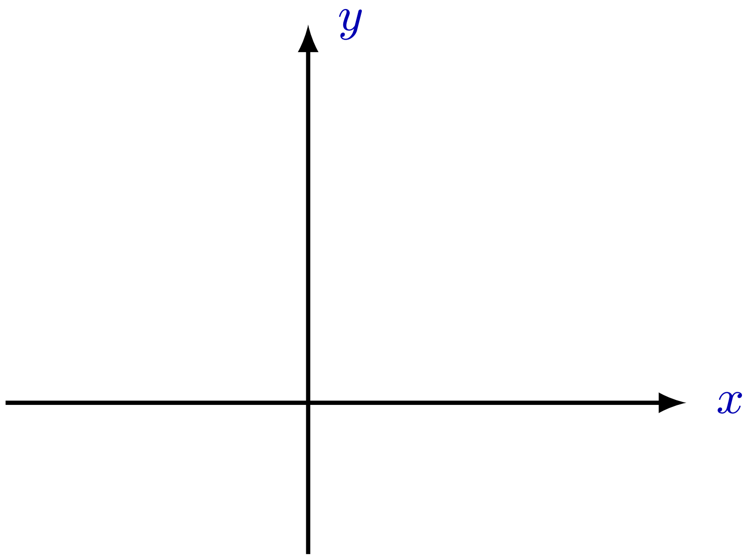

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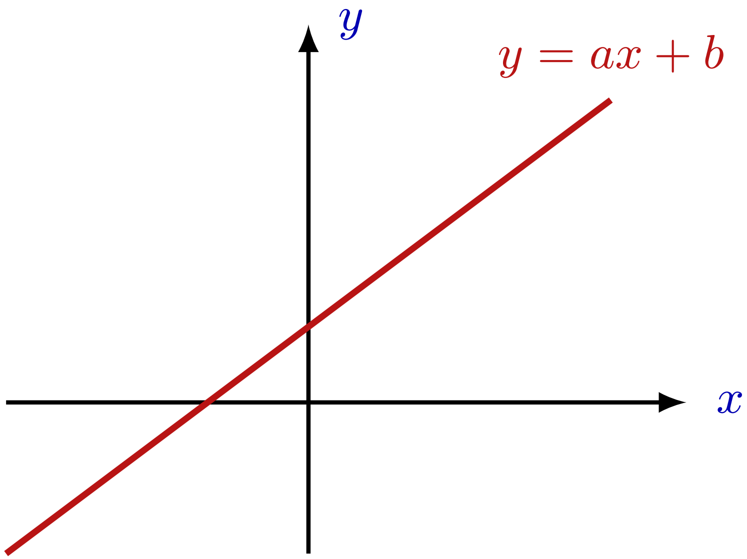

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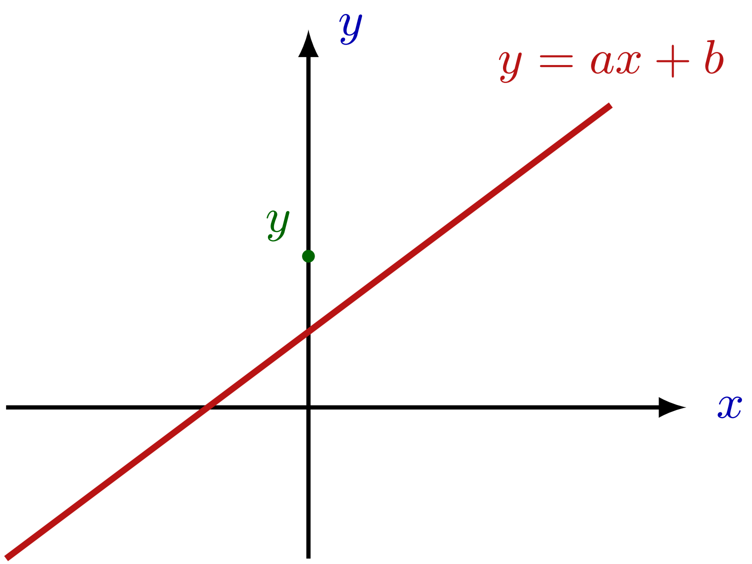

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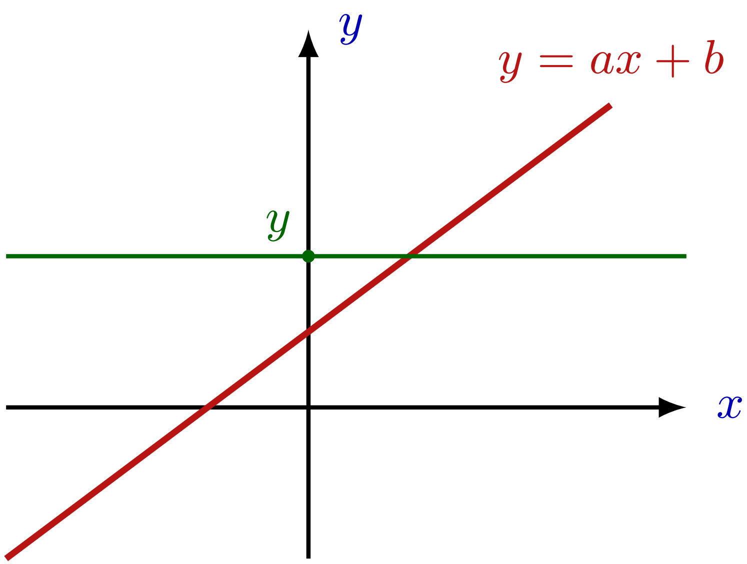

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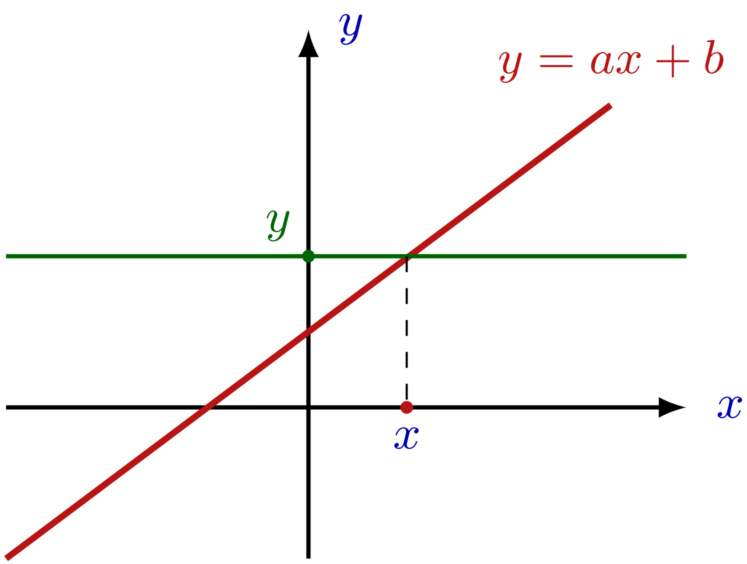

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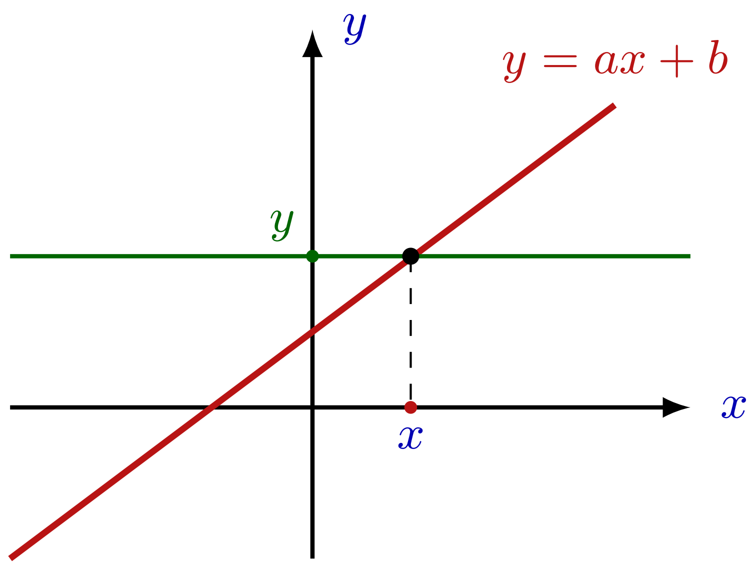

$$\forall\, y \in \mathbb{R} \quad \exists\,!\, x \in \mathbb{R} \quad y = ax + b$$

Inverse map

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Definition.

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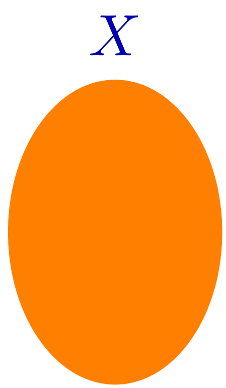

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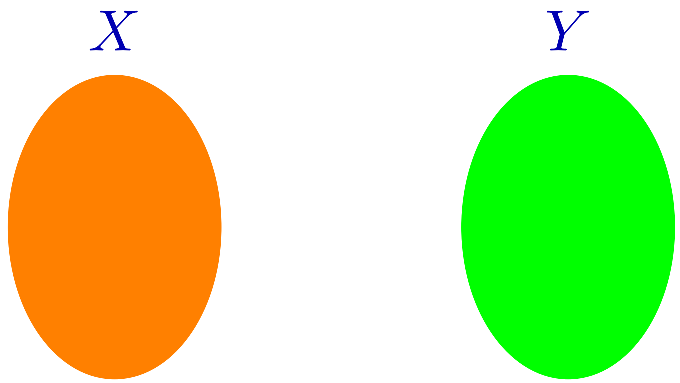

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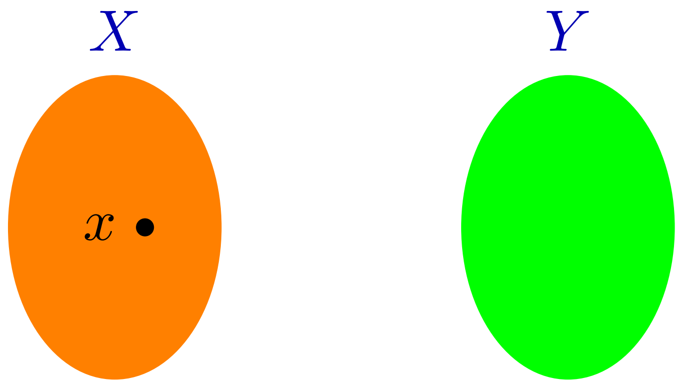

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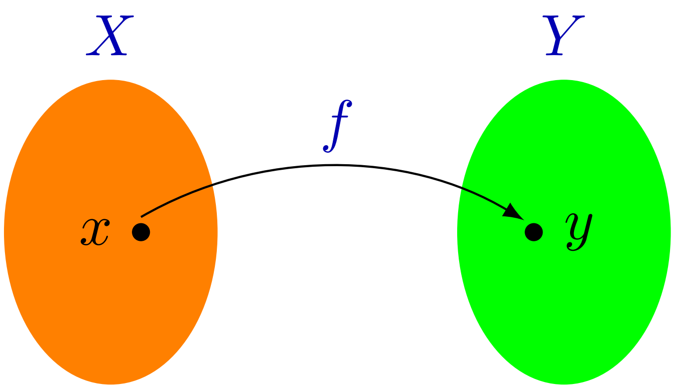

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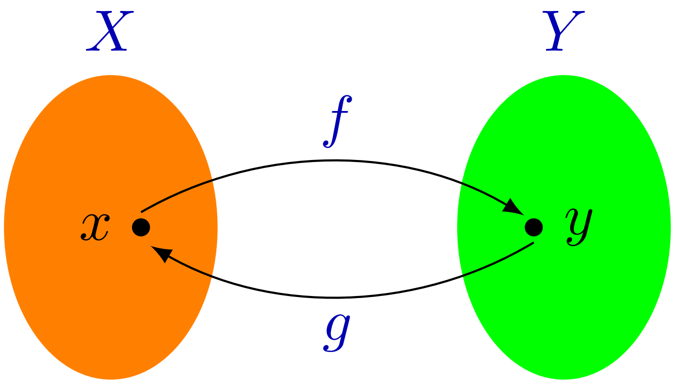

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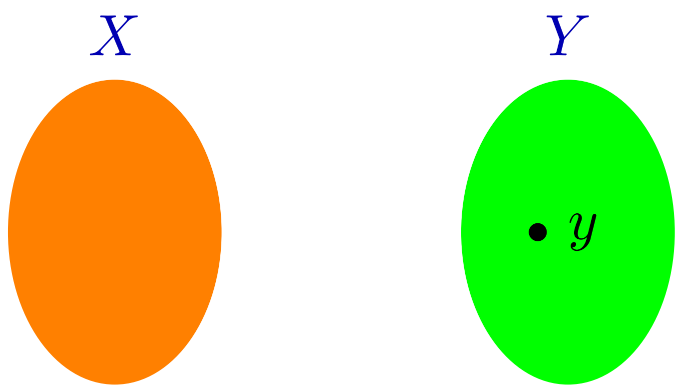

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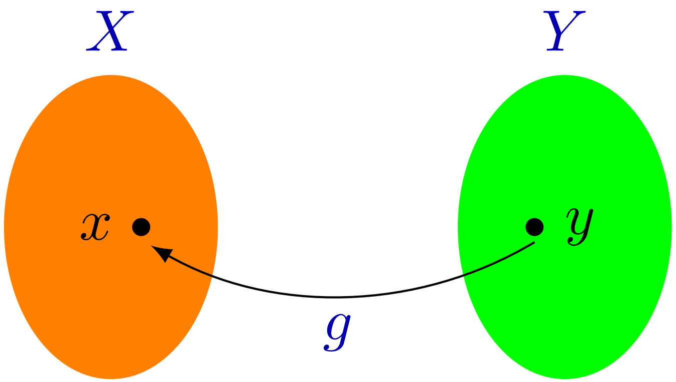

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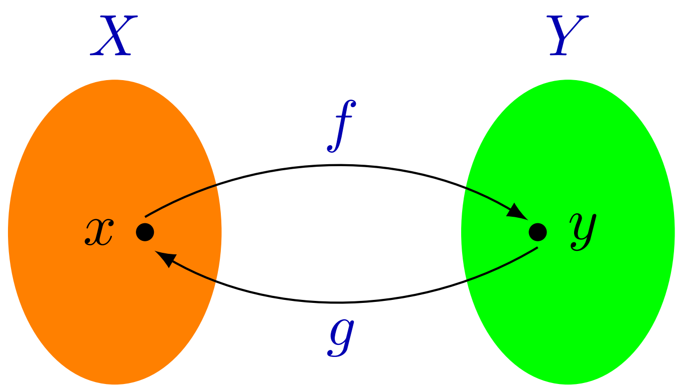

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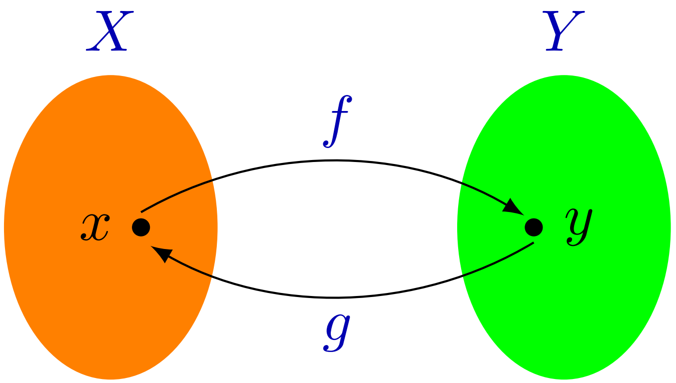

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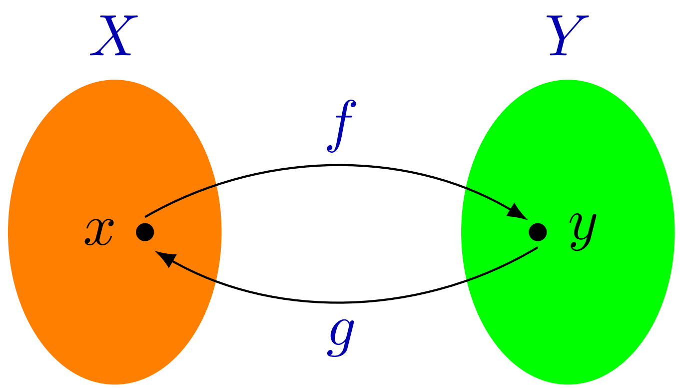

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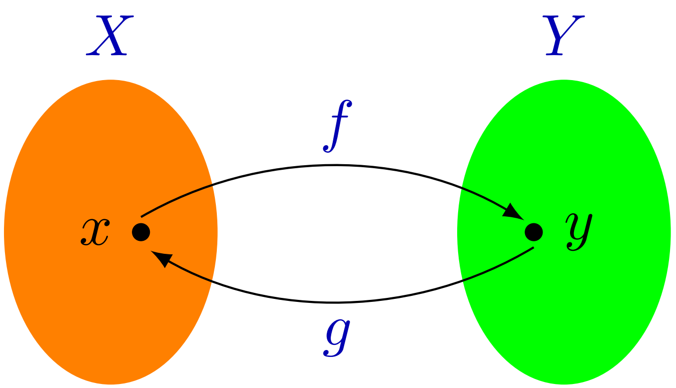

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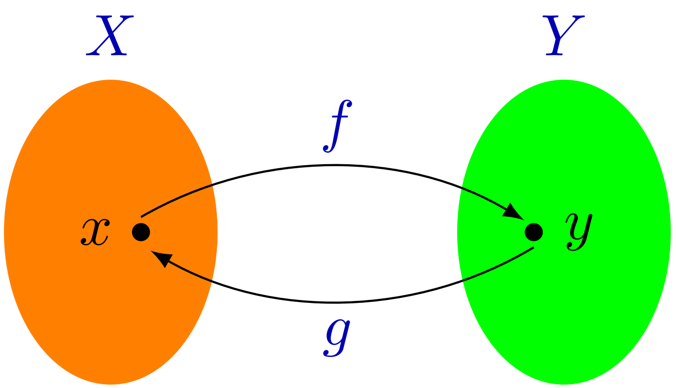

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Warning. Not all maps are invertible!

Inverse is unique

Inverse is unique

Theorem.

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Theorem. If an inverse map exists,

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Since the inverse map is **unique**, it deserves a **functional notation**.

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The inverse for f is denoted by f^{-1}.

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Therefore, $g = h$ and the inverse map is unique. □

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Bijection = invertible map

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Theorem.

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To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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By this, f is surjective.

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Theorem. A map is invertible iff it is a bijection.

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To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

By this, f is surjective.

We have proved that f is injective and surjective,

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

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We have proved that f is injective and surjective, therefore, f is **bijective**.

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We have proved that f is injective and surjective, therefore, f is **bijective**.

The half of the proof is done!

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$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

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$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

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Therefore, by the definition of the inverse,

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Thus, f is **invertible**.

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By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

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Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Thus, f is **invertible**. And the other half of the proof is done!

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$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

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⚠ **Warning.**

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$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Thus, f is **invertible**. And the other half of the proof is done! □

⚠ **Warning.** The symbol f^{-1} is used in two ways.

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

Define a map $g : Y \to X$ by the formula $g(y) = x$, where $y = f(x)$.

Let us prove that g is the inverse for f .

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

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⚠ **Warning.** The symbol f^{-1} is used in two ways.

1. f^{-1} denotes the inverse map for f if f is invertible.

2. $f^{-1}(B)$ denotes the preimage of a set B under under any f (not necessarily invertible).

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In these identities, what is f from the point of view of f^{-1}?

f is the inverse for f^{-1}!

Therefore, f^{-1} is invertible (and by this, is a bijection) and $(f^{-1})^{-1} = f$.

□

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□

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Theorem. A monotonic function is injective.

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Therefore, f is injective. $\square$

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To avoid this ambiguity, always use $\arctan x$

as a notation for the inverse function for $\tan x$.

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Example 4. What is $\arccos$?