Lecture 7

Definitions in Mathematics

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We will deal mostly with **binary** relations on a **single** set.

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$\mathcal{P}(X \times X)$ is a huge set!

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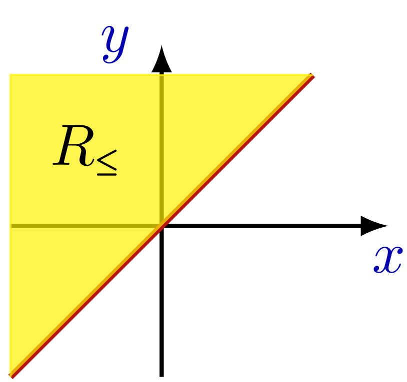

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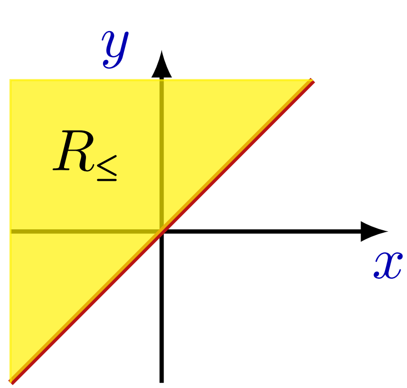

$$\forall x, y \in \mathbb{R} \quad \underbrace{(x,y) \in R_\leq}_{x \leq y} \text{ or } \underbrace{(y,x) \in R_\leq}_{y \leq x}\,.$$

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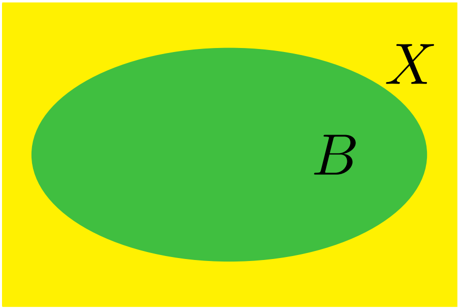

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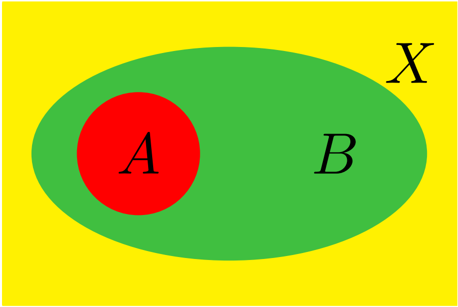

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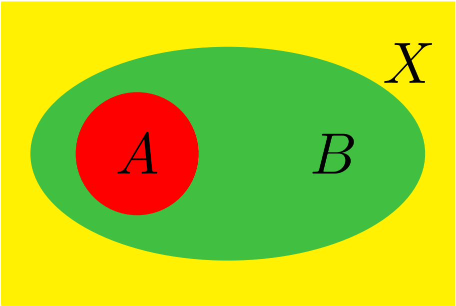

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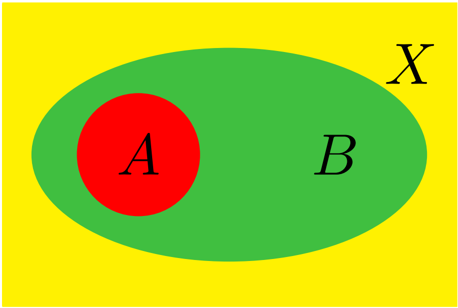

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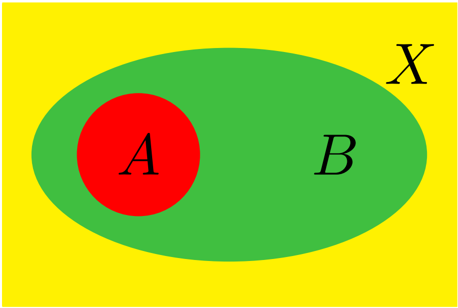

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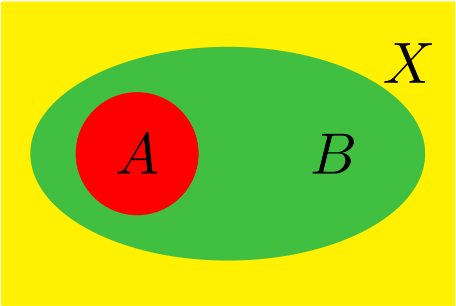

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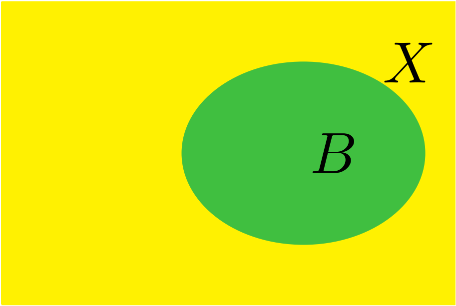

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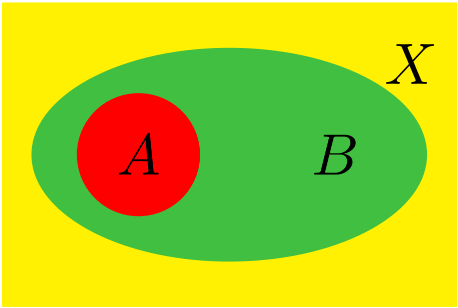

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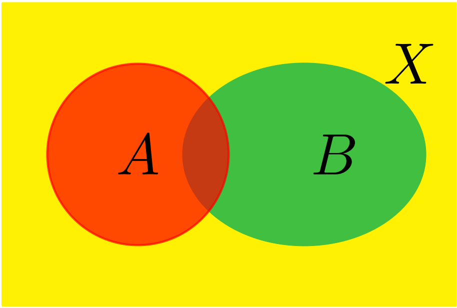

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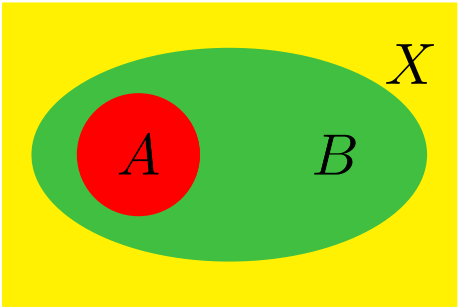

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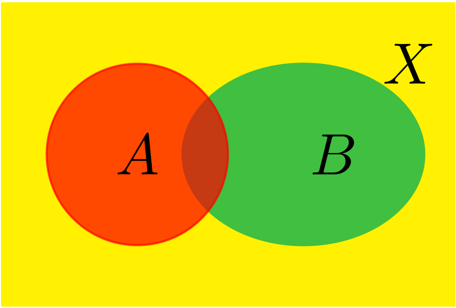

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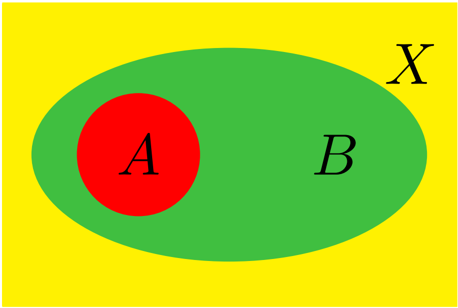

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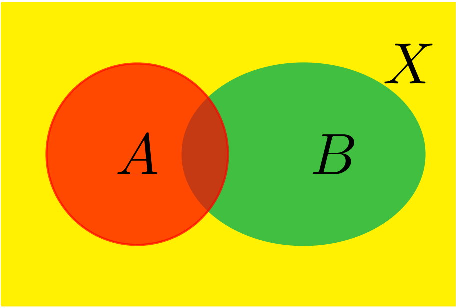

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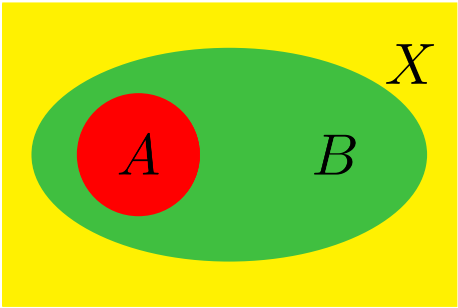

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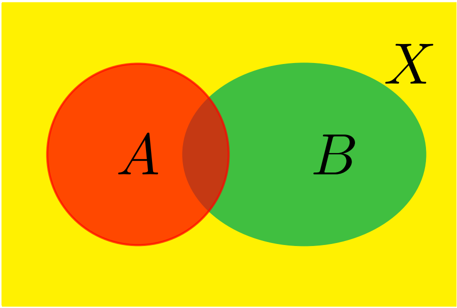

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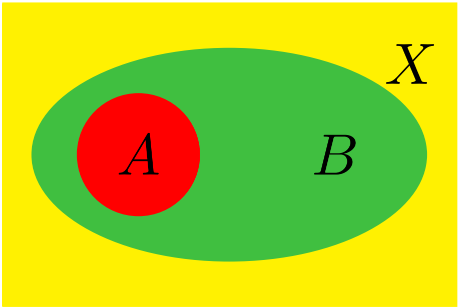

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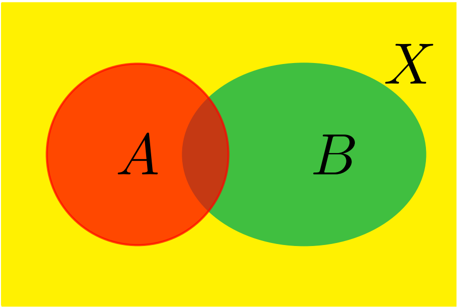

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Is it true that $\forall A, B \in \mathcal{P}(X)$ $\quad \underbrace{(A, B) \in R_\subset}_{A \subset B}$ or $\underbrace{(B, A) \in R_\subset}_{B \subset A}$?

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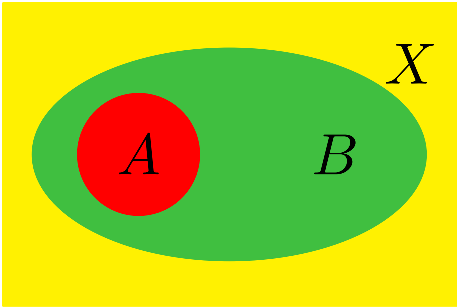

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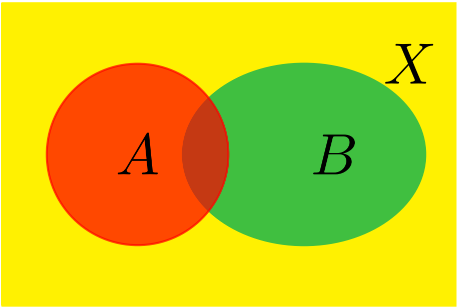

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Criteria for divisibility by 3 and 9

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Properties of relations

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$\leq$ on $\mathbb{R}$	$\equiv \mod 3$ on $\mathbb{Z}$	$\subset$ on $\mathcal{P}(X)$	divisibility on $\mathbb{N}$
reflexive $x \leq x$	reflexive $a \equiv a \mod 3$	reflexive $A \subset A$	reflexive $a \mid a$
antisymmetric $x \leq y \wedge y \leq x \implies x = y$	symmetric $a \equiv b \mod 3 \implies b \equiv a \mod 3$	antisymmetric $A \subset B \wedge B \subset A \implies A = B$	antisymmetric $a \mid b \wedge b \mid a \implies a = b$
transitive $x \leq y \wedge y \leq z \implies x \leq z$	transitive $a \equiv b \mod 3 \wedge b \equiv c \mod 3 \implies a \equiv c \mod 3$	transitive $A \subset B \wedge B \subset C \implies A \subset C$	transitive $a \mid b \wedge b \mid c \implies a \mid c$
total $\forall x, y \in \mathbb{R}$ $x \leq y \vee y \leq x$			

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- **Ordering relations:**

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Non-strict total (linear) order

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Example 3. **Similarity** of triangles

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Examples of equivalence relations

Examples of equivalence relations

Example 4.

Examples of equivalence relations

Example 4. Relation "to be **parallel**" is an equivalence relation

Examples of equivalence relations

Example 4. Relation "to be **parallel**" is an equivalence relation on the set of lines on a plane.

Examples of equivalence relations

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Example 5 (from linear algebra).

Examples of equivalence relations

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Example 5 (from linear algebra).
Two matrices $A, B \in Mat_n(\mathbb{R})$ are called **similar**

Examples of equivalence relations

Example 4. Relation "to be **parallel**" is an equivalence relation on the set of lines on a plane.

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Two matrices $A, B \in Mat_n(\mathbb{R})$ are called **similar** if there exists an invertible matrix C such that $B = C^{-1}AC$.

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Example 6. Friendship

Examples of equivalence relations

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What about congruence of real numbers modulo π?

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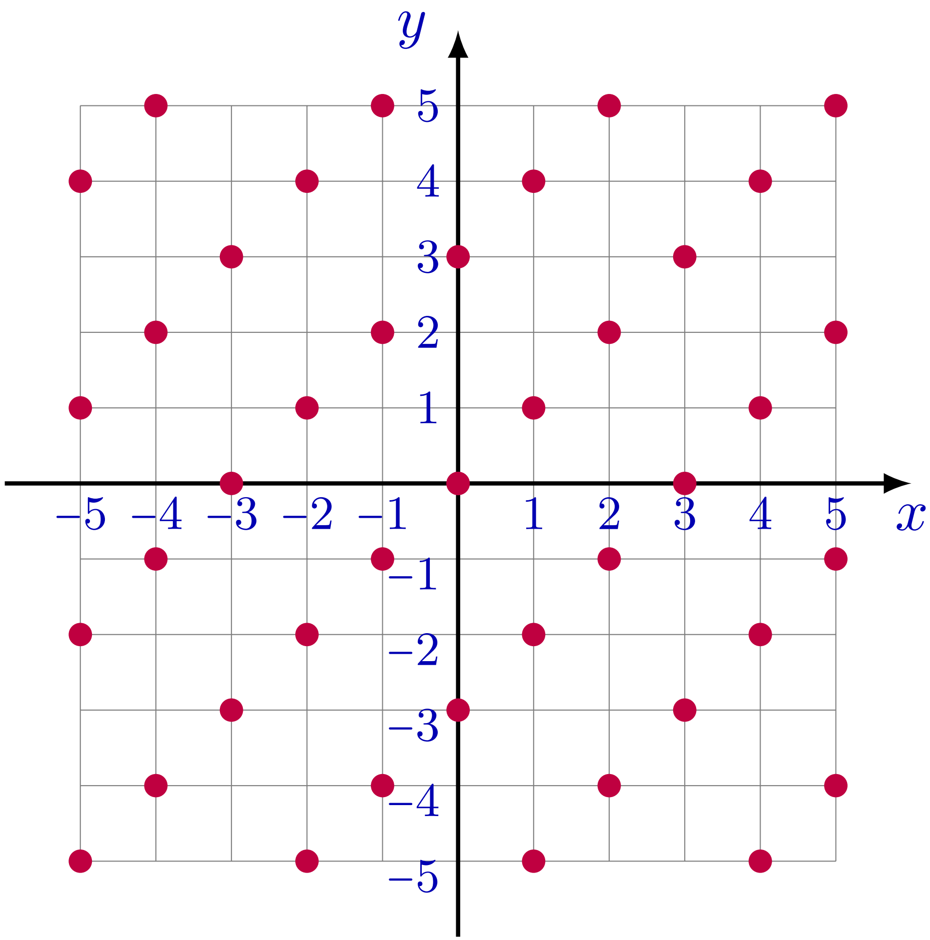

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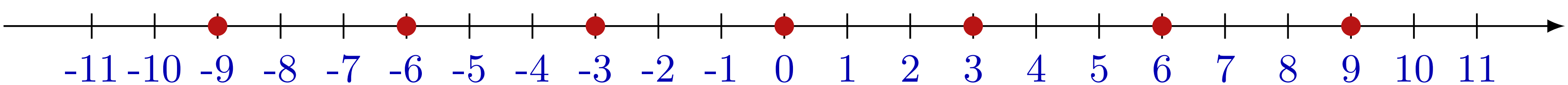

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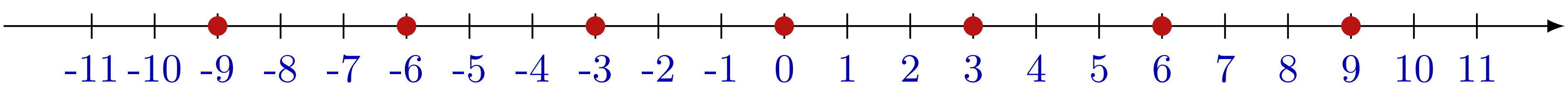

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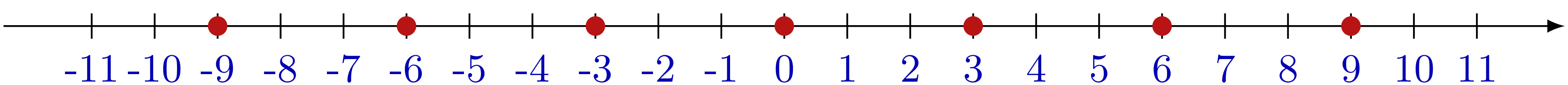

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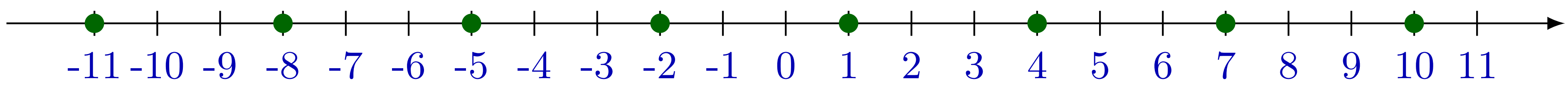

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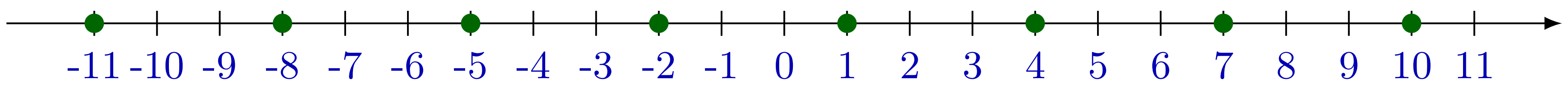

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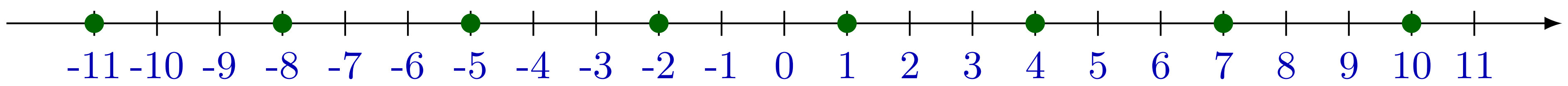

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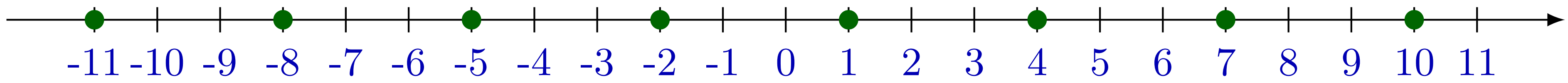

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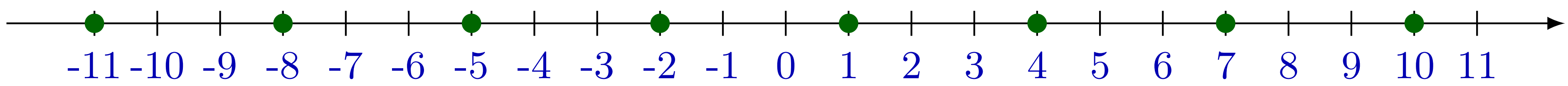

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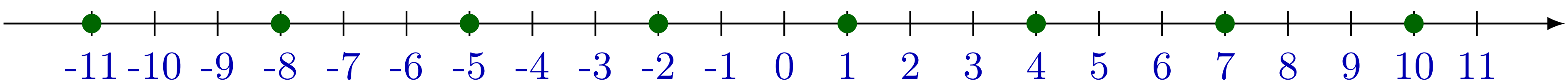

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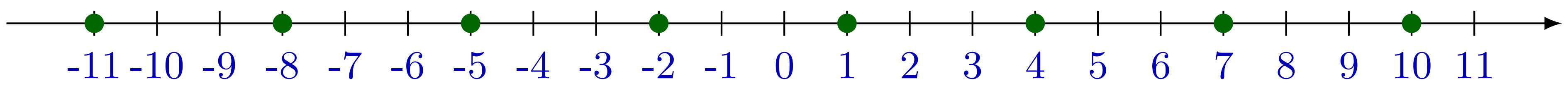

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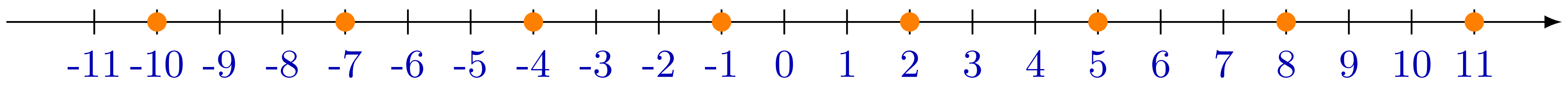

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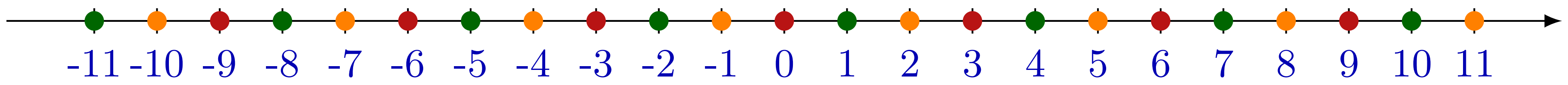

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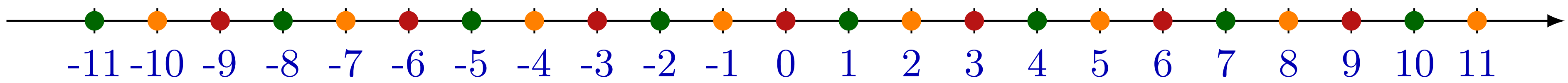

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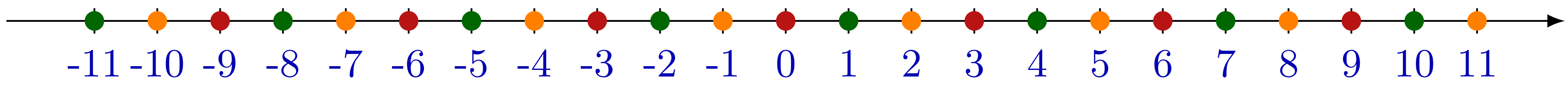

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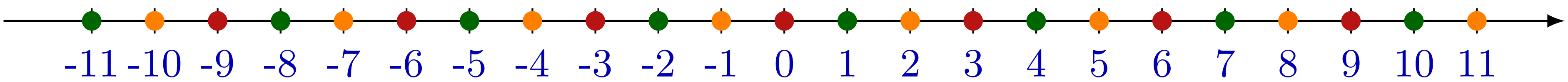

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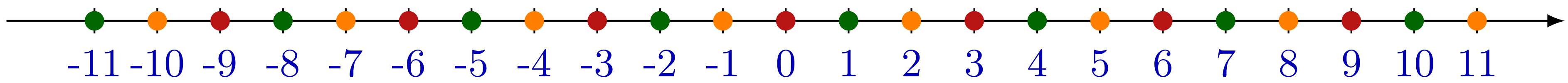

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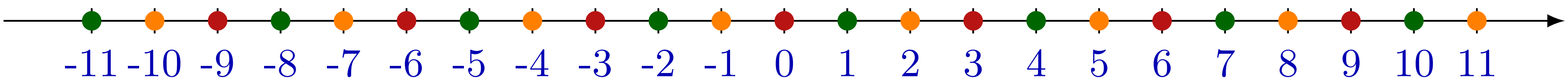

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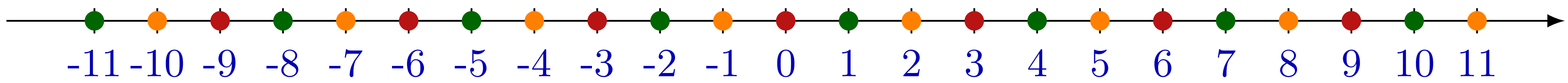

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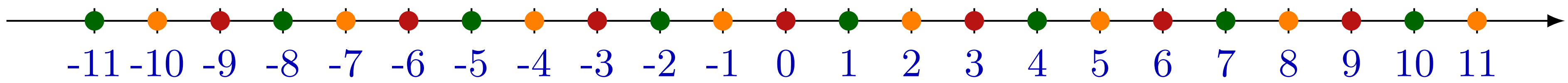

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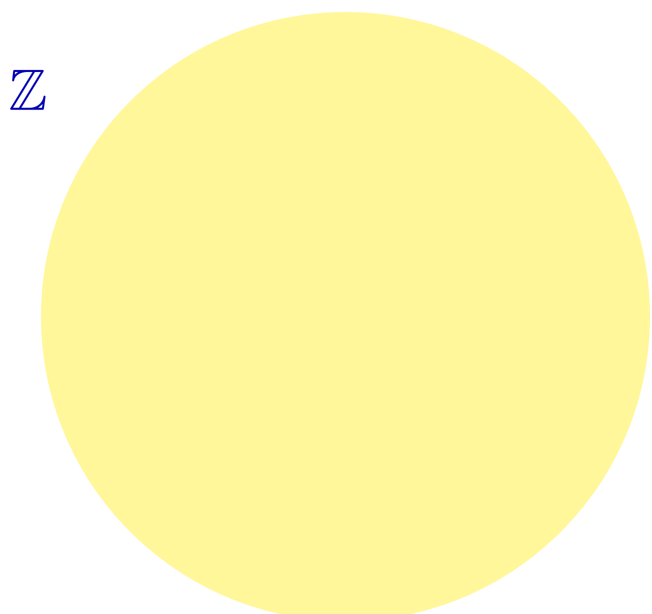

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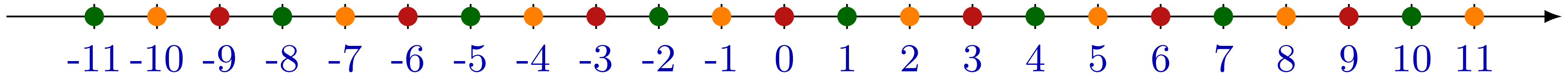

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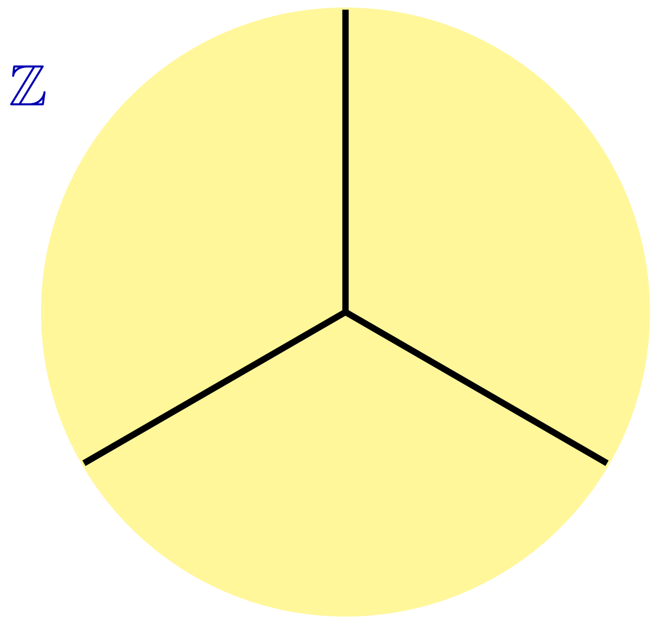

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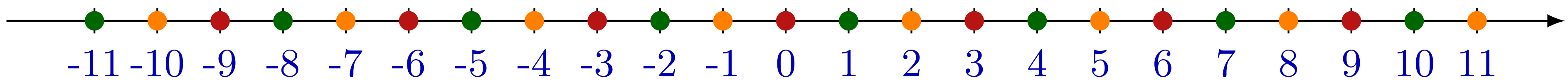

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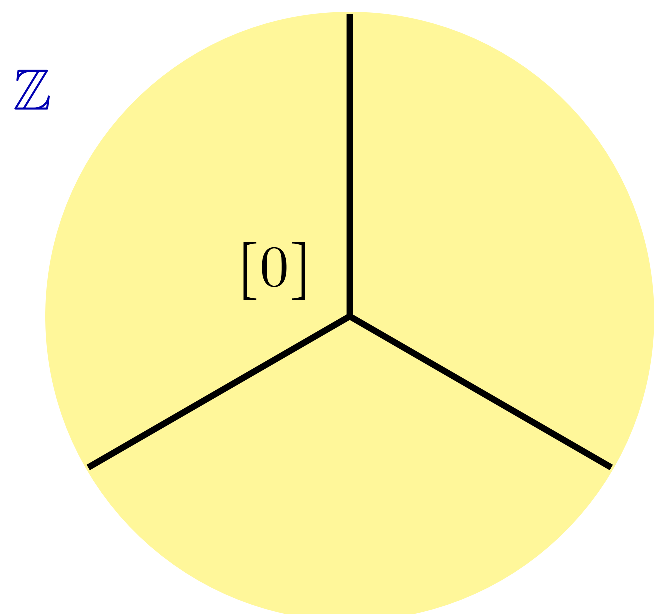

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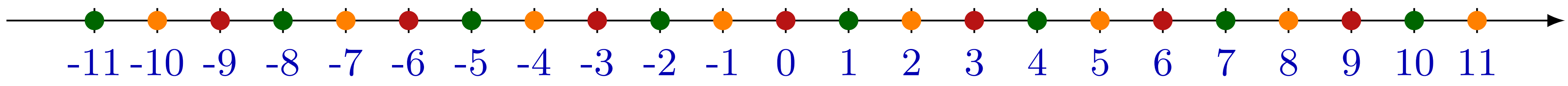

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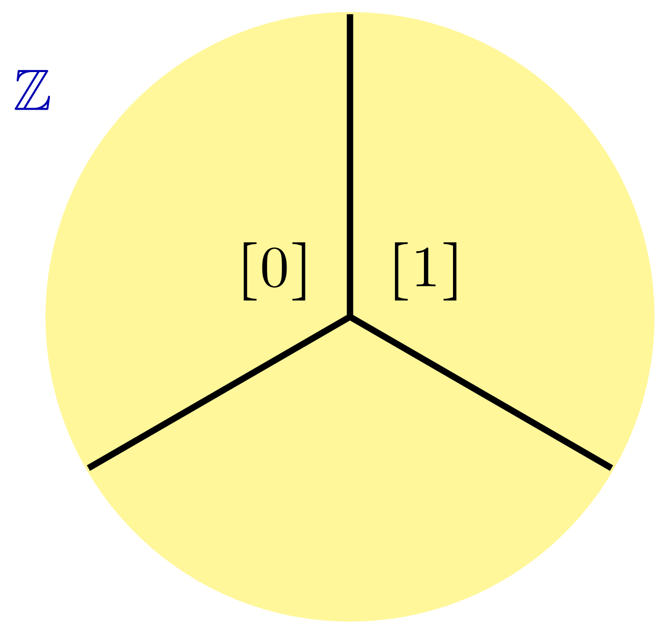

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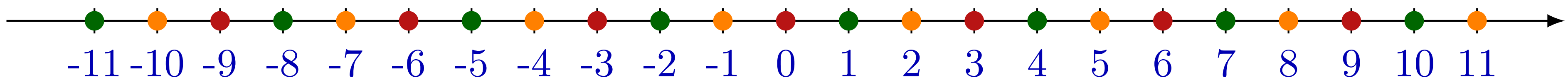

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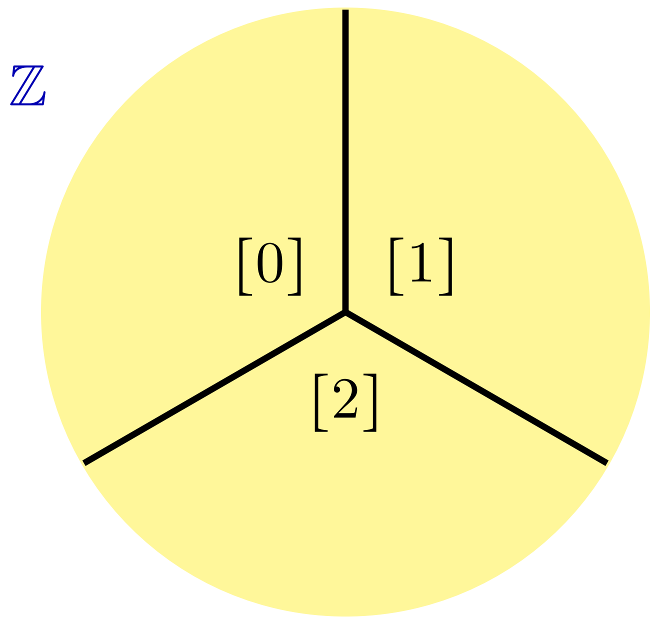

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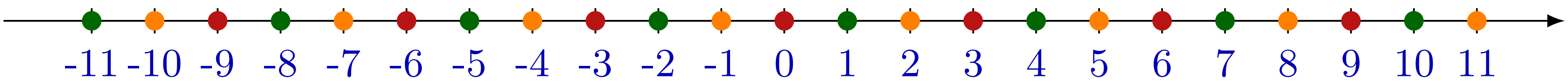

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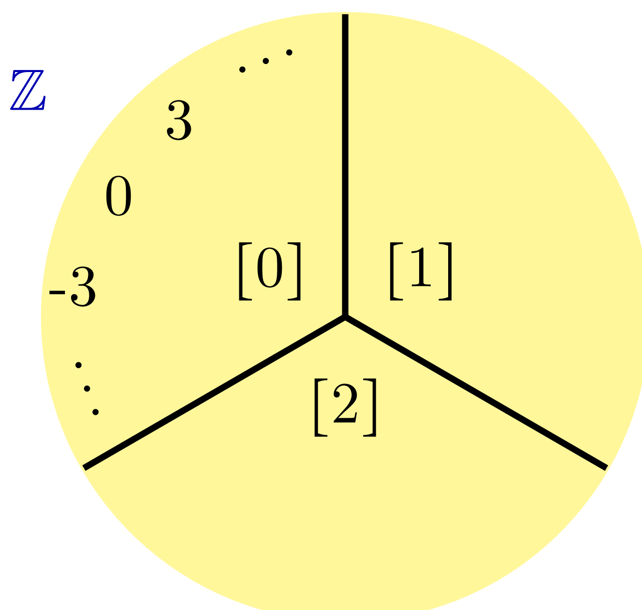

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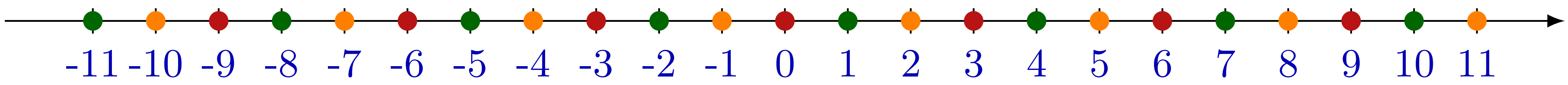

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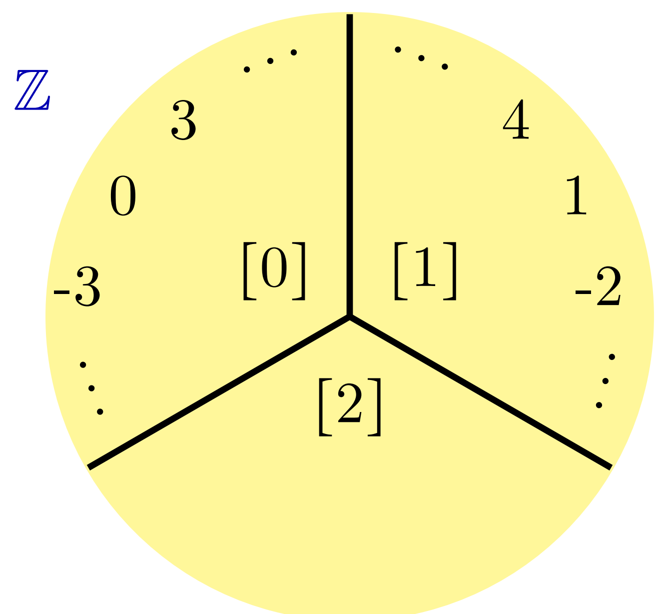

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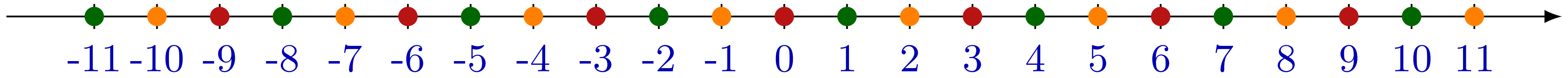

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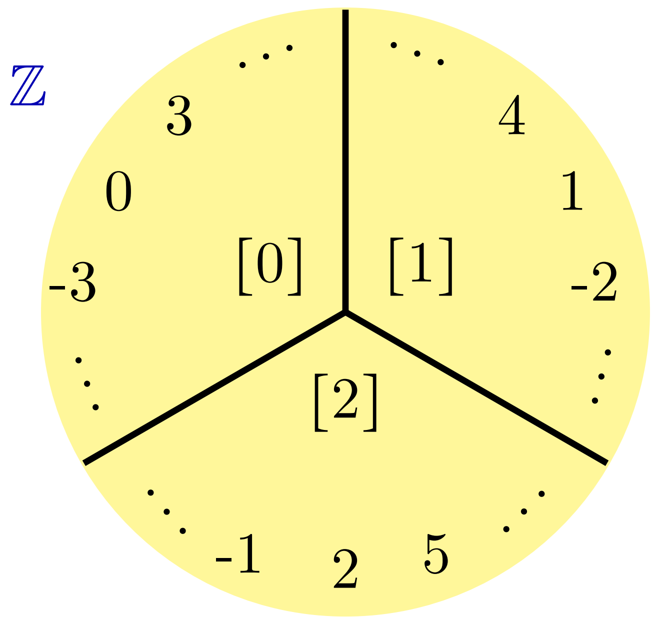

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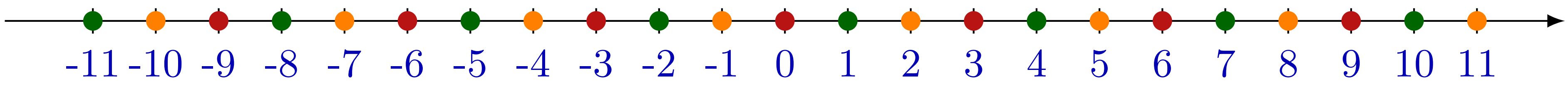

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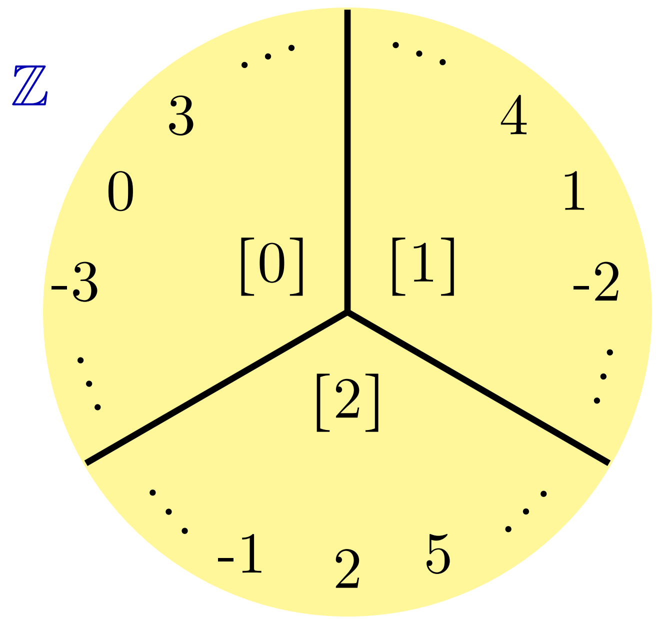

partition of $\mathbb{Z}$

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Together this gives us that $[a] = [b]$. □

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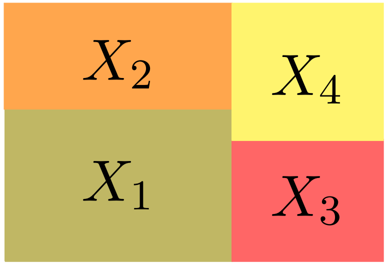

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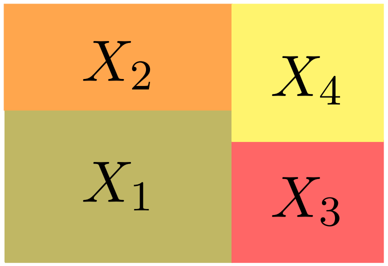

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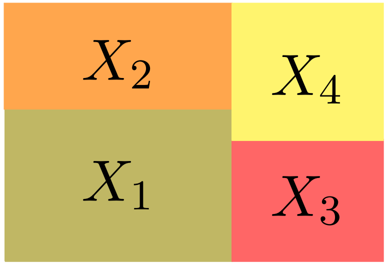

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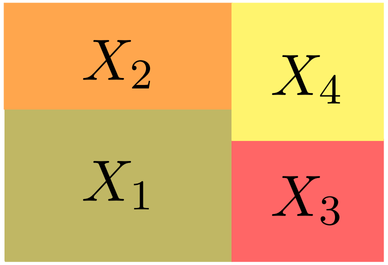

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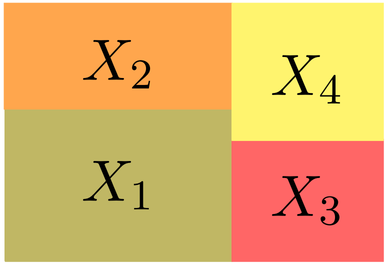

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Moreover, for a partition Σ of X, we denote the quotient set by $X/_{\Sigma}$.

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It is called the **injective quotient** of f .

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1. Prove that $\sim$ is an **equivalence relation** on $\mathbb{R}^2$.

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$$x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$$

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$$x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3) = (y_1 - y_2) + (y_2 - y_3)$$

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$$x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3) = (y_1 - y_2) + (y_2 - y_3) = y_1 - y_3\,,$$

as required for transitivity.

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Since $x_1 - x_2 = y_1 - y_2 \implies x_2 - x_1 = y_2 - y_1$, the symmetry takes place.

- For **transitivity**, we have to show that

$$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$$

$$\underbrace{(x_1, y_1) \sim (x_2, y_2)}_{x_1 - x_2 = y_1 - y_2} \wedge \underbrace{(x_2, y_2) \sim (x_3, y_3)}_{x_2 - x_3 = y_2 - y_3} \implies \underbrace{(x_1, y_1) \sim (x_3, y_3)}_{x_1 - x_3 = y_1 - y_3}.$$

$$x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3) = (y_1 - y_2) + (y_2 - y_3) = y_1 - y_3\,,$$

as required for transitivity.

Therefore, $\sim$ is an equivalence relation.

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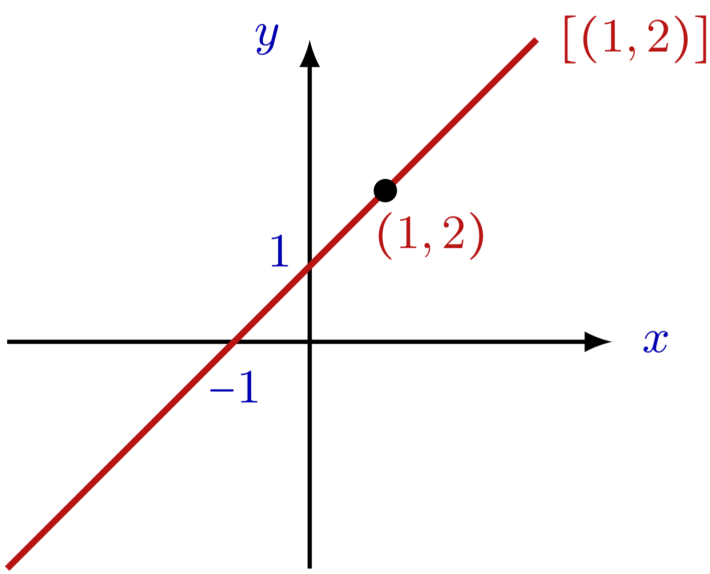

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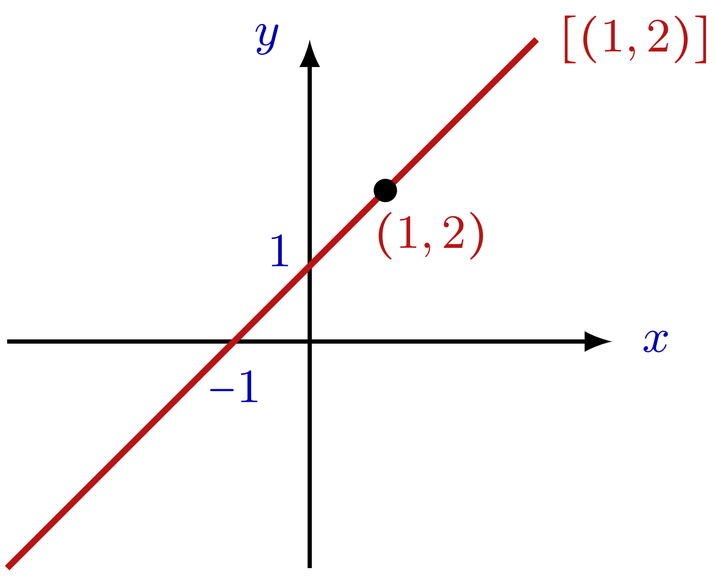

This line contains the point $(1,2) \in \mathbb{R}^2$.

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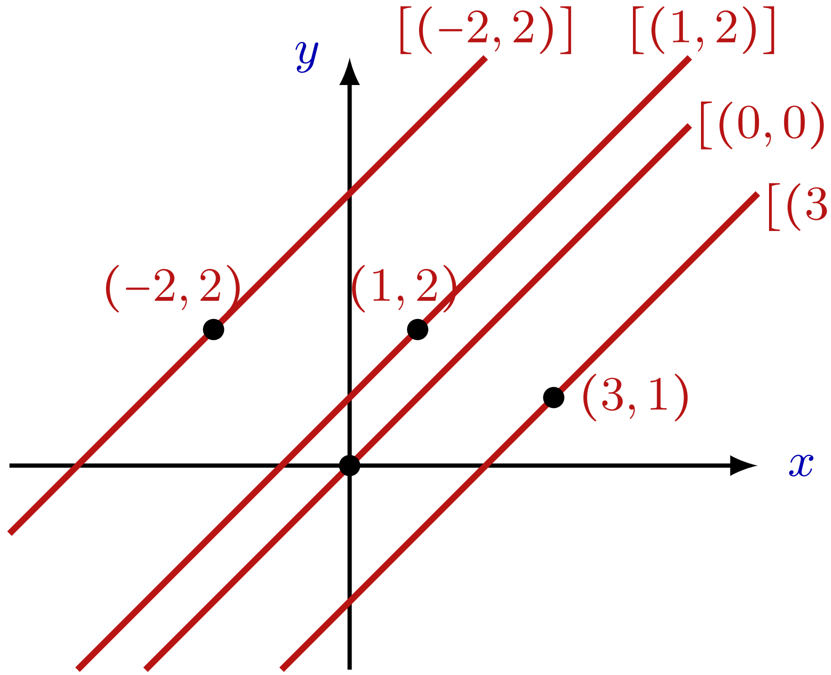

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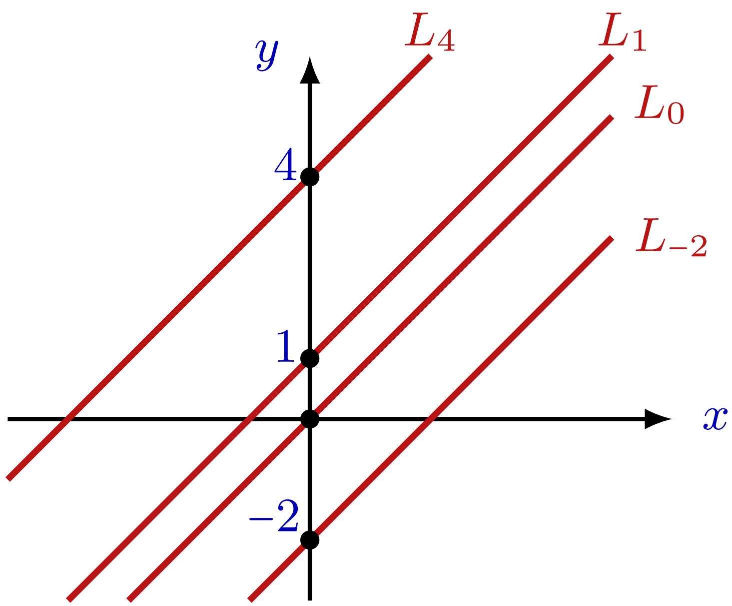

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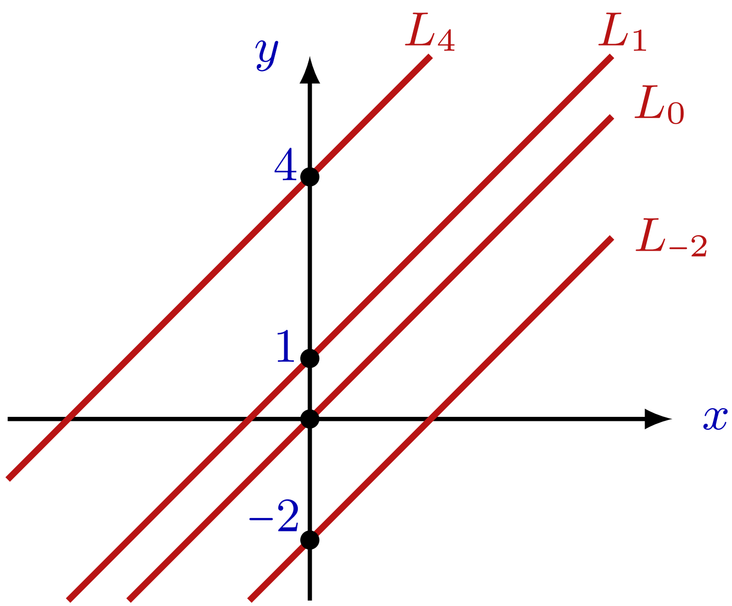

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