Lecture 4

Definitions in Mathematics

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Definition. The vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ are said to be **linearly dependent** if there exist numbers $a_1, a_2, \dots, a_n$, which are not all zeros, such that $a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \dots + a_n\overrightarrow{v_n} = \overrightarrow{0}$.

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Exrecise. Prove that vectors $\overrightarrow{v_1}, \overrightarrow{v}_2, \dots \overrightarrow{v_n}$ are linearly independent iff

$$\forall\ a_1, a_2, \dots, a_n \ \left(a_1\overrightarrow{v}_1 + a_2\overrightarrow{v}_2 + \dots + a_n\overrightarrow{v}_n = \overrightarrow{0} \implies a_1 = a_2 = \dots = a_n = 0\right)$$

Definition of ring (from Algebra)

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6. $\forall\, a \in R \ \exists -a \in R \ \ a + (-a) = 0$ (each element in R has an **additive inverse**)
7. $\forall\, a, b, c \in R \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ ($\cdot$ is **associative**)
8. $\forall\, a, b, c \in R \quad a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$
(multiplication **distributes** over addition)

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• If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a$ ($\cdot$ is **commutative**),
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- If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a$ ($\cdot$ is **commutative**), then R is called a **commutative** ring.
- If, additionally, $\exists\, 1 \in R \quad \forall\, a \in R \quad 1 \cdot a = a \cdot 1 = a$

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- If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a$ ($\cdot$ is **commutative**),
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- If, additionally, $\exists\, 1 \in R \quad \forall\, a \in R \quad 1 \cdot a = a \cdot 1 = a$

(there exists a **multiplicative identity**), then R is called a ring with **unity**.

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8. $\forall\, a, b, c \in R \quad a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$
(multiplication **distributes** over addition)

- If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a$ ($\cdot$ is **commutative**),
then R is called a **commutative** ring.
- If, additionally, $\exists\, 1 \in R \quad \forall\, a \in R \quad 1 \cdot a = a \cdot 1 = a$
(there exists a **multiplicative identity**), then R is called a ring with **unity**.

The properties are called the **axioms** of a ring.

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Important: To prove that each of the listed above objects is a ring, we have to verify all ring axioms.

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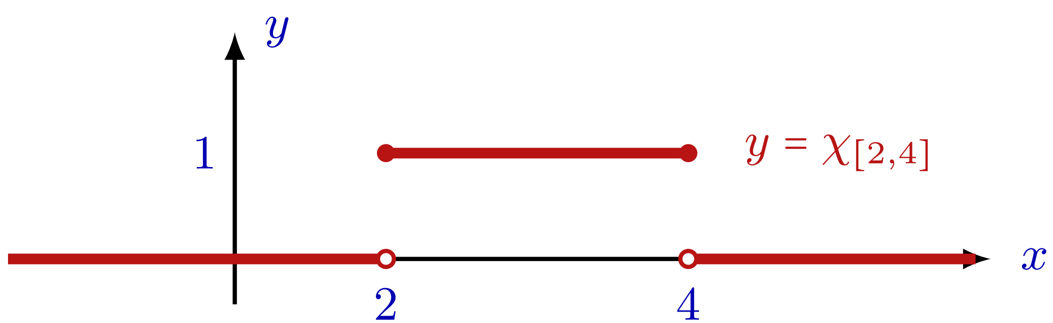

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Overall, $A \subset B \iff \mathcal{P}(A) \subset \mathcal{P}(B)$ □

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Exercise 2. Formulate and prove a similar identity for $(g \circ f)^*$.

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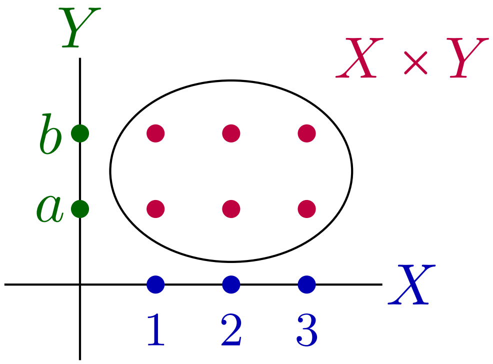

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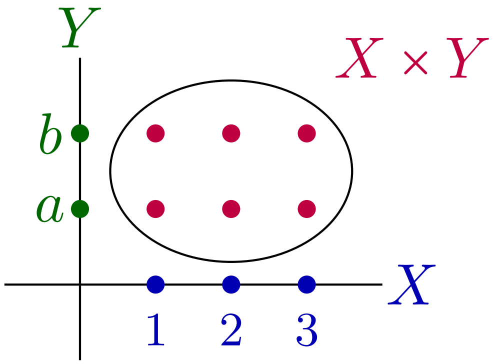

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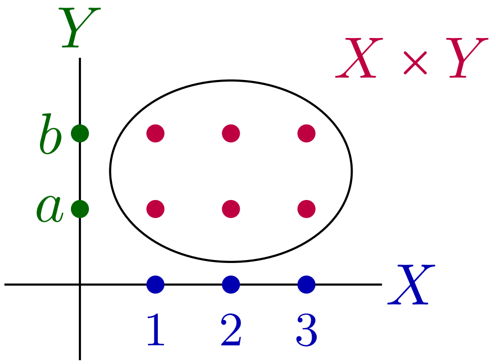

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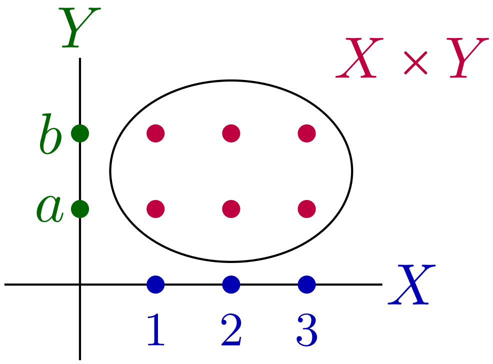

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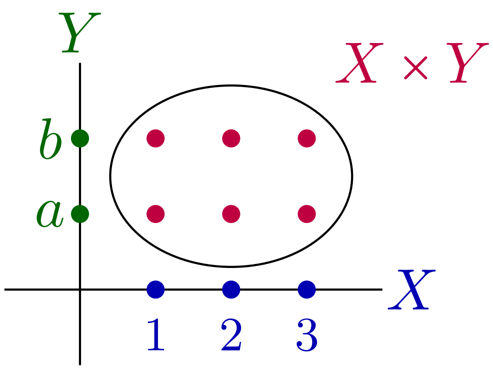

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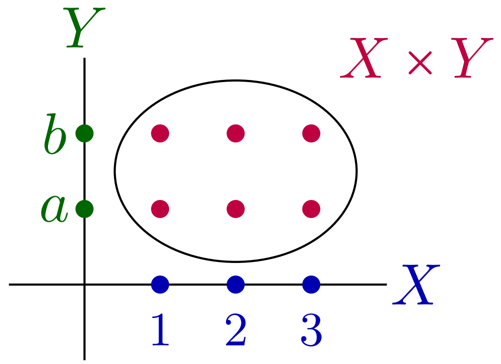

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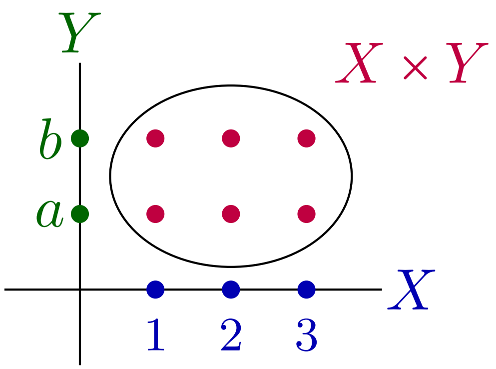

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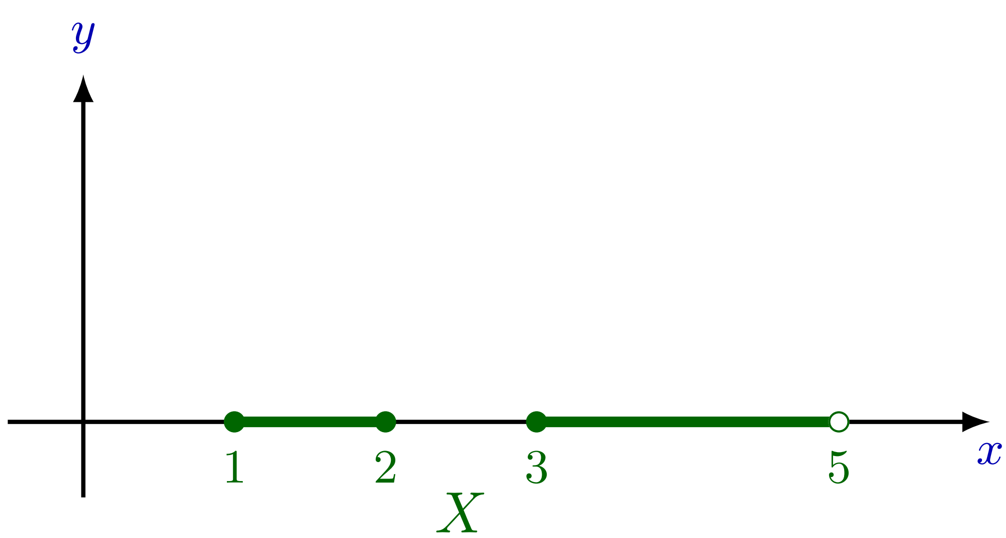

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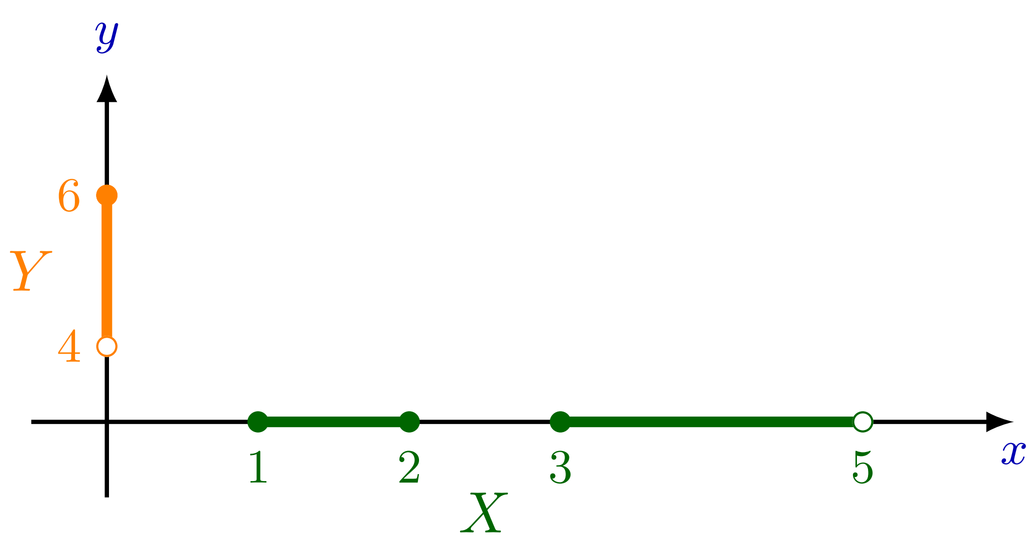

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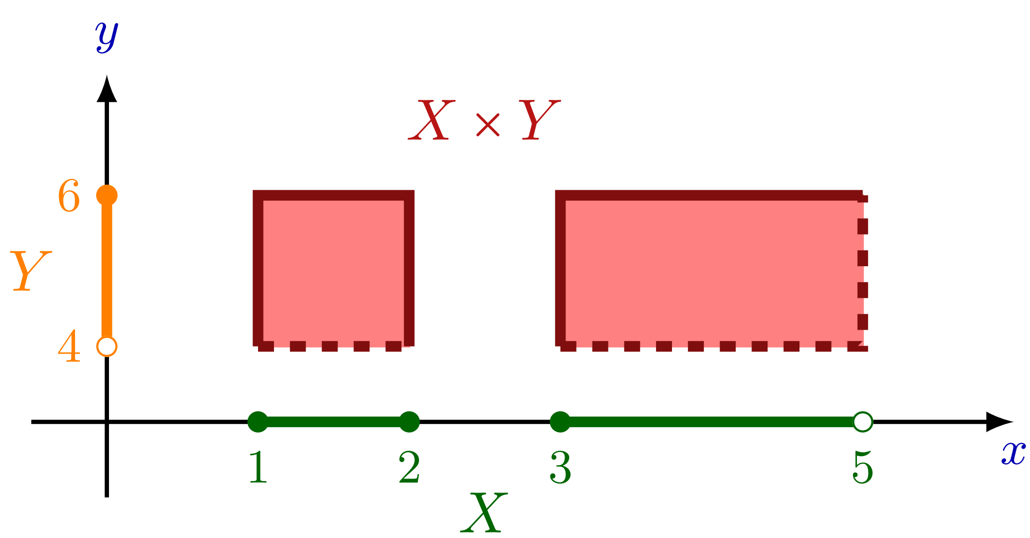

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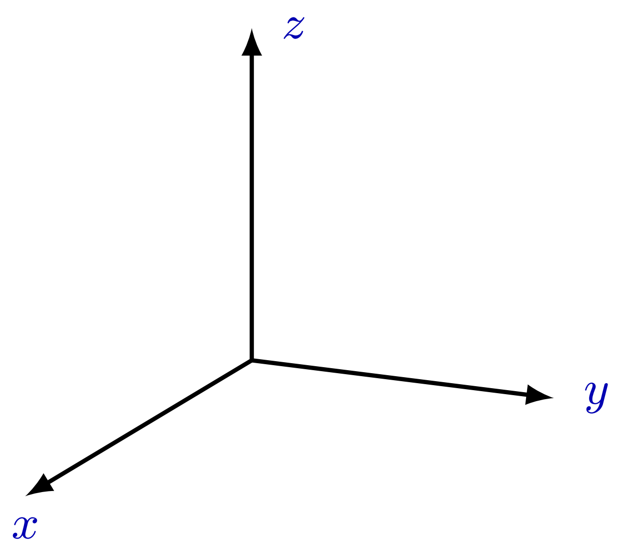

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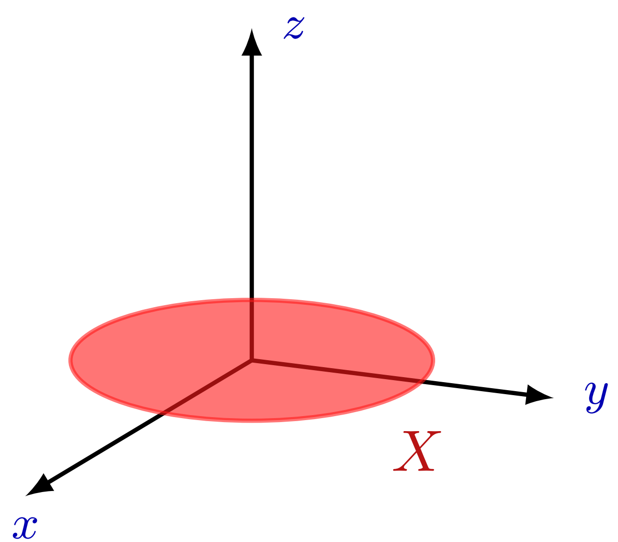

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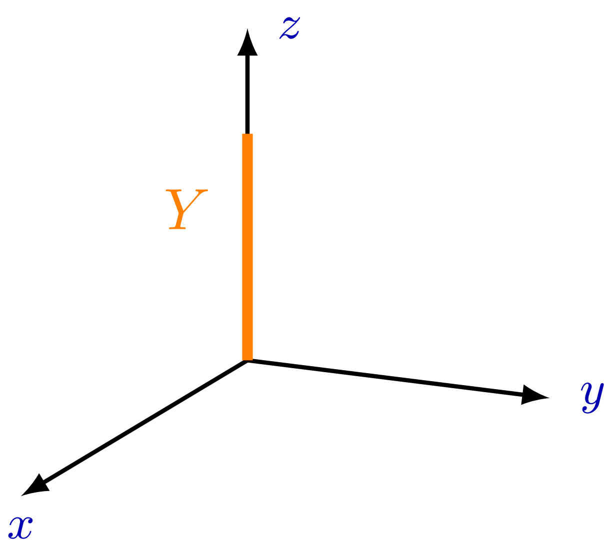

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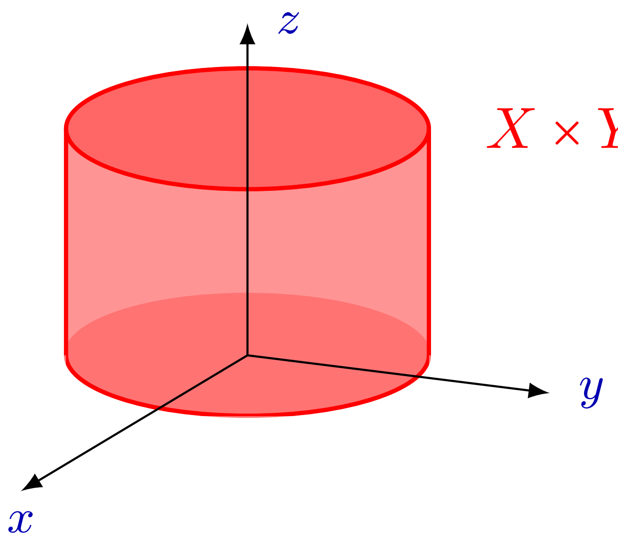

$X \times Y$ is a solid cylinder in $\mathbb{R}^3$

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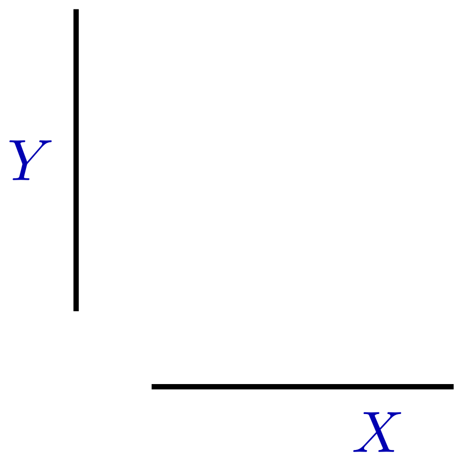

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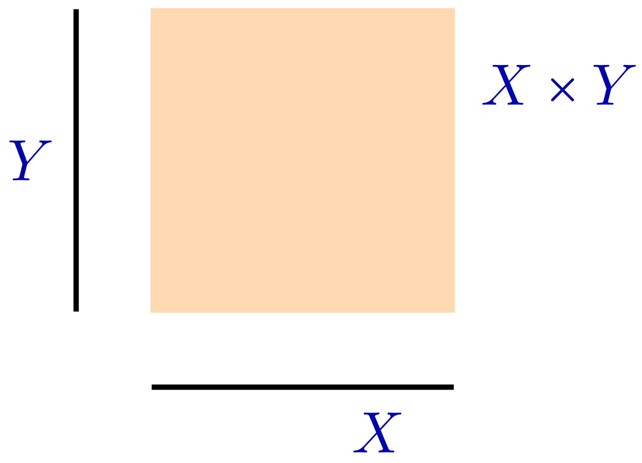

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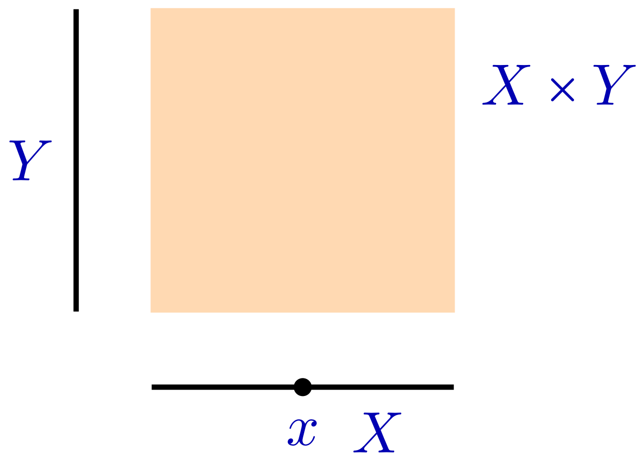

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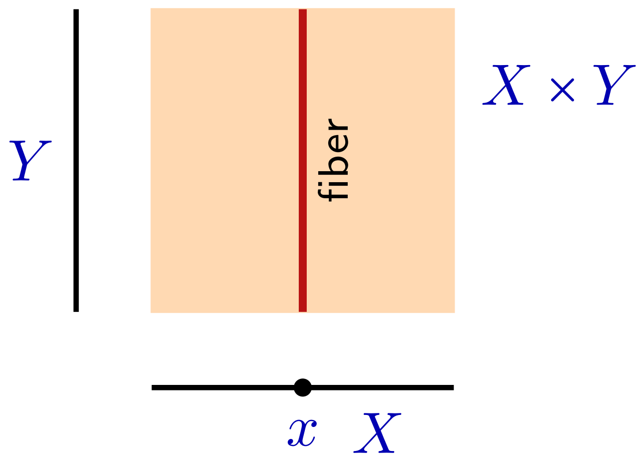

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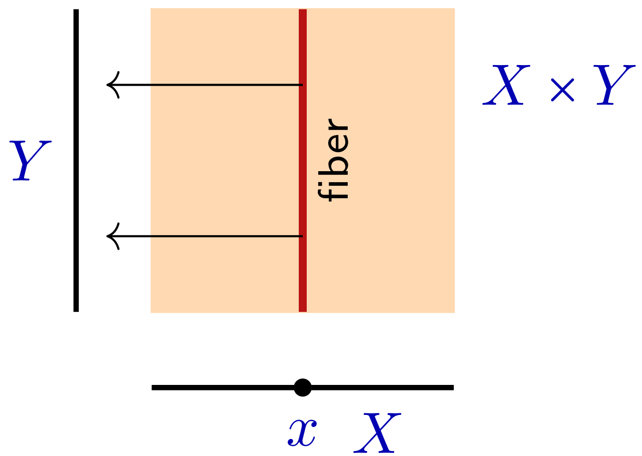

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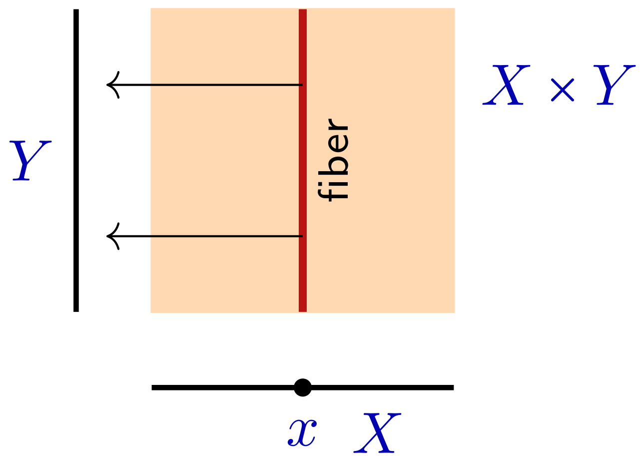

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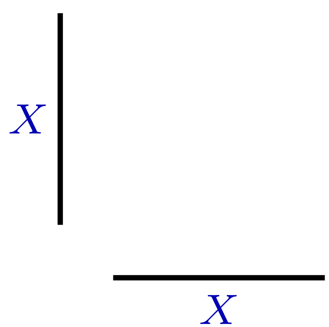

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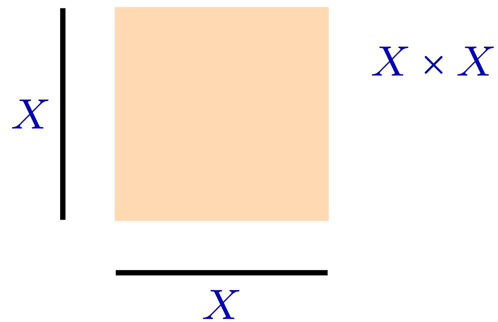

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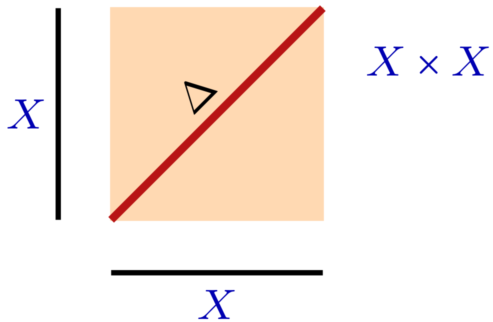

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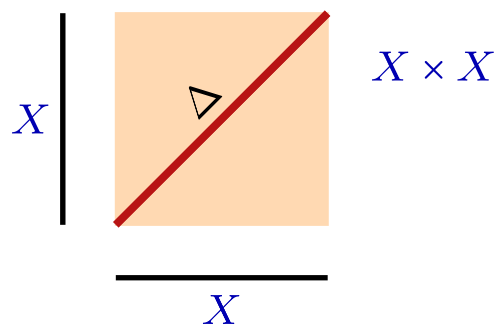

The diagonal is the image of $\mathrm{id}_X \odot \mathrm{id}_X$.

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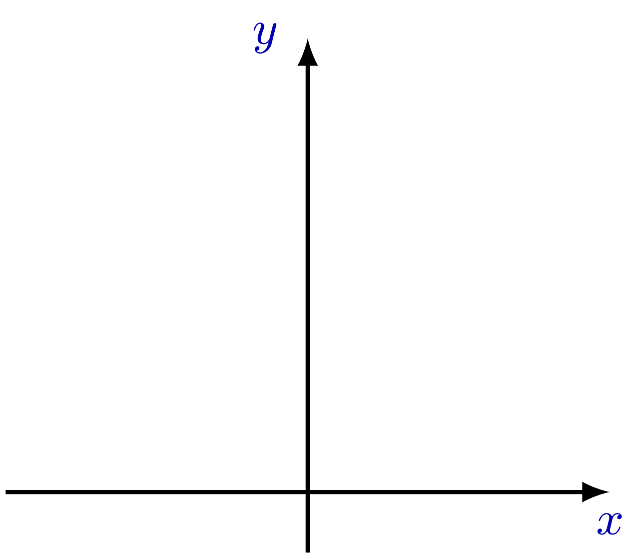

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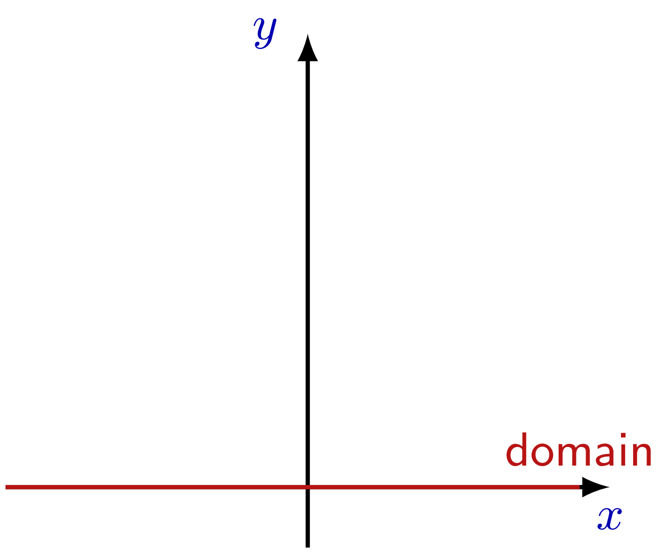

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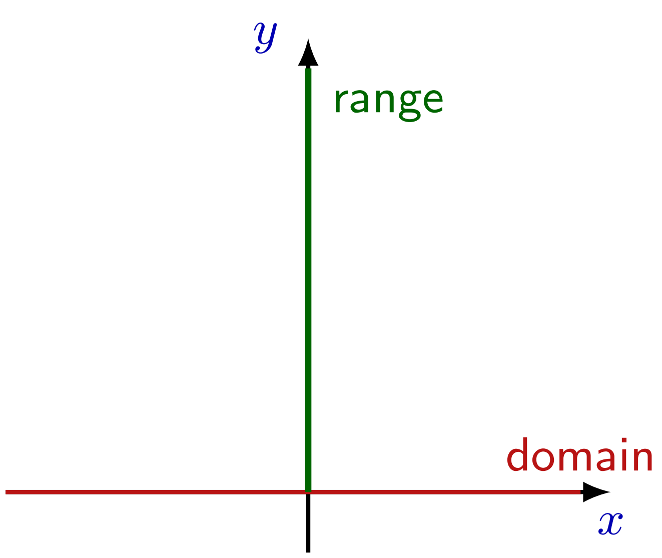

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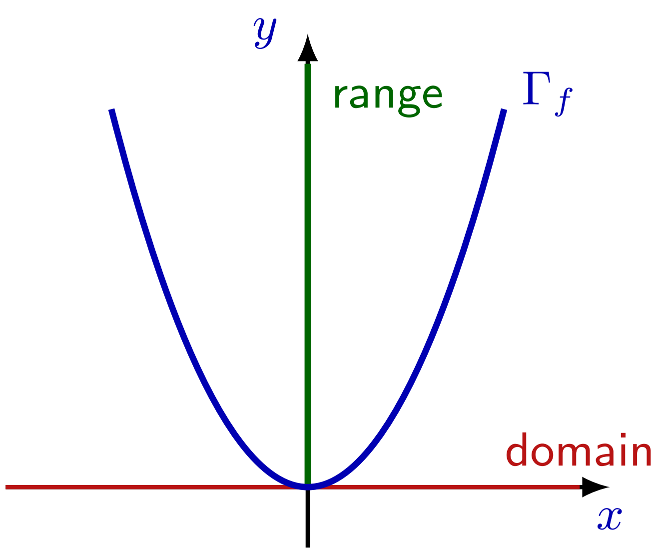

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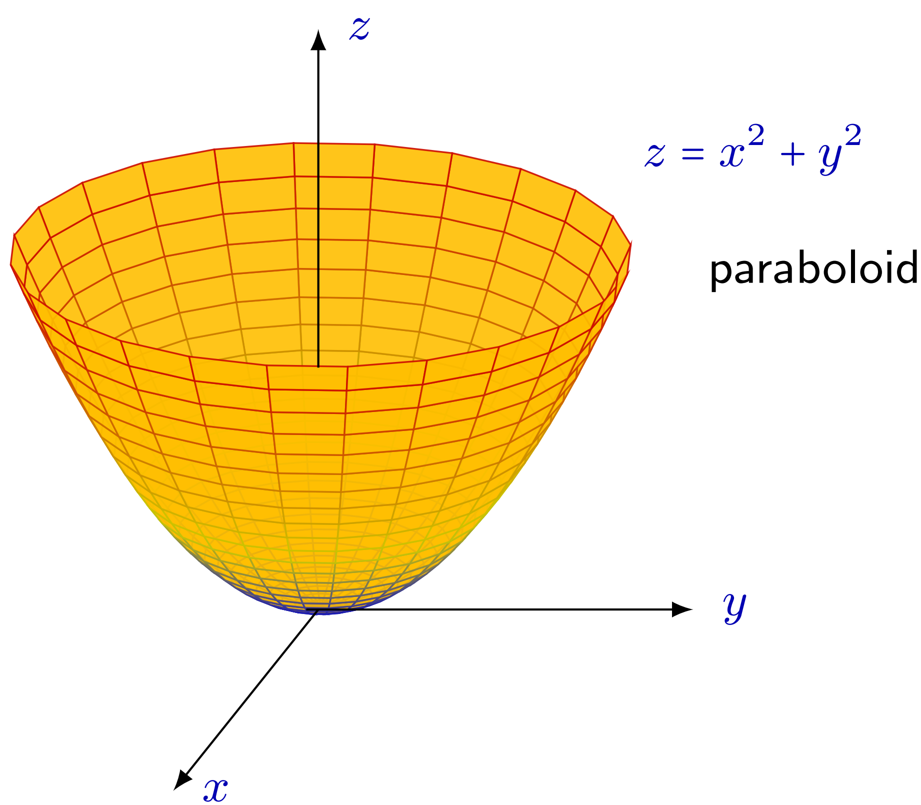

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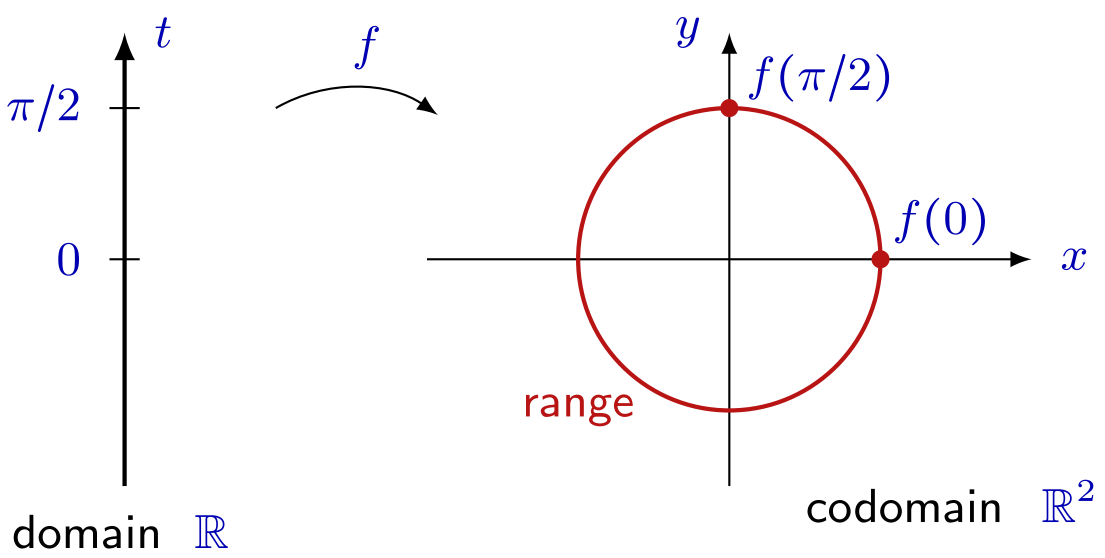

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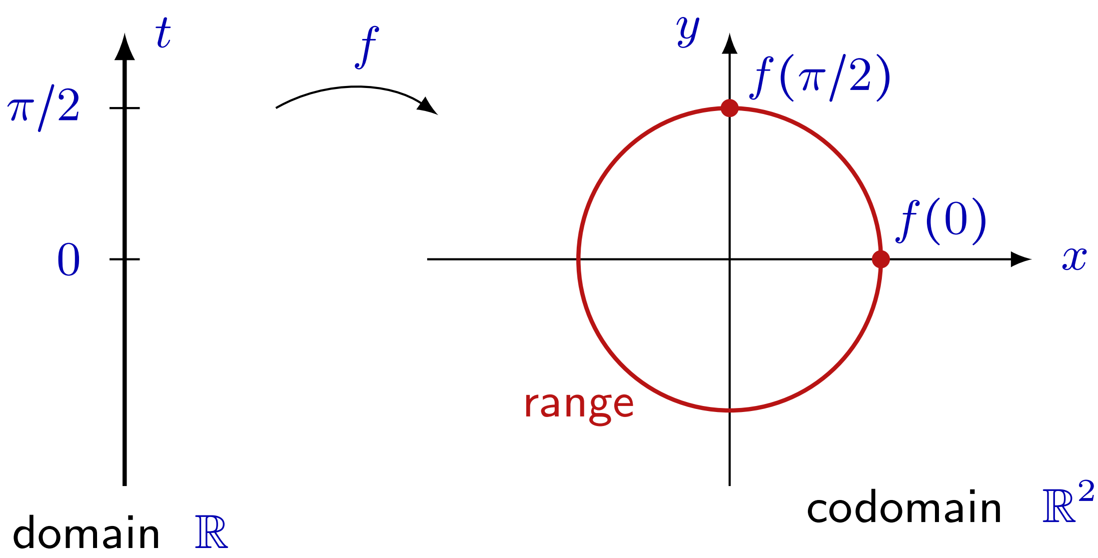

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f reels up the line on the circle.

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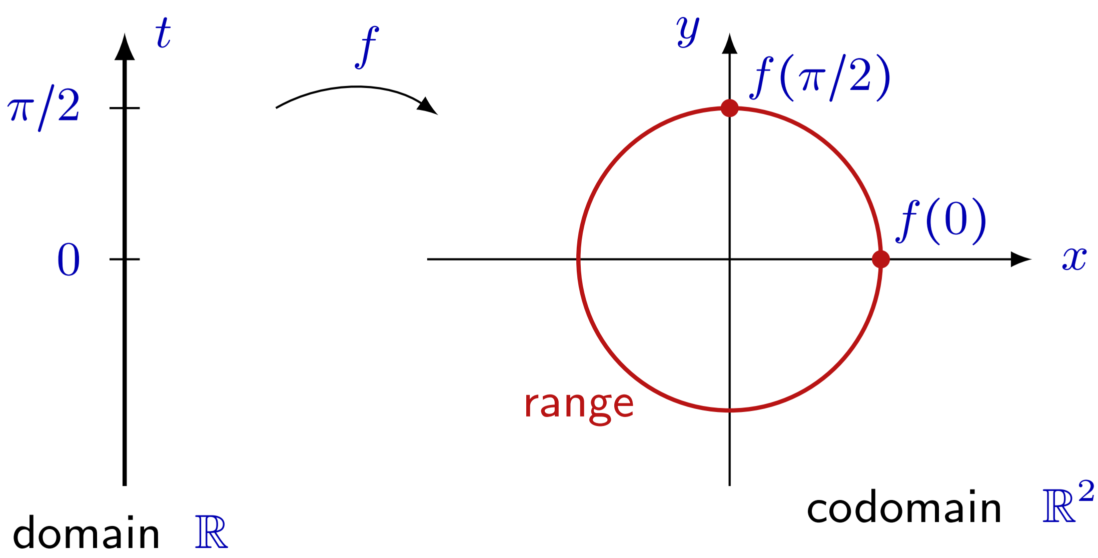

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The graph of f is the set $\Gamma_f=\{(t,\cos t,\sin t)\in\mathbb{R}\times\mathbb{R}^2\}$

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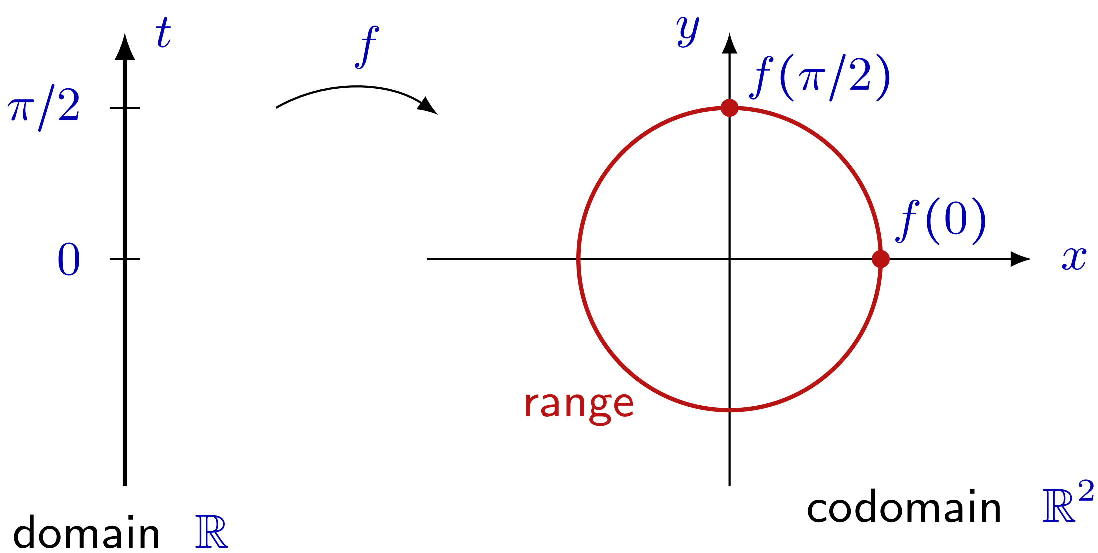

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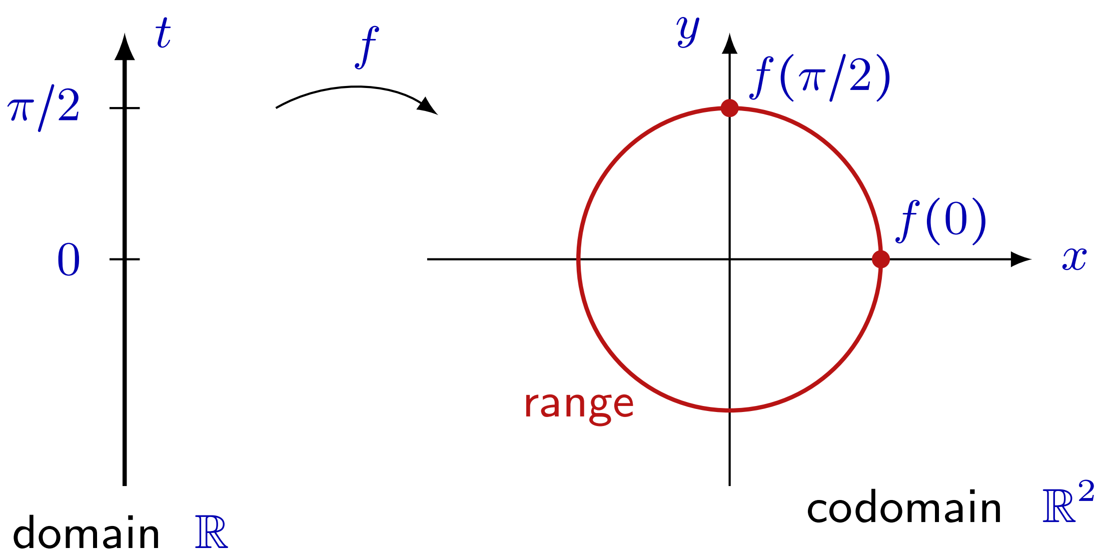

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Γ_f is a curve in $\mathbb{R}^3$. It is called helix.

Helix

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The graph of $f : \mathbb{R} \to \mathbb{R}^2$, $\quad f(t) = (\cos t, \sin t)$

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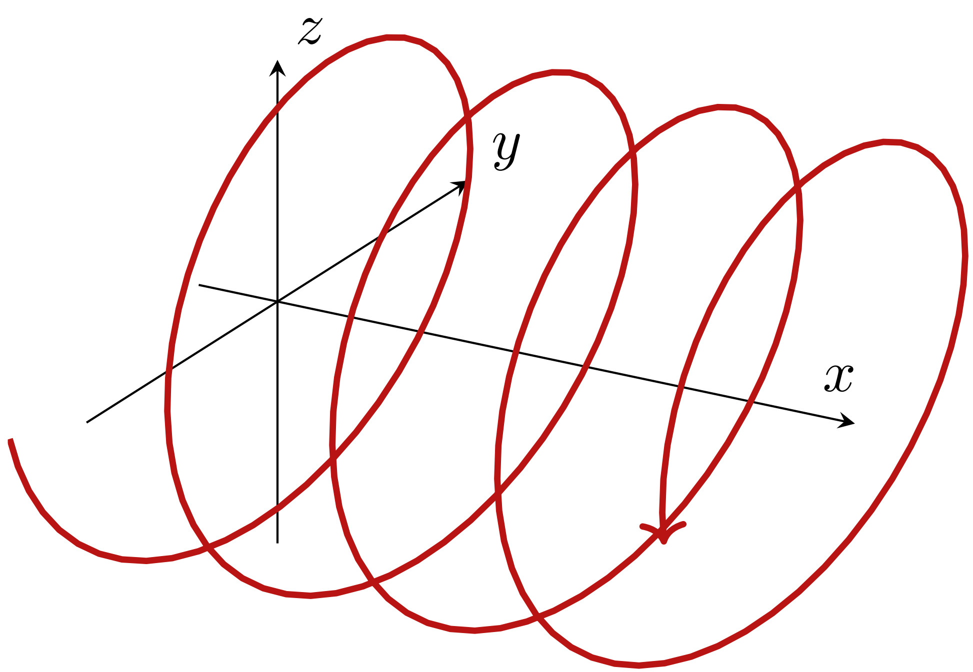

Metric

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Definition.

Metric

Definition. A **metric**

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Definition. A **metric** (or distance function) on a set X

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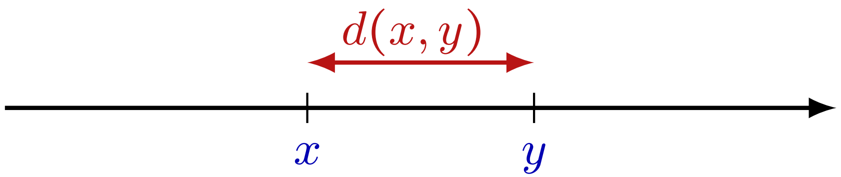

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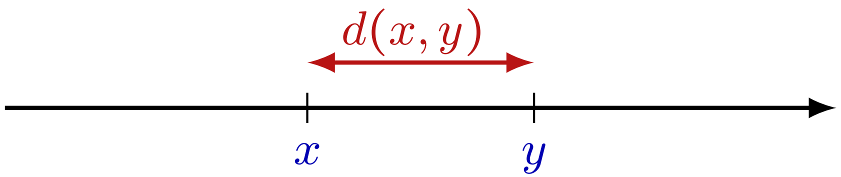

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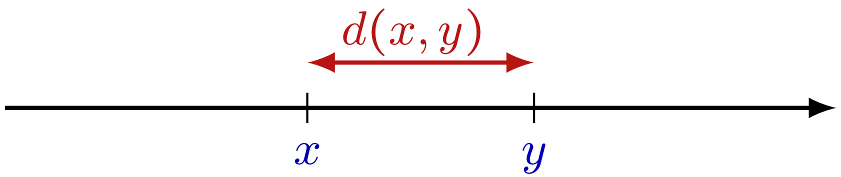

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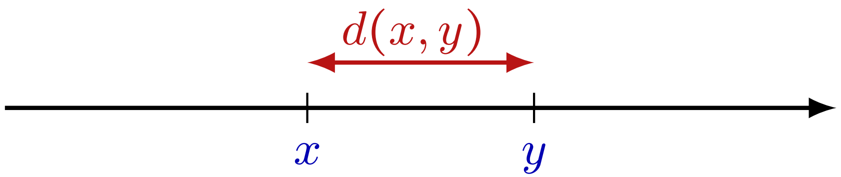

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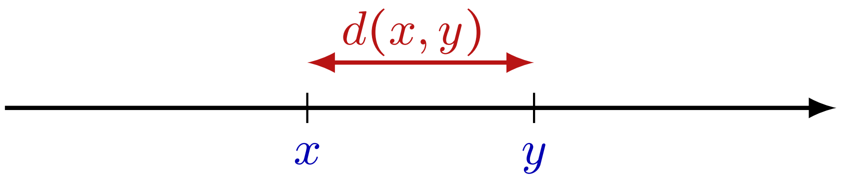

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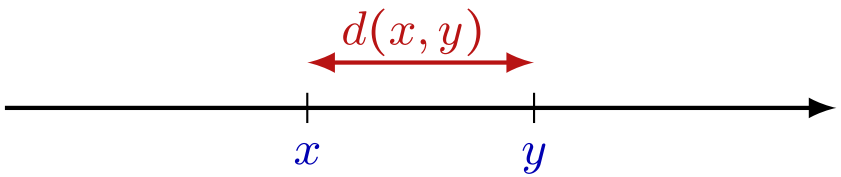

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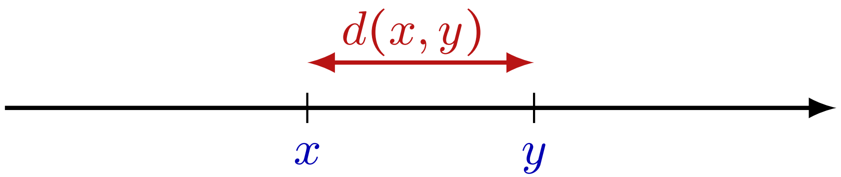

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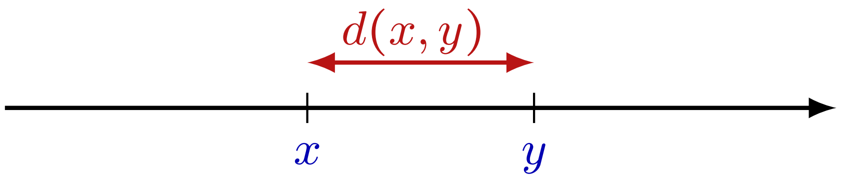

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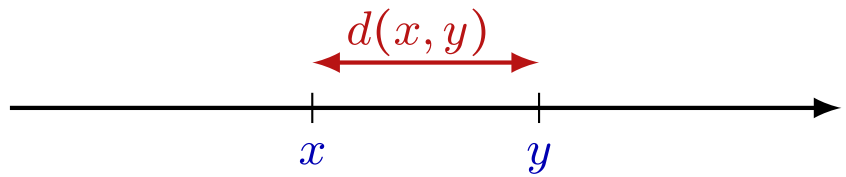

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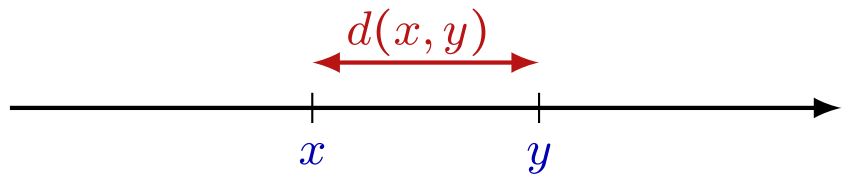

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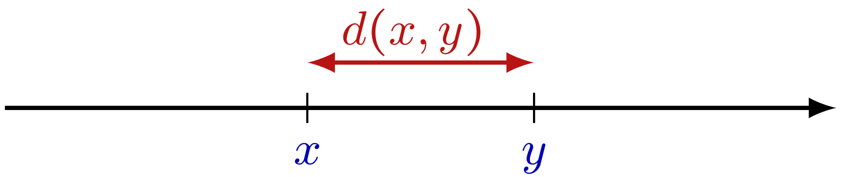

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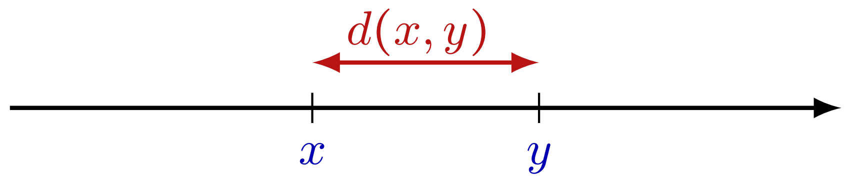

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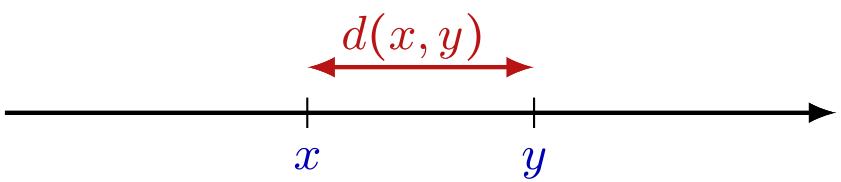

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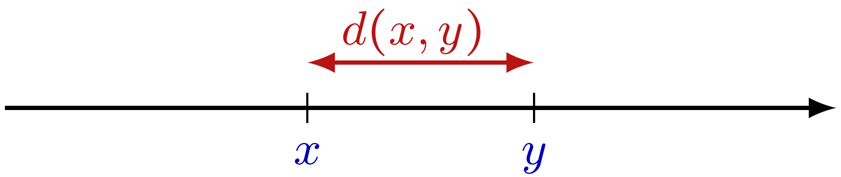

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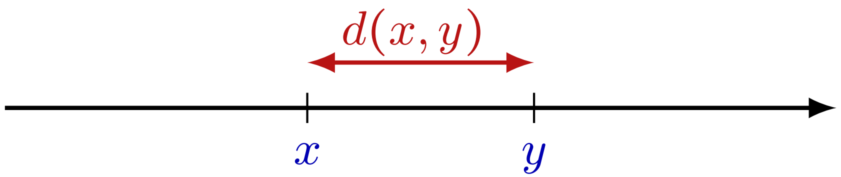

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($|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$)

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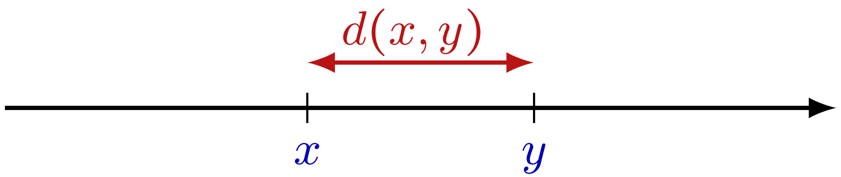

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Therefore, all axioms are satisfied and the map d is a metric.

Euclidean metric on a plane

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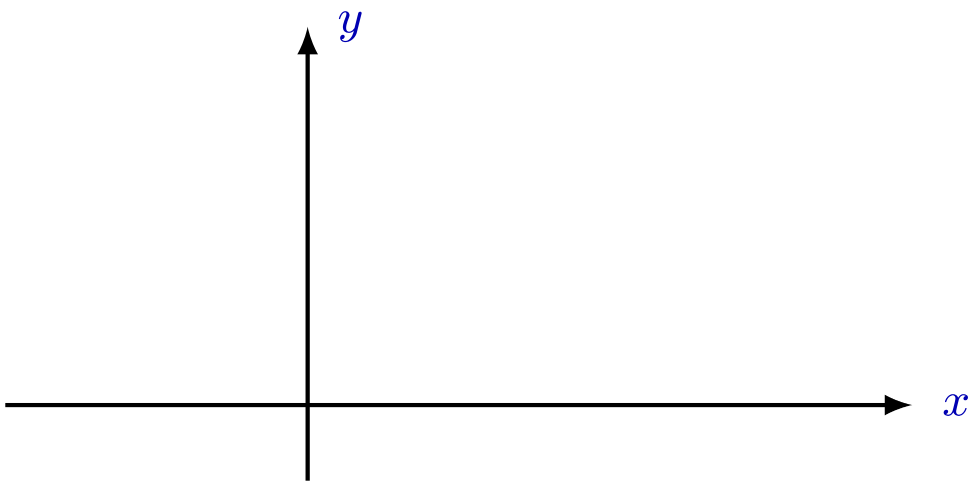

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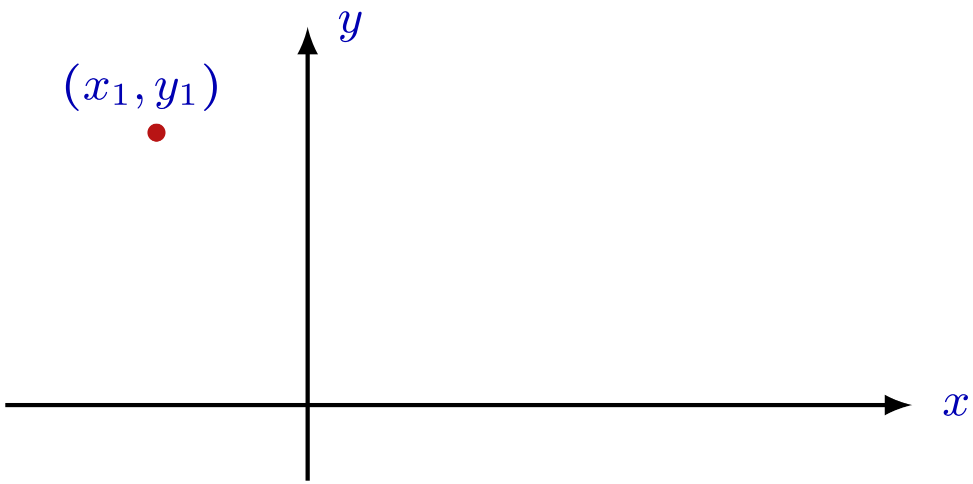

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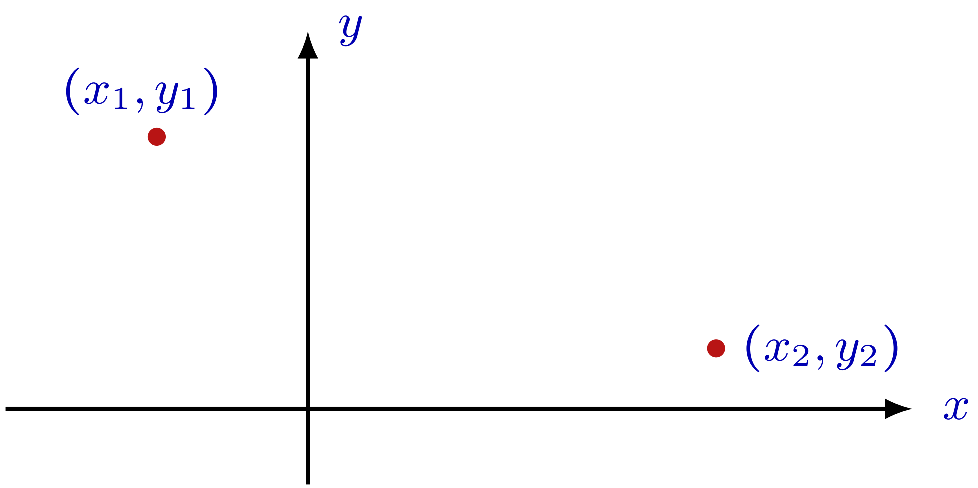

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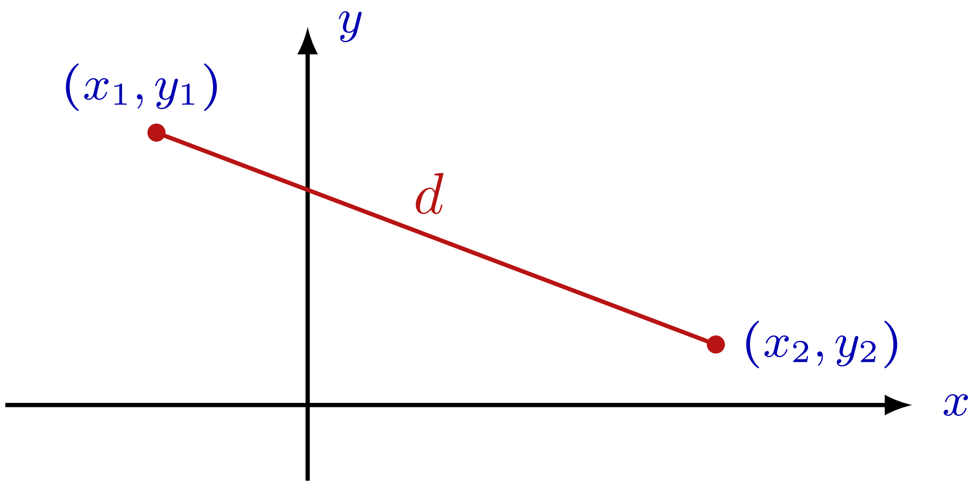

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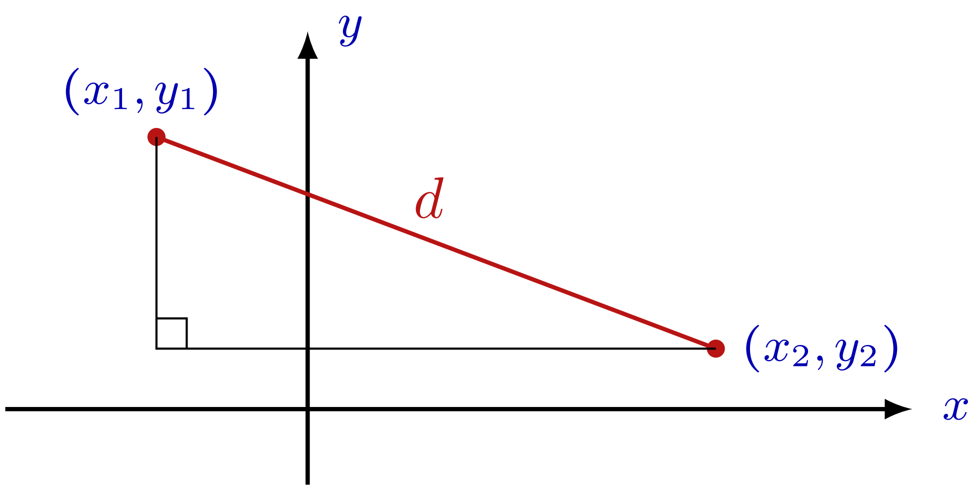

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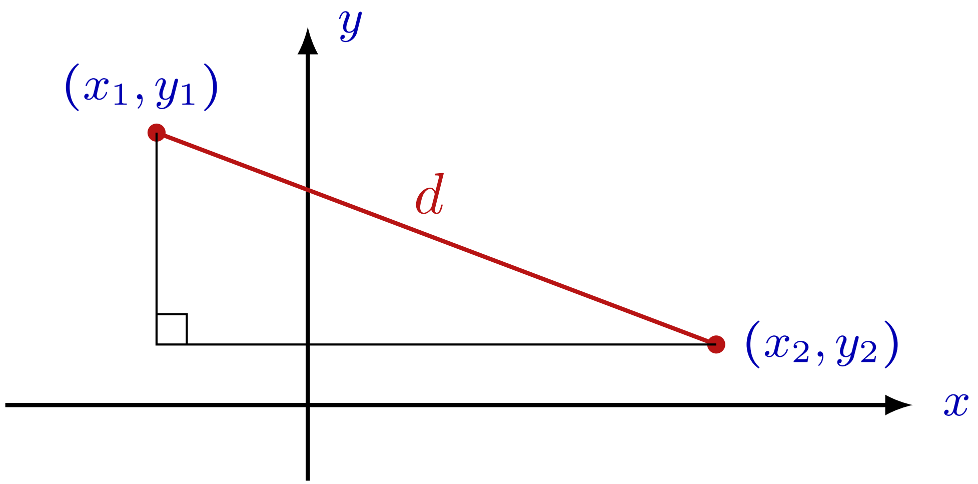

This metric is called **Euclidean**.

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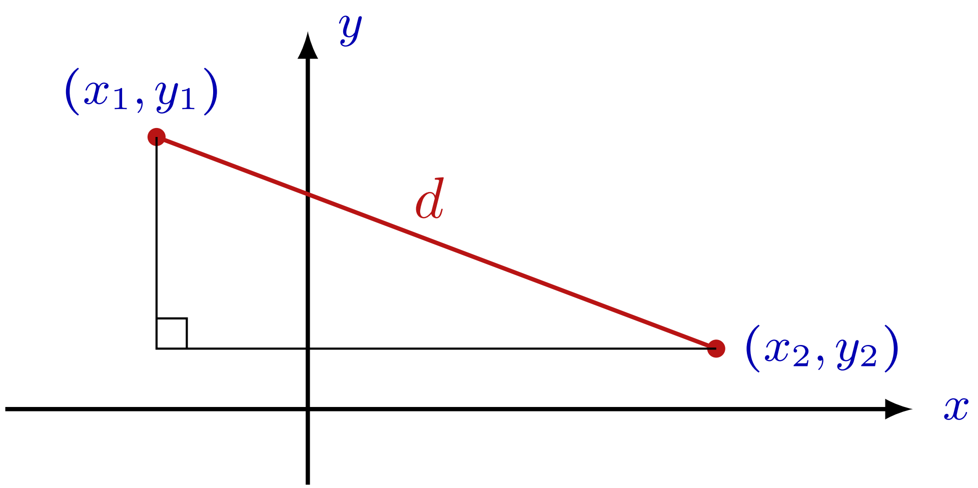
Theorem. A map $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, defined by

$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, is a metric.



This metric is called **Euclidean**.

Proof will be given in a course of Linear Algebra.

Taxi driver metric on a plane

Taxi driver metric on a plane

is defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

for any (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$

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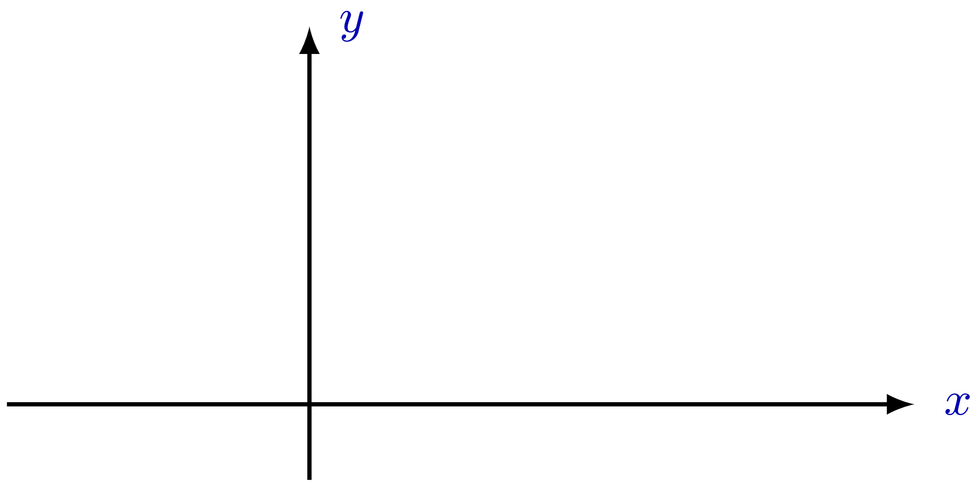

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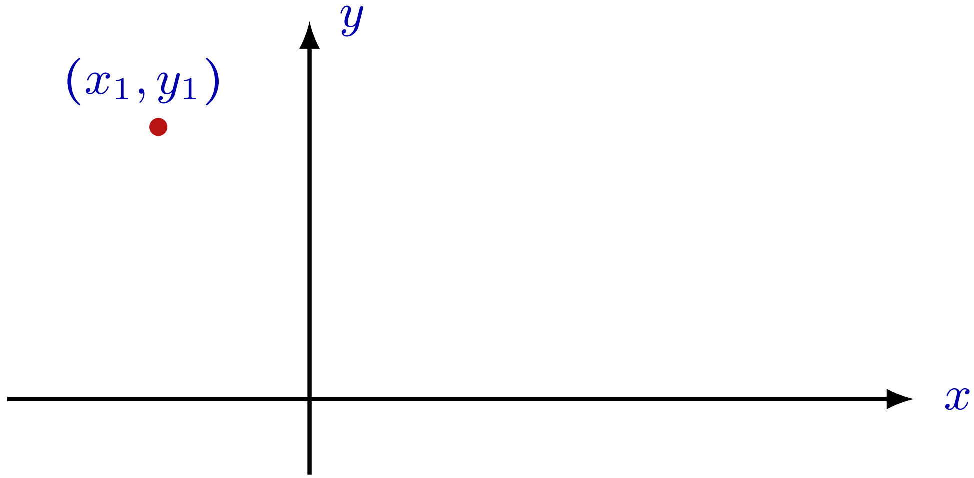
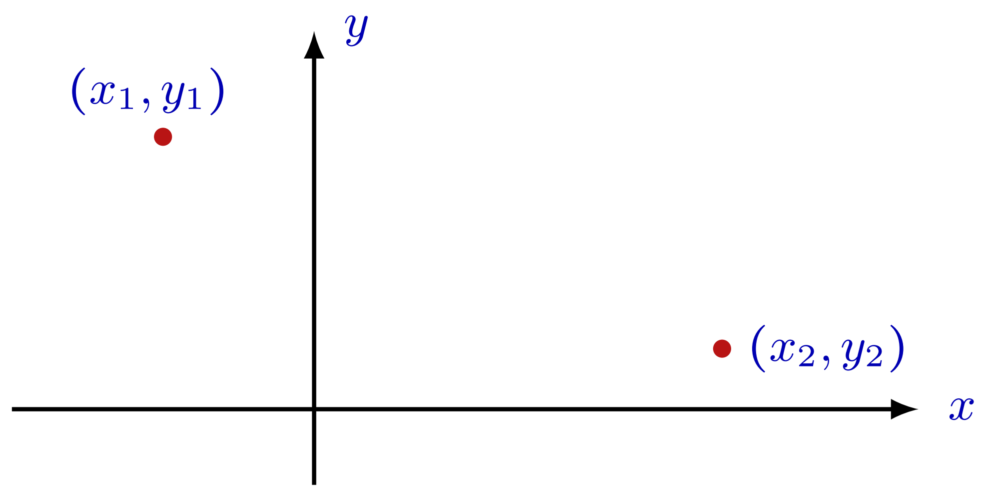


Taxi driver metric on a plane

is defined by

$$d((x_1,y_1),(x_2,y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

for any (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2$



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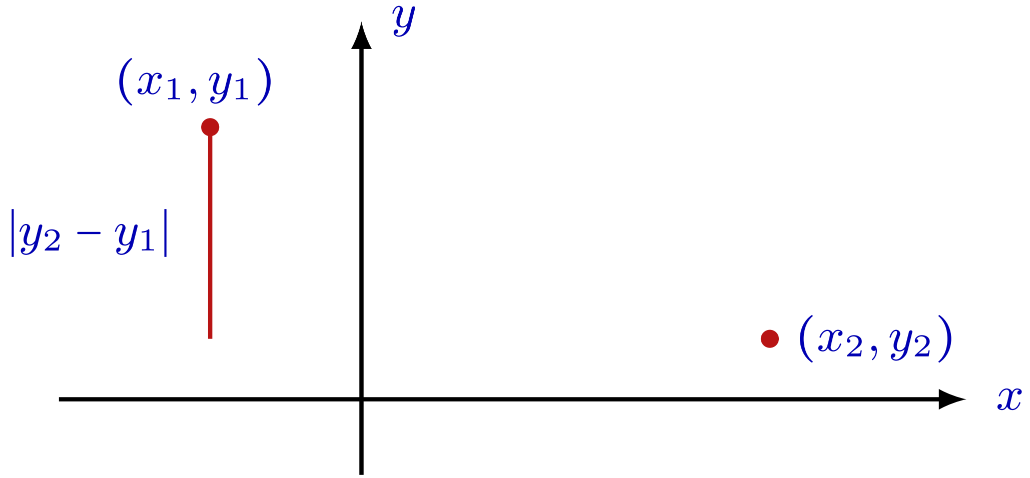

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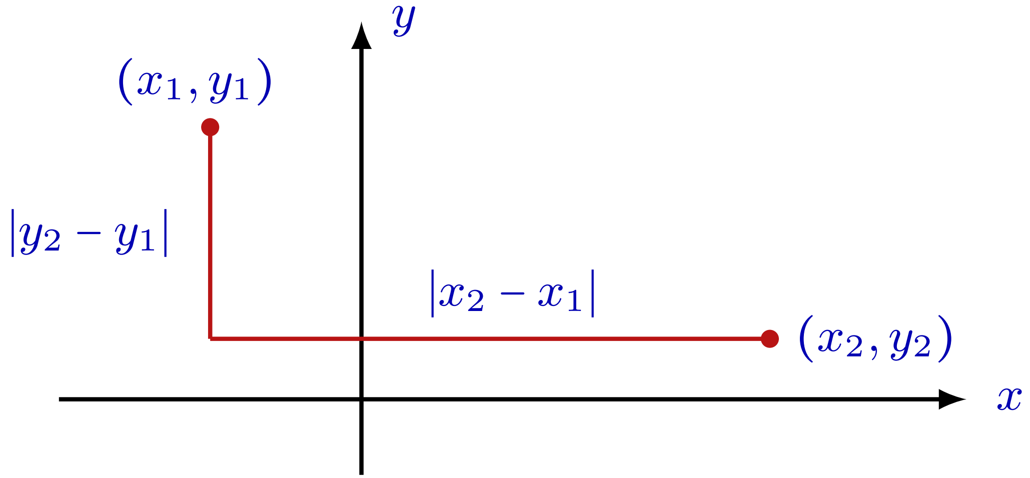

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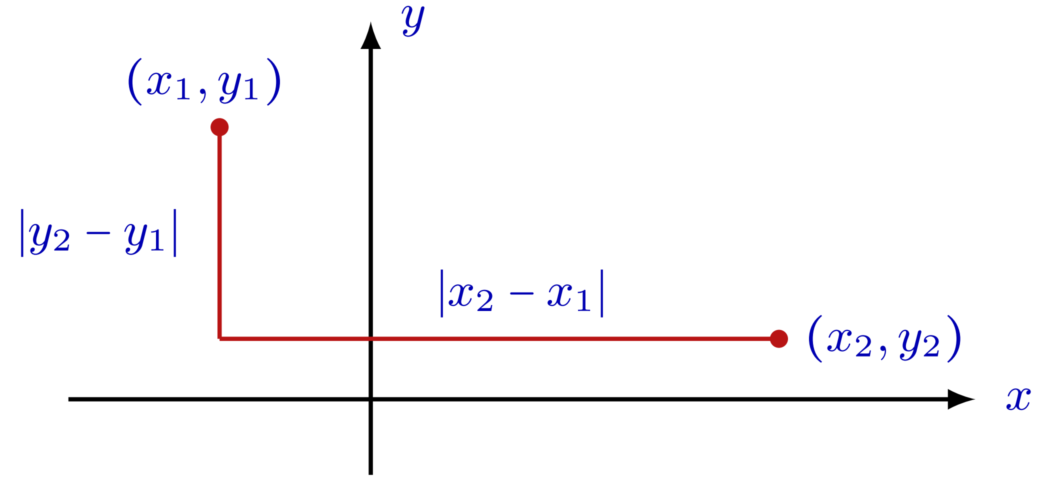


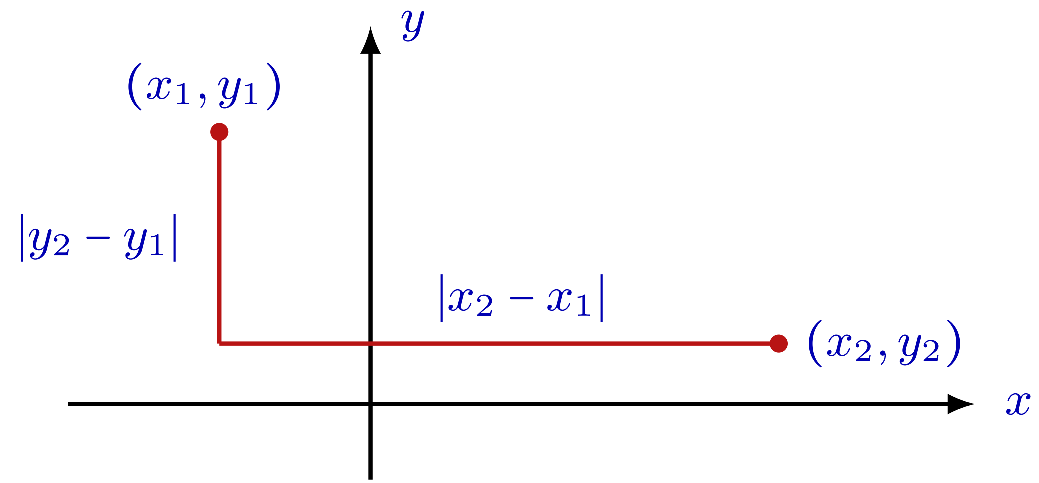
It's easy to check that this is a metric indeed.

Taxi driver metric on a plane

is defined by

$$d((x_1,y_1),(x_2,y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

for any $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$



It's easy to check that this is a metric indeed.

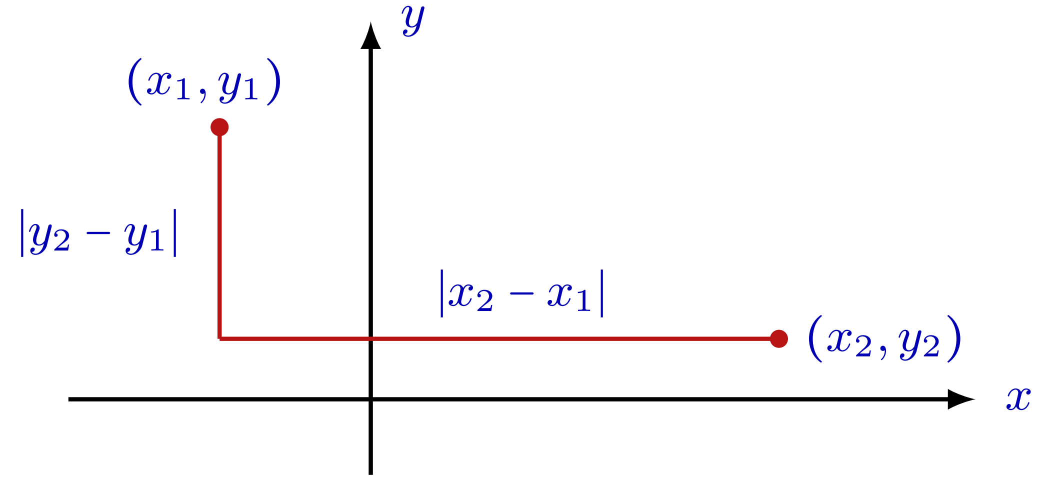
The plane with Euclidean metric

Taxi driver metric on a plane

is defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$



It's easy to check that this is a metric indeed.

The plane with Euclidean metric

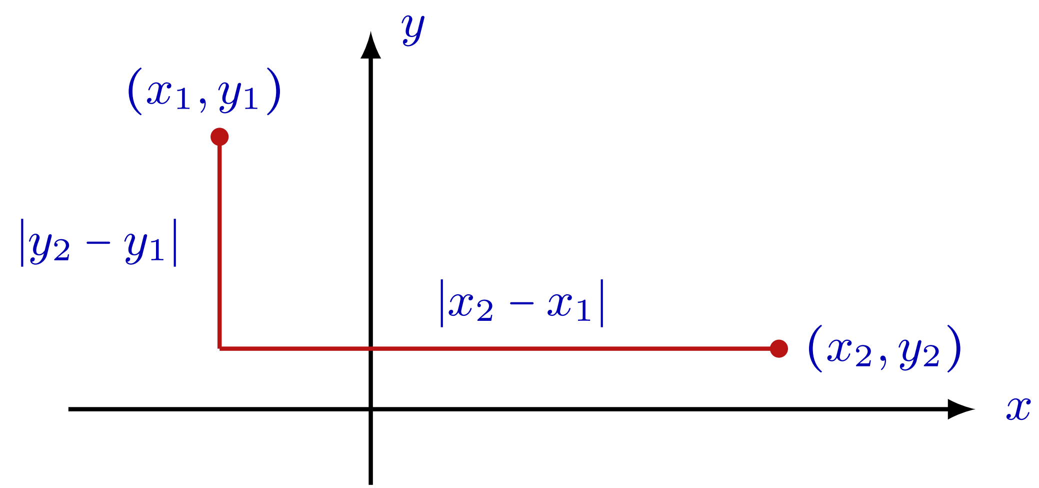
and the plane with taxi driver metric

Taxi driver metric on a plane

is defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$



It's easy to check that this is a metric indeed.

The plane with Euclidean metric and the plane with taxi driver metric are **different** metric spaces.