Lecture 5. Quaternions

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The lecture on quaternions was given by Professor Alexander Kirillov. Below you can find a concise list of definitions and statements on this topic.

5.1 Quaternions as quartuples of real numbers

The set of quaternions is denoted by \( \mathbb{H} \). This is a very concrete mathematical object. As a vector space over \( \mathbb{R} \), it has the standard basis \( 1, i, j, k \). A quaternion expanded in this standard basis is \( a + bi + cj + dk \), where \( a, b, c, d \in \mathbb{R} \). The quaternion addition is component-wise.

\[
(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k.
\]

Quaternions can be multiplied by each other. The multiplication is associative (i.e., \((xy)z = x(yz)\) for any quaternions \( x, y, z \in \mathbb{H} \)) and distributive (i.e., \((x + y)z = xz + yz\)). The generators are subject to relations

\[
i^2 = j^2 = k^2 = ijk = -1.
\]

The quaternion products of the generators are calculated according to the formulas

\[
ij = k, \ jk = i, \ kj = -i, \ ki = j \text{ and } ik = -j \tag{1}
\]

These formulas can be deduced from the relations \( i^2 = j^2 = k^2 = ijk = -1 \) and associativity of multiplication. For example, multiply the last relation \( ijk = -1 \) by \( k \) from the right hand side: \( ijk^2 = -k \). Since \( k^2 = -1 \), then \(-ij = -k\). Multiply both sides by \(-1\). This results the first formula \( ij = k \) that we wanted to prove.

Take the square of it: \( iji = k^2 = -1 \). Multiply by \( i \) from the left and by \( j \) from the right: \( i(iji)j = -ij \). The right hand side is \(-ij = -k\). The left hand side: \( i^2ji = (-1)ji(-1) = ji \). Hence \( ji = -k \).

**Exercise:** prove the rest of formulas (1).

Notice that the quaternion multiplication is not commutative: \( ij \neq ji \).
5.2 Scalars and vectors.

The field \( \mathbb{R} \) of real numbers is contained in \( \mathbb{H} \) as \( \{ a + 0i + 0j + 0k \mid a \in \mathbb{R} \} \).
A quaternion of the form \( a + 0i + 0j + 0k \), is called \textit{real}. A quaternion of the form \( 0 + bi + cj + dk \), where \( b, c, d \in \mathbb{R} \) is called \textit{pure imaginary}.

If \( q = a + bi + cj + dk \) is any quaternion, then \( a \) is called its \textit{scalar part} or \textit{real part} and denoted by \( \text{Re} \ q \) and \( bi + cj + dk \) is called its \textit{vector part} and denoted by \( \text{Ve} \ q \). The set of pure imaginary quaternions \( bi + cj + dk \) is identified with the real 3-space \( \mathbb{R}^3 \).

5.3 Multiplication of quaternions and multiplications of vectors.

A real quaternion commutes with any quaternion. Multiplication of quaternions is composed of all the standard multiplications of factors which are real numbers and vectors: multiplications of real numbers, multiplication of a vector by a real number and dot and cross products of vectors. It is not accident: the very notion of vector and all the operations with vectors were introduced by Hamilton after invention of quaternions. (Many mathematicians nowadays are not aware about this.)

\textbf{Quaternion product of vectors.} Let \( p = ui + vj + wk \) and \( q = xi + yj + zk \) be vector quaternions. Then \( pq = -p \cdot q + p \times q \).

\textit{Proof.} Indeed,

\[
pq = (ui + vj + wk)(xi + yj + zk)
= -(ux + vy + wz) + (vz - wy)i + (wx - uz)j + (wy - vx)k
= -p \cdot q + p \times q.
\]

\textbf{Product of arbitrary quaternions via other products.} For any \( p, q \in \mathbb{H} \)

\[
pq = (\text{Re} p + \text{Ve} p)(\text{Re} q + \text{Ve} q) = \text{Re} p \text{Re} q + \text{Re} p \text{Ve} q + \text{Ve} p \text{Re} q + \text{Ve} p \text{Ve} q
= \text{Re} p \text{Re} q + \text{Re} p \text{Ve} q + \text{Re} q \text{Ve} p - \text{Ve} p \cdot \text{Ve} q + \text{Ve} p \times \text{Ve} q
= \text{Re} p \text{Re} q - \text{Ve} p \cdot \text{Ve} q + \text{Re} p \text{Ve} q + \text{Re} q \text{Ve} p + \text{Ve} p \times \text{Ve} q
\]

\textbf{5.4 Conjugation.}

The map \( \mathbb{H} \rightarrow \mathbb{H} : q \mapsto \bar{q} = \text{Re} q - \text{Ve} q \) is called \textit{conjugation}. The conjugation is an antiautomorphism of \( \mathbb{H} \) in the sense that
• it is invertible (in fact, it coincides with its inverse: \(\overline{q} = q\)),
• preserve addition, i.e., \((\overline{p + q}) = \overline{p} + \overline{q}\),
• and maps the product to the product of exchanged factors: \((\overline{pq}) = \overline{q}\overline{p}\).

The latter property is verified as follows:
\[
\overline{pq} = (\text{Re}p\text{Re}q - \text{Ve}p \cdot \text{Ve}q + \text{Re}p\text{Ve}q + \text{Re}q\text{Ve}p + \text{Ve}p \times \text{Ve}q)
\]
\[
= \text{Re}p\text{Re}q - \text{Ve}p \cdot \text{Ve}q - \text{Re}p\text{Ve}q - \text{Re}q\text{Ve}p - \text{Ve}p \times \text{Ve}q
\]
\[
= \text{Re}p\text{Re}q - (-\text{Ve}p) \cdot (-\text{Ve}q) + \text{Re}p(-\text{Ve}q) + \text{Re}q(-\text{Ve}p) + (-\text{Ve}q) \times (-\text{Ve}p)
\]
\[
= (\text{Re}q - \text{Ve}q)(\text{Re}p - \text{Ve}p) = \overline{q}\overline{p}.
\]
(Here we used well-known properties of dot and cross products.)

5.5 Norm.

The product \(\overline{qq}\) is a real number for any quaternion \(q\).
Indeed, \((\overline{qq}) = \overline{q}(\overline{q}) = \overline{qq}\).

If \( q = a + bi + cj + dk \), then \( q^*q = a^2 + b^2 + c^2 + d^2 \).
Indeed, \( \overline{q}q = \text{Re}q^2 - (-\text{Ve}q) \cdot \text{Ve}q + \text{Re}q\text{Ve}q + \text{Re}q(-\text{Ve}q) + (-\text{Ve}q) \times \text{Ve}q = \text{Re}q^2 + \text{Ve}q \cdot \text{Ve}q = a^2 + b^2 + c^2 + d^2 \).

Corollary. For any quaternion \( q \), the product \( \overline{qq} \) is non-negative real number. It is zero if and only if \( q \) is zero.

The number \( \sqrt{q^*q} \) is called the norm of \( q \) and denoted by \(|q|\). This is the Euclidean distance from \( q \) to the origin in \( \mathbb{H} \rightarrow \mathbb{R}^4 \).

The norm is a multiplicative homomorphism \( \mathbb{H} \rightarrow \mathbb{R} \). This means that \(|pq| = |p||q|\) for any quaternions \( p \) and \( q \).

Proof. \(|pq| = \sqrt{pq(pq)} = \sqrt{pq\overline{pq}} = \sqrt{p(qq)p} = \sqrt{p\overline{q}q\overline{p}} = \sqrt{p\overline{p}}\sqrt{q\overline{q}} = |p||q|\).

5.6 Unit quaternions.

A quaternion \( q \) with \(|q| = 1\) is called a unit quaternion. The set of all unit quaternions \( \{q \in \mathbb{H} \mid |q| = 1\} \) is a sphere of radius one in the 4-space \( \mathbb{H} \). It is denoted by \( S^3 \).

The set of unit quaternions is closed under quaternion multiplication, because the norm of the product of quaternions is the product of norms of the factors. The inverse to a unit quaternion \( q \in S^3 \) coincides with \( \overline{q} \).

Indeed, \(|q| = \sqrt{(qq)} = 1\), hence \( q\overline{q} = 1 \) and \( \overline{q} = q^{-1} \).

Unit vector quaternions form the unit 2-sphere \( S^2 \) in \( \mathbb{R}^3 \). It is contained in \( S^3 \) as an equator. The unit vectors are very special quaternions.
Theorem 1. Each unit quaternion can be presented as a product of two unit vectors. Moreover, if \( q \) is a unit quaternion and \( v \) is a unit vector perpendicular to \( qv \), then there exist unit vectors \( w_+ \) and \( w_- \) such that \( q = vw_+ = w_-v \).

**Proof.** Let \( q \in S^3 \) be a unit quaternion. Then \( q = \text{Re} q + \text{Ve} q \) with \( 1 = |q|^2 = \text{Re} q^2 + |\text{Ve} q|^2 \). Choose \( \alpha \in [0, \pi] \) such that \( \text{Re} q = \cos \alpha \) and \( |\text{Ve} q| = \sin \alpha \). Then \( q = \cos \alpha + u \sin \alpha \) for some unit vector \( u \).

Take any unit vector \( v \) perpendicular to \( u \). Then \( w_+ = -v \cos \alpha + (u \times v) \sin \alpha \) and \( w_- = -v \cos \alpha - (u \times v) \sin \alpha \) are also unit vectors perpendiculars to \( u \), with the required properties: \( vw_+ = q \) and \( w_-v = q \). Indeed, 
\[
vw_+ = v(-v \cos \alpha + (u \times v) \sin \alpha) = -v(-v \cos \alpha + v \times (u \times v) \sin \alpha) = \cos \alpha + u \sin \alpha = q
\]
and \( w_-v = (-v \cos \alpha - (u \times v) \sin \alpha)v = -(v \cos \alpha) \cdot v - (u \times v) \times v \sin \alpha = \cos \alpha + u \sin \alpha = q \) \( \blacksquare \)

**Remark.** Any unit vector quaternion \( u \) has order four, its multiplicative inverse coincides with the additive inverse: \( u^{-1} = -u \).

Indeed, let \( u \) be unit vector. Then \( u^2 = -u \cdot u + u \times u = -1 \), hence \( u^3 = -u \) and \( u^4 = (u^2)^2 = (-1)^2 = 1 \).

By Theorem 1 any unit quaternion \( q \) admits presentation as product of two unit vector quaternions: \( q = vw \).

A unit quaternion can be presented as a sort of *quotient* of two unit vectors: first, present \(-q\) as product of two unit vector quaternions: \( -q = vw \), then re-write this as \( q = -vw = (-v)w = v^{-1}w \). This presentation goes back to W.R.Hamilton, the inventor of quaternions. In his book [1], Hamilton introduced quaternions as quotients of vectors.

### 5.7 The action of unit quaternions in the 3-space.

A unit quaternion \( q \) defines a map \( \rho_q : \mathbb{H} \to \mathbb{H} \) by formula \( \rho_q(p) = qpq^{-1} = q\overline{p}q \). We say that the group \( S^3 \) of unit quaternions acts in \( \mathbb{H} \).

This action commutes with the conjugation \( p \mapsto \overline{p} \).

Indeed,
\[
\rho_q(p) = qpq = \overline{(q)(\overline{p})q} = \overline{(qpq)} = \overline{(\rho_q(p))}.
\]

Therefore the action of \( S^3 \) in \( \mathbb{H} \) preserves all the structures defined by the conjugation. In particular, it preserves the norm and the decomposition into scalar and vector parts. Indeed,
\[
\rho_q(\text{Ve} p) = \rho_q \left( \frac{p - \overline{p}}{2} \right) = \frac{\rho_q(p) - \rho_q(\overline{p})}{2} = \frac{\rho_q(p) - \rho_q(\overline{p})}{2} = \text{Ve}(\rho_q(p)),
\]
\[
\rho_q(\text{Re } p) = \rho_q \left( \frac{p + \bar{p}}{2} \right) = \frac{\rho_q(p) + \rho_q(\bar{p})}{2} = \frac{\rho_q(p) + \rho_q(p)}{2} = \text{Re}(\rho_q(p)),
\]
\[
|\rho_q(p)| = \sqrt{\rho_q(p)\rho_q(p)} = \sqrt{\rho_q(p)\rho_q(p)} = \sqrt{(qp\bar{q})(q\bar{p})} = \sqrt{qp\bar{q}} = |p|.
\]

In particular, the space \(\mathbb{R}^3\) of vector quaternions is invariant, and \(S^3\) acts on \(\mathbb{R}^3\) by isometries.

**Theorem 2.** A unit vector quaternion \(v\) acts in \(\mathbb{R}^3\) as the symmetry about the line generated by \(v\).

**Proof.** The statement that we are going to prove admits the following reformulation: for the linear operator \(\mathbb{R}^3 \to \mathbb{R}^3: u \mapsto uv\bar{v}\), the vector \(v\) is mapped to itself and each unit vector \(u\) orthogonal to \(v\) is mapped to the opposite vector \(-u\).

Let us verify the first statement. Since \(v\) is a unit vector, \(v\bar{v} = |v|^2 = 1\). Therefore \(vv\bar{v} = v\).

Now let us verify the second statement. Since \(u\) is a unit vector orthogonal to \(v\), \(vu = v \times u - v \cdot u = v \times u\). Therefore, \(vu\bar{v} = -(v \times u - v \cdot u)v = -(v \times u)v\). Vector \(v \times u\) is orthogonal to \(v\). Therefore \(-v \times u)v = -(v \times u) \times v + (v \times u) \cdot v = -(v \times u) \times v = -v\). The latter equality holds true, because \((a \times b) \times a = b\) for any orthogonal unit vectors \(a, b\) (e.g., \((i \times j) \times i = k \times i = j\)).

**Theorem 3** (Euler-Rodrigues-Hamilton). Let \(q\) be any unit quaternion. Represent it as \(q = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2}\). Then the map \(\mathbb{R}^3 \to \mathbb{R}^3: p \mapsto qp\bar{q}\) is the rotation of \(\mathbb{R}^3\) about the axis generated by a unit vector \(u\) by the angle \(\theta\).

**Proof.** By Theorem 1 any unit quaternion \(q\) can be presented as a product of unit vectors \(v\) and \(w\). In this proof it will be more convenient to use a modification of this presentation, the fraction presentation \(q = \frac{1}{2}v - w\) discussed above.

By Theorem 2 a unit vector acts as a symmetry about the line generated by this vector. Thus, \(\rho_q\) is the composition of the symmetries \(\rho_{-v}\) and \(\rho_w\). The composition of symmetries about lines is a rotation by the angle equal the half of the angle between the lines. On the other hand, \(q = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2}\). Thus \(q = \cos \frac{\theta}{2}\), where \(\theta\) is the rotation angle.

The vector part \(q_v\) of the product of two unit vectors \(v\) and \(-w\) is collinear to \(v \times (-w)\). The cross product of vectors is perpendicular to the vectors.
On the other hand, we know that composition of symmetries about lines is a rotation about the axis perpendicular to the lines. Thus the vector \(q_v\) is collinear to the axis of the rotation \(\rho_q\). The length \(|q_v|\) is \(\sin \frac{\theta}{2}\), because \(|q| = 1\) and \(q_s = \cos \frac{\theta}{2}\). Therefore \(q_v = u \sin \frac{\theta}{2}\) for some unit vector \(u\) collinear to the axis of rotation.

The quaternion \(q\) can be written down as \(a + bi + cj + dk\). It is defined by the rotation up to multiplication by \(-1\). Its components \(a, b, c, d\) are called the **Euler parameters** for this rotation. They are calculated as follows: \(a = \cos \frac{\theta}{2}\), \(b = u_x \sin \frac{\theta}{2}\), \(c = u_y \sin \frac{\theta}{2}\) and \(d = u_z \sin \frac{\theta}{2}\), where \(u_x, u_y\) and \(u_z\) are coordinates of the unit vector \(u\) directed along the rotation axis.

**References**