Lecture 5. Quaternions

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The lecture on quaternions was given by Professor Alexander Kirillov. Below you can find a concise list of definitions and statements on this topic.

5.1 Quaternions as quartiples of real numbers

The set of quaternions is denoted by $\mathbb{H}$. This is a very concrete mathematical object. As a vector space over $\mathbb{R}$, it has the standard basis $1, i, j, k$. A quaternion expanded in this standard basis is $a+bi+cj+dk$, where $a, b, c, d \in \mathbb{R}$. The quaternion addition is component-wise.

$$(a+bi+cj+dk)+(a'+b'i+c'j+d'k) = (a+a')+(b+b')i+(c+c')j+(d+d')k.$$

Quaternions can be multiplied by each other. The multiplication is associative (i.e., $(xy)z = x(yz)$ for any quaternions $x, y, z \in \mathbb{H}$) and distributive (i.e., $(x + y)z = xz + yz$). The generators are subject to relations $i^2 = j^2 = k^2 = ijk = -1$.

The quaternion products of the generators are calculated according to the formulas

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j \text{ and } ik = -j \tag{1}$$

These formulas can be deduced from the relations $i^2 = j^2 = k^2 = ijk = -1$ and associativity of multiplication. For example, multiply the last relation $ijk = -1$ by k from the right hand side: $ijk^2 = -k$. Since $k^2 = -1$, then $-ij = -k$. Multiply both sides by -1. This results the first formula $ij = k$ that we wanted to prove.

Take the square of it: $ijij = k^2 = -1$. Multiply by i from the left and by j from the right: $i(ijij)j = -ij$. The right hand side is $-ij = -k$. The left hand side: $i^2jij^2 = (-1)ji(-1) = ji$. Hence $ji = -k$.
Exercise: prove the rest of formulas (1).

Notice that the quaternion multiplication is not commutative: $ij \neq ji$.

5.2 Scalars and vectors.

The field $\mathbb{R}$ of real numbers is contained in $\mathbb{H}$ as $\{a+0i+0j+0k \mid a \in \mathbb{R}\}$. A quaternion of the form $a+0i+0j+0k$, is called *real*. A quaternion of the form $0+bi+cj+dk$, where $b,c,d \in \mathbb{R}$ is called *pure imaginary*.

If $q = a+bi+cj+dk$ is any quaternion, then a is called its *scalar part* or *real part* and denoted by $\operatorname{Re} q$ and $bi+cj+dk$ is called its *vector part* and denoted by $\operatorname{Ve} q$. The set of pure imaginary quaternions $bi+cj+dk$ is identified with the real 3-space $\mathbb{R}^3$.

5.3 Multiplication of quaternions and multiplications of vectors.

A real quaternion commutes with any quaternion. Multiplication of quaternions is composed of all the standard multiplications of factors which are real numbers and vectors: multiplications of real numbers, multiplication of a vector by a real number and dot and cross products of vectors. It is not accident: the very notion of vector and all the operations with vectors were introduced by Hamilton after invention of quaternions. (Many mathematicians nowadays are not aware about this.)

Quaternion product of vectors. *Let $p = ui+vj+wk$ and $q = xi+yj+zk$ be vector quaternions. Then $pq = -p\cdot q + p\times q$.*

Proof. Indeed,

$$\begin{aligned} pq &= (ui+vj+wk)(xi+yj+zk) \\ &= -(ux+vy+wz)+(vz-wy)i+(wx-uz)j+(uy-vx)k \\ &= -p\cdot q + p\times q. \end{aligned}$$

□

Product of arbitrary quaternions via other products. *For any $p,q \in \mathbb{H}$*

$$\begin{aligned} pq &= (\operatorname{Re} p+\operatorname{Ve} p)(\operatorname{Re} q+\operatorname{Ve} q) = \operatorname{Re} p\operatorname{Re} q+\operatorname{Re} p\operatorname{Ve} q+\operatorname{Ve} p\operatorname{Re} q+\operatorname{Ve} p\operatorname{Ve} q \\ &= \operatorname{Re} p\operatorname{Re} q+\operatorname{Re} p\operatorname{Ve} q+\operatorname{Re} q\operatorname{Ve} p-\operatorname{Ve} p\cdot\operatorname{Ve} q+\operatorname{Ve} p\times\operatorname{Ve} q \\ &= \operatorname{Re} p\operatorname{Re} q-\operatorname{Ve} p\cdot\operatorname{Ve} q+\operatorname{Re} p\operatorname{Ve} q+\operatorname{Re} q\operatorname{Ve} p+\operatorname{Ve} p\times\operatorname{Ve} q \quad \square \end{aligned}$$

5.4 Conjugation.

The map $\mathbb{H}\to\mathbb{H} : q \mapsto \bar{q} = \operatorname{Re} q-\operatorname{Ve} q$ is called *conjugation*. The conjugation is an antiautomorphism of $\mathbb{H}$ in the sense that

- it is invertible (in fact, it coincides with its inverse: $\overline{(\bar{q})} = q$),
- preserve addition, i.e., $\overline{(p+q)} = \bar{p} + \bar{q}$,
- and maps the product to the product of exchanged factors: $\overline{(pq)} = \bar{q}\bar{p}$.

The latter property is verified as follows:

$$\overline{(pq)} = \overline{(\operatorname{Re} p \operatorname{Re} q - \operatorname{Ve} p \cdot \operatorname{Ve} q + \operatorname{Re} p \operatorname{Ve} q + \operatorname{Re} q \operatorname{Ve} p + \operatorname{Ve} p \times \operatorname{Ve} q)}$$
$$= \operatorname{Re} p \operatorname{Re} q - \operatorname{Ve} p \cdot \operatorname{Ve} q - \operatorname{Re} p \operatorname{Ve} q - \operatorname{Re} q \operatorname{Ve} p - \operatorname{Ve} p \times \operatorname{Ve} q$$
$$= \operatorname{Re} p \operatorname{Re} q - (-\operatorname{Ve} p) \cdot (-\operatorname{Ve} q) + \operatorname{Re} p(-\operatorname{Ve} q) + \operatorname{Re} q(-\operatorname{Ve} p) + (-\operatorname{Ve} q) \times (-\operatorname{Ve} p)$$
$$= (\operatorname{Re} q - \operatorname{Ve} q)(\operatorname{Re} p - \operatorname{Ve} p) = \bar{q}\bar{p}.$$

(Here we used well-known properties of dot and cross products.)

5.5 Norm.

The product $\bar{q}q$ is a real number for any quaternion q.
Indeed, $\overline{(\bar{q}q)} = \bar{q}\overline{(\bar{q})} = \bar{q}q$. □

If $q = a + bi + cj + dk$, then $q^*q = a^2 + b^2 + c^2 + d^2$.
Indeed, $\bar{q}q = \operatorname{Re} q^2 - (-\operatorname{Ve} q) \cdot \operatorname{Ve} q + \operatorname{Re} q \operatorname{Ve} q + \operatorname{Re} q(-\operatorname{Ve} q) + (-\operatorname{Ve} q) \times \operatorname{Ve} q = \operatorname{Re} q^2 + \operatorname{Ve} q \cdot \operatorname{Ve} q = a^2 + b^2 + c^2 + d^2$. □

Corollary. *For any quaternion q, the product $\bar{q}q$ is non-negative real number. It is zero if and only if q is zero.* □

The number $\sqrt{q^*q}$ is called the *norm* of q and denoted by $|q|$. This is the Euclidean distance from q to the origin in $\mathbb{H} = \mathbb{R}^4$.

The norm is a multiplicative homomorphism $\mathbb{H} \to \mathbb{R}$. This means that $|pq| = |p||q|$ for any quaternions p and q.

Proof. $|pq| = \sqrt{pq\overline{(pq)}} = \sqrt{pq\bar{q}\bar{p}} = \sqrt{p(q\bar{q})\bar{p}} = \sqrt{p\bar{p}}\sqrt{q\bar{q}} = |p||q|$. □

5.6 Unit quaternions.

A quaternion q with $|q| = 1$ is called a *unit quaternion*. The set of all unit quaternions $\{q \in \mathbb{H} \mid |q| = 1\}$ is a sphere of radius one in the 4-space $\mathbb{H}$. It is denoted by S^3.

The set of unit quaternions is closed under quaternion multiplication, because the norm of the product of quaternions is the product of norms of the factors. The inverse to a unit quaternion $q \in S^3$ coincides with $\bar{q}$. Indeed, $|q| = \sqrt{(q\bar{q})} = 1$, hence $q\bar{q} = 1$ and $\bar{q} = q^{-1}$.

Unit vector quaternions form the unit 2-sphere S^2 in $\mathbb{R}^3$. It is contained in S^3 as an equator. The unit vectors are very special quaternions.

Theorem 1. *Each unit quaternion can be presented as a product of two unit vectors. Moreover, if q is a unit quaternion and v is a unit vector perpendicular to q_v, then there exist unit vectors w_+ and w_- such that $q = vw_+ = w_-v$.*

Proof. Let $q \in S^3$ be a unit quaternion. Then $q = \operatorname{Re} q + \operatorname{Ve} q$ with $1 = |q|^2 = \operatorname{Re} q^2 + |\operatorname{Ve} q|^2$. Choose $\alpha \in [0, \pi]$ such that $\operatorname{Re} q = \cos\alpha$ and $|\operatorname{Ve} q| = \sin\alpha$. Then $q = \cos\alpha + u\sin\alpha$ for some unit vector u.

Take any unit vector v perpendicular to u. Then $w_+ = -v\cos\alpha + (u \times v)\sin\alpha$ and $w_- = -v\cos\alpha - (u\times v)\sin\alpha$ are also unit vectors perpendiculars to u, with the required properties: $vw_+ = q$ and $w_-v = q$. Indeed, $vw_+ = v(-v\cos\alpha + (u\times v)\sin\alpha) = -v\cdot(-v\cos\alpha) + v\times(u\times v)\sin\alpha = \cos\alpha + u\sin\alpha = q$ and $w_-v = (-v\cos\alpha - (u\times v)\sin\alpha)v = -(-v\cos\alpha)\cdot v - (u\times v)\times v\sin\alpha = \cos\alpha + u\sin\alpha = q$ $\square$

Remark. *Any unit vector quaternion u has order four, its multiplicative inverse coincides with the additive inverse: $u^{-1} = -u$.*
Indeed, let $\mathbf{u}$ be unit vector. Then $u^2 = -u\cdot u + u\times u = -1$, hence $u^3 = -u$ and $u^4 = (u^2)^2 = (-1)^2 = 1$. $\square$

By Theorem 1 any unit quaternion q admits presentation as product of two unit vector quaternions: $q = vw$.

A unit quaternion can be presented as a sort of *quotient* of two unit vectors: first, present $-q$ as product of two unit vector quaternions: $-q = vw$, then re-write this as $q = -vw = (-v)w = v^{-1}w$. This presentation goes back to W.R.Hamilton, the inventor of quaternions. In his book [1], Hamilton introduced quaternions as quotients of vectors.

5.7 The action of unit quaternions in the 3-space.

A unit quaternion q defines a map $\rho_q : \mathbb{H} \to \mathbb{H}$ by formula $\rho_q(p) = qpq^{-1} = qp\bar{q}$. We say that *the group S^3 of unit quaternions acts in $\mathbb{H}$.*

This action commutes with the conjgation $p \mapsto \bar{p}$.

Indeed,

$$\rho_q(\bar{p}) = q\bar{p}\bar{q} = \overline{((\bar{q})(\bar{p})\bar{q})} = \overline{(qp\bar{q})} = \overline{(\rho_q(p))}.$$

Therefore the action of S^3 in $\mathbb{H}$ preserves all the structures defined by the congugation. In particular, it preserves the norm and the decomposition into scalar and vector parts. Indeed,

$$\rho_q(\operatorname{Ve} p) = \rho_q\left(\frac{p - \bar{p}}{2}\right) = \frac{\rho_q(p) - \rho_q(\bar{p})}{2} = \frac{\rho_q(p) - \overline{\rho_q(p)}}{2} = \operatorname{Ve}(\rho_q(p)),$$

$$\rho_q(\operatorname{Re} p) = \rho_q\left(\frac{p+\bar{p}}{2}\right) = \frac{\rho_q(p)+\rho_q(\bar{p})}{2} = \frac{\rho_q(p)+\overline{\rho_q(p)}}{2} = \operatorname{Re}(\rho_q(p)),$$

$$|\rho_q(p)| = \sqrt{\rho_q(p)\overline{(\rho_q(p))}} = \sqrt{\rho_q(p)\rho_q(\bar{p})}$$
$$= \sqrt{(qp\bar{q})(q\bar{p}\bar{q})} = \sqrt{qp\bar{p}\bar{q}} = |p|.$$

In particular, the space $\mathbb{R}^3$ of vector quaternions is invariant, and S^3 acts on $\mathbb{R}^3$ by isometries.

Theorem 2. *A unit vector quaternion v acts in $\mathbb{R}^3$ as the symmetry about the line generated by v.*

Proof. The statement that we are going to prove admits the following reformulation: for the linear operator $\mathbb{R}^3 \to \mathbb{R}^3 : u \mapsto vuv^*$, the vector v is mapped to itself and each unit vector u orthogonal to v is mapped to the opposite vector $-u$.

Let us verify the first statement. Since v is a unit vector, $v\bar{v} = |v|^2 = 1$. Therefore $vv\bar{v} = v$.

Now let us verify the second statement. Since u is a unit vector orthogonal to v, $vu = v \times u - v \cdot u = v \times u$. Therefore, $vu\bar{v} = -vuv = -(v \times u - v \cdot u)v = -(v \times u)v$. Vector $v \times u$ is orthogonal to v. Therefore $-(v \times u)v = -(v \times u) \times v + (v \times u) \cdot v = -(v \times u) \times v = -u$. The latter equality holds true, because $(a \times b) \times a = b$ for any orthogonal unit vectors a, b (e.g., $(i \times j) \times i = k \times i = j$). □

Theorem 3 (Euler-Rodrigues-Hamilton)**.** *Let q be any unit quaternion. Represent it as $q = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$. Then the map $\mathbb{R}^3 \to \mathbb{R}^3 : p \mapsto qp\bar{q}$ is the rotation of $\mathbb{R}^3$ about the axis generated by a unit vector u by the angle θ.*

Proof. By Theorem 1 any unit quaternion q can be presented as a product of unit vectors v and w. In this proof it will be more convenient to use a modification of this presentation, the fraction presentation $q = v^{-1}w = -vw$ discussed above.

By Theorem 2 a unit vector acts as a symmetry about the line generated by this vector. Thus, ρ_q is the composition of the symmetries ρ_{-v} and ρ_w. The composition of symmetries about lines is a rotation by the angle equal the half of the angle between the lines. On the other hand, $q_s = (v(-w))_s = -v \cdot (-w) = v \cdot w = \cos\alpha$, where α is the angle between the vectors v and w. Thus $q_s = \cos\frac{\theta}{2}$, where θ is the rotation angle.

The vector part q_v of the product of two unit vectors v and $-w$ is collinear to $v \times (-w)$. The cross product of vectors is perpendicular to the vectors.

On the other hand, we know that composition of symmetries about lines is a rotation about the axis perpendicular to the lines. Thus the vector q_v is collinear to the axis of the rotation ρ_q. The length $|q_v|$ is $|\sin\frac{\theta}{2}|$, because $|q| = 1$ and $q_s = \cos\frac{\theta}{2}$. Therefore $q_v = u \sin\frac{\theta}{2}$ for some unit vector u collinear to the axis of rotation. $\square$

The quaternion q can be written down as $a + bi + cj + dk$. It is defined by the rotation up to multiplication by -1. Its components a, b, c, d are called the *Euler parameters* for this rotation. They are calculated as follows: $a = \cos\frac{\theta}{2}$, $b = u_x \sin\frac{\theta}{2}$, $c = u_y \sin\frac{\theta}{2}$ and $d = u_z \sin\frac{\theta}{2}$, where u_x, u_y and u_z are coordinates of the unit vector u directed along the rotation axis.

References

[1] *Elements of quaternions,* by the late Sir William Rowan Hamilton, LL. P., M.R.I.A., edited by his son, William Edwin Hamilton, A.B.T.C.D., C.E. London : Longmans, Green, & Co. 1866.