Lecture 4. Rational points on the circle

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February 3, 2016

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Nice formulas! Homogeneous! Divide by u^2

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$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{(1-t)^2+4t^2}{(1+t^2)^2} = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^4}{(1+t^2)^2} = 1.$$

Trigonometric meaning

Recall that at some moment we got

$$\cos\alpha = \frac{u^2 - v^2}{u^2 + v^2} \quad \text{and} \quad \sin\alpha = \frac{2uv}{u^2 + v^2}\,.$$

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If $u = \cos\beta$ and $v = \sin\beta$, then

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This is the trigonometric meaning of our solution.

Riddle:

Draw on a picture all the heros: α, $\cos\alpha$, $\sin\alpha$, β, and $t = \tan\beta$.