Lecture 2. Traversing Nets of Curves

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2.1 Seven bridges of Königsberg

Our story goes back to Leonard Euler (1707-1783). In 1736 he solved the seven bridges of Königsberg problem.

The story below about this is cited from Wikipedia

`https://en.wikipedia.org/wiki/Seven_Bridges_of_Konigsberg`

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to devise a walk through the city that would cross each bridge once and only once, with the provisos that: the islands could only be reached by the bridges and every bridge once accessed must be crossed to its other end. The starting and ending points of the walk need not be the same.

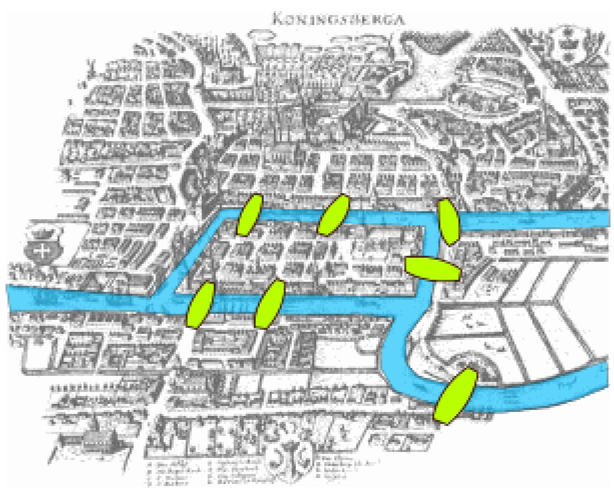
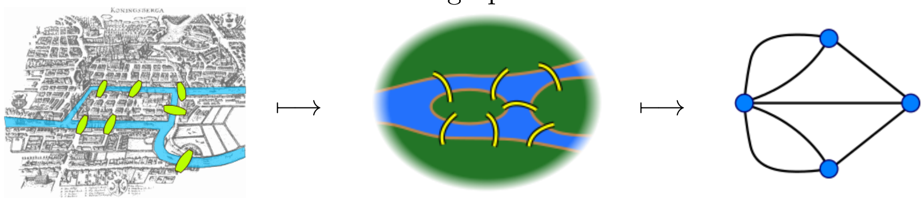


Figure 1: Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges.

Euler proved that the problem has no solution. The difficulty was the development of a technique of analysis and of subsequent tests that established this assertion with mathematical rigor.

First, Euler pointed out that the choice of route inside each land mass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of graph theory), eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land mass with an abstract "vertex", and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is called a graph.



Since only the connection information is relevant, the shape of pictorial representations of a graph may be distorted in any way, without changing the graph itself. Only the existence (or absence) of an edge between each pair of vertices is significant. For example, it does not matter whether the edges drawn are straight or curved, or whether one node is to the left or right of another.

Next, Euler observed that (except at the endpoints of the walk), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now, if every bridge has been traversed exactly once, it follows that, for each land mass (except for the ones chosen for the start and finish), the number of bridges touching that land mass must be even (half of them, in the particular traversal, will be traversed "toward" the landmass; the other half, "away" from it). However, all four of the land masses in the original problem are touched by an odd number of bridges (one is touched by 5 bridges, and each of the other three is touched by 3). Since, at most, two land masses can serve as the endpoints of a walk, the proposition of a walk traversing each bridge once leads to a contradiction.

In modern language, Euler shows that the possibility of a walk through a graph, traversing each edge exactly once, depends on the degrees of the nodes. The degree of a node is the number of edges touching it. Euler's argument shows that a necessary condition for the walk of the desired form is that the graph be connected and have exactly zero or two nodes of odd degree. This condition turns out also to be sufficient.

Such a walk is now called an Eulerian path or Euler walk in his honor. Further, if there are nodes of odd degree, then any Eulerian path will start at one of them and end at the other. Since the graph corresponding to historical Königsberg has four nodes of odd degree, it cannot have an Eulerian path.

An alternative form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an Eulerian circuit or an Euler tour.

Exercise. State a condition under which an Eulerian circuit exists.

2.2 Statement of the problem by Rademacher and Toeplits

Rademacher and Toeplits considered a different and more general problem:

"A streetcar company decides to reorganize its system of routes without changing the existing tracks. It wishes to do this in such a way that each section of track will be used by just one route. Passengers will be allowed to transfer from route to route until they finally reach their destinations. The problem is: *how many routes must the company operate in order to serve all sections without having more than one route on any section?*"

A network of streetcar lines consists of finite number of tracks connecting transfer points.

2.3 Traditional mathematical reformulation

A network of streetcar lines admit the following geometric description. On the plane, there are finitely many points called *verticies*. Some of them are connected with arcs. Each arc connects two vertices or a vertex to itself (then tis arc is called a *loop*). Two arcs are allowed to intersect only at their endpoints, vertices that they connect. The arcs are called *edges*. Altogether vertices and edges are called a *graph*.

In this reformulation, routes from the Rademacher and Toeplits problem turn into Eulerian paths. An *Eulerian path* is a sequence of edges e_1, e_2, ..., e_n in which

- each edge appears once: if $e_i = e_j$, then $i = j$.
- any two subsequent edges have a common vertex,

- if e_i has two vertices, $i \neq 1$ and $i \neq n$, then one of the vertices of e_i is a vertex of the preceding edge e_{i-1} and the other vertex of e_i is a vertex of the next edge e_{i+1}.

An Eulerian path $e_1, e_2, \dots, e_n$ is said to be *closed* if $e_2, \dots, e_n, e_1$, is also an Eulerian path. This means that e_1 and e_n have a common vertex and, if e_n is not a loop, the common vertex of e_1 and e_n differs from the common vertex of e_n and e_{n-1}.

A collection of Eulerian paths in a graph G, which contain all edges of G and no two of which have a common edge, is called *Eulerian path system*.

The problem formulated above in 2.2 turns into the following one: *for a given graph what is the least number of paths in an Eulerian path system?*

More detailed problem: *Find an Eulerian path system on a given graph with the least number of paths*.

2.4 A game for your little brother

Draw a graph on a paper. Ask your little brother to redraw the graph without lifting the pencil from the paper, and tracing over each edge exactly once? If this turns to be impossible, ask to redraw the graph with the least possible number of liftings the pencil from the paper.

It's a game in which you will surely excel after reading the rest of this section.

2.5 Low estimate for the number of paths

Let v be a vertex in a graph G. Among edges adjacent to v there may be loops. Recall that loops are edges which connect a vertex to itself. The number of edges adjacent to v plus the number of loops adjacent to v is called the *degree* of v and denoted by $d(v)$. In other words, $d(v)$ is the number of edges adjacent to v and count with multiplicities: a loop is count with multiplicity two (it approaches v twice) while edges that are not loops are count once.

A vertex of odd degree is said to be *odd*, a vertex of even degree is said to *even*.

Theorem 1. *For any graph, the number of paths in an Eulerian path system is greater than or equal to half the number of odd vertices.*

Proof. Each odd vertex is an end-point of at least one non-closed path from the Eulerian path system. A non-closed Eulerian path has two end-points.

Hence the number of non-closed paths in the Eulerian path system is at least the half of the number of odd vertices. □

Of course, it may happen that a graph has only even vertices, and then the estimate is not exact: the number of odd vertices is zero, but the number of paths of a Eulerian path system cannot be zero. Thus, Theorem 1 can be improved as follows.

Theorem 2. *For any graph, the number of paths in an Eulerian path system is greater than or equal to half the number of odd vertices, if the number odd vertices is not zero. Otherwise it is greater than or equal to one.*

2.6 Constructing Eulerian path systems

Let G be a graph. The simplest (and biggest) of its Eulerian path systems consists of paths each of which is formed by a single edge. The number of paths in this system equals the number of all edges in G.

Any other Eulerian path system can be obtained from this one by merging paths which have common end-points. If a vertex v is the final point of the path $e_1, \dots, e_n$, and initial point of the path $f_1, \dots, f_m$ which belong to the same Eulerian path system, then $e_1, \dots, e_n, f_1, \dots, f_m$ is an Eulerian path, and replacement of the two paths by this new one gives a new Eulerian path system. In the new system the number of paths is less by one.