Notes for MAT 150
Introduction to Advanced Mathematics

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1 Preface

1.1 The world of numbers and the likes. Mathematics studies objects, like numbers, which do belong neither to the physical world, nor to the world of social relations. They form their own world. We will call it a *mathematical universe*. Its objects can be modelled in many ways in the physical world, but the physical world is not their natural habitat.

On the other hand, sooner or later a study of anything starts to use mathematics. In its mature stage, any science uses mathematics, looks at its own objects through the glass of mathematical models.

Nonetheless, mathematics is not just a tool set of models for other sciences. Mathematical universe is a real consistent world. Any attempt to divide it to unrelated pieces fails. It deserves to be studied both for its usefulness and beauty.

1.2 The language for mathematics.

> In the beginning was the Word,
> and the Word was with God,
> and the Word was God.
>
> The Bible

The mathematical universe is not open directly to human senses. Any study of mathematics, any penetration to the mathematical universe, starts with words.

Introductory stories about the mathematical universe are delivered in a common language (no matter which), but soon the language starts to change, to develop, because it has to adjust to mathematical realities.

A similar story happens to a language used by any other science. Each science works out specific notions, and they require new words. Adding one by one these extra words converts a common language into a jargon specific to the science.

The mathematical language goes far beyond this. Mathematics requires a new layer of language which is necessary for proofs. To a great extent, proofs in mathematics take the role of human senses and devices that enhance the senses.

Here the language of proofs has to be understood in a broad sense. It includes the system of notions and constructions for making definitions and statements, as well as words necessary for adequate presentations of mathematical objects. All in all, this stuff forms a part of mathematics.

1.3 Why different and why difficult. For a student, the mathematical language is a challenge to learn. At first glance, it is not needed. The most important words in the mathematical language are borrowed from the common vocabulary. Apparently, mathematicians speak the same plain English (sometimes broken) as everybody else. The differences are subtle, almost invisible for a non-mathematician.

A mathematician is used to speak a mathematically meaningful language, so that the difference is not noticed by a mathematician, either. It is not noticed until a non-mathematician tries to rephrase what was said. A mathematician may be surprised by the change of meaning.

The most profound and common difficulty in studying of mathematics comes from poor skills in mathematical language. In the low level mathematical courses (up to Calculus and Linear Algebra) the language is not that important, because these courses are targeted at solving of standard problems. Proofs are avoided. The success in any mathematics course above Calculus depends first of all on understanding of the mathematical language.

One can easily see this looking at grades. As soon as a student gets the language, all the mathematical courses become equally easy. The grades are uniform across the subjects and do not depend even on student's efforts.

In this course, you will learn the basics of the mathematical language.

1.4 Our path to mastering language. We will build the mathematical language on top of the natural one, the plain English. When one studies another natural language, the very first task is to learn words. Contrary to this, we will use common English words, but discuss one by one those words which have specific meaning in the mathematical language. Usually, we will consider whole groups of words, discuss synonyms and versions of usage.

The order in which we will study words is quite unusual. Usually, when one starts to learn a foreign language, the bulk of new words that are learned on the first stage are nouns and verbs. In our study of the mathematical language, after study of two groups of nouns and verbs, we will dive into a detailed study of conjunctions.

2 Sets

2.1 Why sets? In any intellectual activity, one of the most profound actions is gathering objects in groups. The gathering is performed in mind and is not supposed to be accompanied with any action in the physical world. As soon as a group has been created and assigned a name, it can be a subject of thoughts and arguments and, in particular, it can be included into other groups.

Mathematics has an elaborated system of notions, which organizes and regulates creating those groups and manipulating them. The system is called the *naive set theory.* This name is slightly misleading, because this is rather a language than a theory.

2.2 The first words. The first words in the mathematical language are *set* and *element*. By a *set* we understand an arbitrary collection of various objects. An object included into the collection is called an *element* of the set.

A few words are used for the relation between a set and its elements. An element *belongs* to a set. A set *contains* an element. A set *consists* of its elements. Elements *form* the set.

In order to diversify the wording, the word *set* is replaced by the word *collection*. Sometimes other words, such as *class*, *family*, and *group*, are used in the same sense, but this is not quite safe, because each of these words is associated in modern mathematics with a more special meaning, and hence should be used instead of the word *set* with a caution.

2.3 The first formulas. If x is an element of a set A, then we write $x \in A$.

The sign $\in$ is a version of the Greek letter epsilon, which corresponds to the first letter of the Latin word *element*. To make the notation more flexible, the formula $x \in A$ is also allowed to be written backwards, that is in the form $A \ni x$.

This disrespect to the origin of the notation is payed off by emphasizing a meaningful similarity of $\in$ and $\ni$ to the inequality symbols $<$ and $>$.

To state that x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

2.4 Equality of sets. A set is determined by its elements. The set is nothing but a collection of its elements. This manifests most sharply in the following principle (called *Axiom of Extensionality*):

Two sets are considered equal if and only if they have the same elements.

In this sense, the word *set* has slightly disparaging meaning. When something is called a set, this shows, maybe unintentionally, a lack of interest to whatever organization of the elements of this set.

For example, when one says that a line is a set of points, it implies that two lines coincide if and only if they consist of the same points. On the other hand, it means that all relations between points on a line (e.g., the distance between points, the order of points on the line, etc.) are not included into the notion of line.

2.5 Built in a commitment to take them easy. You may think of sets as of boxes that can be built effortlessly around elements, just to distinguish them from the rest of the world. The cost of this lightness is that such a box is not more than the collection of elements placed inside.

This is a little more than just a name: it's a declaration of our intention to think about this collection of things as of an entity and not to go into details about the nature of its members-elements. Elements, in turn, may also be sets, but as long as we consider them elements, they play the role of atoms, with their own original nature ignored.

2.6 Overuse of the words. In modern Mathematics, the words *set* and *element* are very common and appear in most texts. They are even overused, that is they are used at instances when it is not appropriate.

For example, it is not good to use the word *element* as a replacement for other, more meaningful word.

When you call something an *element*, then the *set*,
whose element this one is, should be clear.

The word *element* makes sense only in combination with the word *set*, unless we deal with a non-mathematical term (like a *chemical element*), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an *infinitesimal element*; lines, planes, and other geometric images are also called *elements* in old texts). Euclid's famous book on Geometry is called *Elements*, too.

2.7 The empty set. Thus, an element may not be without a set. However, a set may have no elements. Actually, there is such a set. This set is unique, because a set is completely determined by its elements. It is called the *empty set* denoted[1] by $\varnothing$.

2.8 Basic sets of numbers. In addition to $\varnothing$, there are some other sets so important that they have their own special names and denoted by special symbols.

- The set of all positive integers, i.e., 1, 2, 3, 4,..., etc., is denoted by $\mathbb{N}$.
- The set of all integers, both positive, negative, and the zero, is denoted by $\mathbb{Z}$.
- The set of all rational numbers (join to the integers all the numbers that are presented by fractions, like 2/3 and $\frac{-7}{5}$) is denoted by $\mathbb{Q}$.
- The set of all real numbers (obtained by adjoining to rational numbers the numbers like $\sqrt{2}$ and $\pi = 3.14\dots$) is denoted by $\mathbb{R}$.
- The set of all complex numbers is denoted by $\mathbb{C}$.

2.9 Describing a set by a list of its elements. A set presented by a list a, b, ..., x of its elements is denoted by the symbol $\{a, b, \dots, x\}$. In other words, the list of objects enclosed in curly brackets denotes the set whose elements are listed. For example, $\{1, 2, 123\}$ denotes the set consisting of the numbers 1, 2, and 123. The symbol $\{a, x, A\}$ denotes the set consisting of three elements: a, x, and A, whatever objects these three letters denote.

In order to check whether you take this correctly, please, answer to the following questions.

[1] Other symbols, like Λ, are also in use, but $\varnothing$ has become most common one.

2.1. Is it true that $\varnothing = \{\varnothing\}$?

2.2. What is $\{\varnothing\}$? How many elements does it contain?

2.3. Which of the following formulas are correct:
1) $\varnothing \in \{\varnothing, \{\varnothing\}\}$; 2) $\{\varnothing\} \in \{\{\varnothing\}\}$; 3) $\varnothing \in \{\{\varnothing\}\}$?

A set consisting of a single element is called a ***singleton***. This is any set which can be presented as $\{a\}$ for some a.

2.4. Is $\{\{\varnothing\}\}$ a singleton?

2.5. Is it true that $\{1, 2, 3\} = \{3, 2, 1, 2\}$?

At first glance, lists with repetitions are never needed. There even arises a temptation to prohibit usage of lists with repetitions in such notation. However, as it often happens to temptations to prohibit something, this would not be wise. Indeed, quite often one cannot say a priori whether there are repetitions or not. For example, the elements in the list may depend on a parameter, and under certain values of the parameter some entries of the list coincide, while for other values they don't.

2.6. How many elements do the following sets contain?

1) $\{1, 2, 1\}$;	2) $\{1, 2, \{1, 2\}\}$;	3) $\{\{2\}\}$;
4) $\{\{1\}, 1\}$;	5) $\{1, \varnothing\}$;	6) $\{\{\varnothing\}, \varnothing\}$;
7) $\{\{\varnothing\}, \{\varnothing\}\}$;	8) $\{x, 3x - 1\}$, where $x \in \mathbb{R}$.	

3 Subsets and inclusions.

If A and B are sets and every element of A also belongs to B, then we say that A is a ***subset*** of B, or B ***includes*** or ***contains*** A, and write $A \subset B$ or $B \supset A$.

The inclusion signs $\subset$ and $\supset$ resemble the inequality signs $<$ and $>$, but not quite: no number a satisfies the inequality $a < a$, while any set A contains itself:

3.A (Reflexivity of inclusion). *Inclusion $A \subset A$ holds true for any A.*

Proof. Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Therefore, the statement that we must prove can be rephrased as follows: each element of A is an element of A. This is tautologically correct. $\square$

Thus, the inclusion signs are not like inequality signs $<$ and $>$. They are closer to $\leq$ and $\geq$.

Sometimes, being inspired by signs $\leq$ and $\geq$, inclusions are denoted by symbols $\subseteq$ and $\supseteq$ or even $\subseteqq$ and $\supseteqq$, reserving the symbols $\subset$ and $\supset$ for strict inclusions, that prohibit equality, like strict inequalities. We follow the mainstream mathematical notation in which the signs $\subseteq$, $\supseteq$, $\subseteqq$ and $\supseteqq$ are not used and strict inclusions are denoted by $\subsetneq$ and $\supsetneq$ or by $\underset{\neq}{\subset}$ and $\underset{\neq}{\supset}$.

Inclusion $\subset$ and inequality $\leq$ are not only similar, but are closely related:

3.B. *Let A and B be sets, A have a elements, and B have b elements. If $A \subset B$, then $a \leq b$.*

Proof. The question is so simple that it is difficult to find more elementary facts which we could use in the proof. What does it mean that A consists of a elements? This means, say, that we can count elements of A one by one assigning to them numbers 1, 2, 3, and the last element will receive number a. It is known that the result does not depend on the order in which we count. (In fact, later we will develop a set theory which would include a theory of counting, and in which this is proven. However, since we have no doubts in this fact, let us use it here without proof.) Therefore, we can start counting elements of B with counting those in A. Counting the elements in A is done first, and then, if there are some elements of B that are not in A, counting is continued. Thus, the number of elements in A is less than or equal to the number of elements in B. □

3.C (Ubiguity of the empty set). *$\varnothing \subset A$ for any set A. In other words, the empty set is present in each set as a subset.*

Proof. Recall that, by the definition of inclusion, $A \subset B$ means that each element of A is an element of B. Thus, we need to prove that any element of $\varnothing$ belongs to A. This is true because $\varnothing$ does not contain any element. □

It may happen that you are not satisfied with this proof. Arguments about the empty set may confuse at first. To this end, look at

Another proof of 3.C. Let us resort to the question whether the statement which we prove can be wrong. How can it happen that $\varnothing$ is not a subset of A? This is possible only if $\varnothing$ contains an element which is not an element of A. However, $\varnothing$ does not contain such elements because $\varnothing$ contains no elements at all. □

Thus, each set A has two obvious subsets: the empty set $\varnothing$ and A itself. A subset of A different from $\varnothing$ and A is called a ***proper*** subset of A. This word is used when we do not want to consider the obvious subsets (which are ***improper***).

3.D (Transitivity of inclusion). *If A, B, and C are sets, $A \subset B$ and $B \subset C$, then $A \subset C$.*

Proof. We must prove that each element of A is an element of C. Let $x \in A$. Since $A \subset B$, it follows that $x \in B$. Since $B \subset C$, the latter (i.e., $x \in B$) implies $x \in C$. This is what we had to prove. □

3.1 Defining a set by a condition (a set-builder notation) As we know, a set can be described by presenting a list of its elements. This simplest way may be not available or, at least, be not the easiest one. For example, it is easy to say: "the set of all solutions of the following equation" and write down the equation. This is a reasonable description of the set. At least, it is unambiguous. Having accepted it, we may start speaking on the set, studying its properties, and eventually may be lucky to solve the equation and obtain the list of its solutions. Although the latter task may be difficult, this should not prevent us from discussing the set until the time when the equation will be solved. (Solution of some equations took centuries!)

The subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

3.1. Present the following sets by lists of their elements (i.e., in the form $\{a, b, \dots\}$)
(a) $\{x \in \mathbb{N} \mid x < 5\}$, (b) $\{x \in \mathbb{N} \mid x < 0\}$, (c) $\{x \in \mathbb{Z} \mid x < 0\}$.

The set-builder notation unveils a close relation between logic statements and sets. Every statement P about elements of a set A defines a subset $\{x \in A \mid P(x)\}$ of A. On the other hand, any subset $B \subset A$ gives rise to a property of elements of A: namely, the property of belonging to B, that is $x \in B$.

For example, let us figure out what on the side of logic statements corresponds to inclusion. Let B and C are subsets of a set A. Let $B = \{x \in A \mid P(x)\}$ and $C = \{x \in A \mid Q(x)\}$, that is P and Q are the statements defining B and C, respectively. Inclusion $B \subset C$ means that each element of B is an element of C. In other words, if $x \in B$, then $x \in C$, or, in terms of P and Q, if $P(x)$, then $Q(x)$. Thus, the inclusion $B \subset C$ corresponds to implication "if $P(x)$, then $Q(x)$".

3.2 Conditional and biconditional. In everyday English the meaning of the "if ... then" construction is ambiguous. The construction "if P, then Q" always means that if P is true, then Q is true also. Sometimes that is all it means; other times it means something more: that if P is false, Q must be false either. In the first case we say about *conditional statement*, in the second case, about *biconditional*. In ordinary everyday English, usually one decides from the context whether conditional or biconditional sense is intended.

Mathematicians tend to avoid ambiguities. They have agreed to use the construction "if ... then" in the first, conditional sense, as above, so that a statement of the form "If P, then Q" means that if P is true, Q is true also, but if P is false, Q may be either true or false.

There is an important exception from this agreement: in a definition, when a new word is introduced, mathematicians use "if" in the biconditional sense. For example, when defining the notion of subset, we say: "A is a subset of B if each element of A belongs to B" - and this means that whenever we say that A

is a subset of B, each element of A does belong to B. An extra evidence that the word "if" is understood biconditionally are expressions *"is called"*, *"is said to be"* and *"one says that"*, which introduce new words.

However, this is the only exception. In any other mathematical context a sentence "P if Q" is *not* understood biconditionally.

Biconditional statements are not rare guests in the mathematical language. They appear very often, probably, more often than in everyday English. They are presented by the words *"if and only if*. In writing, this expression is often abbreviated to a single word *iff*. So, we write "P iff Q" instead of "P if and only if Q. Another way to express the same: "P is *necessary and sufficient* for Q." There is also a formula-synonym: $P \Longleftrightarrow Q$.

A conditional sentence "if P, then Q" also can be replaced by formula: $P \Rightarrow Q$. Here is a list of other ways to say the same:

- P is sufficient for Q,
- Q is necessary for P,
- P only if Q,
- P implies Q.

3.3 For proving equality of sets, prove two inclusions. Working with sets, we need from time to time to prove that two sets, say A and B, which may have emerged in quite different ways, are equal. The most common way to do this is provided by the following theorem.

3.E (**Test for equality of sets).**

$$A = B \text{ if and only if } A \subset B \text{ and } B \subset A.$$

Proof. We have already seen that $A \subset A$. Hence, if $A = B$, then, indeed, $A \subset B$ and $B \subset A$. On the other hand, $A \subset B$ means that each element of A belongs to B, while $B \subset A$ means that each element of B belongs to A. Hence, A and B have the same elements, i.e., they are equal. □

3.4 Inclusion versus belonging.

3.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Despite this obvious relation between the notions of belonging $\in$ and inclusion $\subset$ and similarity of the symbols $\in$ and $\subset$, the concepts are quite different. Indeed, $A \in B$ means that A is an element in B (i.e., one of the indivisible pieces constituting B), while $A \subset B$ means that A is made of some of the elements of B.

In particular, we have $A \subset A$, while $A \notin A$ for any reasonable A. Thus, *belonging is not reflexive*. One more difference: *belonging is not transitive*, while inclusion is.

3.2 (Non-transitivity of belonging)**.** Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$. Cf. 3.D.

Construction. Take $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{\{1\}\}\}$. $\square$

Remark. It is more difficult to construct sets A, B, and C such that $A \in B$, $B \in C$, and $A \in C$. Though, it is possible. Take, for example, $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{1\}, \{\{1\}\}\}$.

3.3 (Non-reflexivity of belonging)**.** Construct a set A such that $A \notin A$. Cf. 3.A.

Construction. It is easy to construct a set A with $A \notin A$. Take $A = \varnothing$, or $A = \mathbb{N}$, or $A = \{1\}$, ... $\square$

> **3.4. May belonging be reflexive for a set?** Construct a set X such that $X \in X$.
>
> **Construction.** A set X such that $X \in X$ is a strange creature. It would not appear in a real problem, unless you think really globally. Take for X the set of all sets. $\square$
>
> Mathematicians avoid such sets. There are good reasons for this. If we think overly globally, the thoughts may become insane. If we consider the set of all sets, then why not to consider the set Y of all sets X such that $X \notin X$? Does Y belongs to itself? If $Y \in Y$, then $Y \notin Y$ since each element X of Y has the property that $X \notin X$. If $Y \notin Y$, then $Y \in Y$ since Y is the set of ALL sets X such that $X \notin X$. This contradiction shows that our definition of Y does not make sense. An easy way to avoid this paradox is to prohibit consideration of sets with the property $X \in X$. The set of all sets is not a legitimate set.

The *intersection* of sets A and B is the set formed of their common elements,

3.5 Intersection and union. i.e., elements belonging both to A and B. It is denoted by $A \cap B$ and described by the formula

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Sets A and B are said to be *disjoint* if $A \cap B = \varnothing$. In other words, sets are disjoint if they have no common elements.

The *union* of sets A and B is
the set formed by all elements that belong to at least one of the two sets.

The union of A and B is denoted by $A \cup B$. It is described by the formula

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the conjunction *or* should be understood in the inclusive way: the statement "$x \in A$ or $x \in B$" means that x belongs to *at least one* of the sets A and B, and, maybe, to both of them. This agrees with the usage of the word "or" commonly accepted in mathematics.

In everyday English, the word "or" is ambiguous. Sometimes the statement "P or Q" means "P or Q, or both" and sometimes it means "P or Q, but not both". The intended meaning usually is recovered from the context.

Mathematicians tend to keep their language free of ambiguities. In particular, they have agreed to use the word "or" only in the first sense, so that the statement "P or Q" always means "P or Q, or both." If one means "P or Q, but not both," then one has to include the phrase "but not both" explicitly.

3.5 (Commutativity of $\cap$ and $\cup$)**.** For any two sets A and B, we have

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A.$$

In the above figure, the first equality of Theorem 3.G is illustrated by a sort of comics. Such comics are called *Venn diagrams* or *Euler circles*. They are quite useful, and we strongly recommend to try to draw them for each formula involving sets. (At least, for formulas with at most three sets, since in this case they can serve as proofs! (Guess why?)).

3.6. Prove that for any set A we have

$$A \cap A = A, \qquad A \cup A = A, \qquad A \cup \varnothing = A, \quad \text{and} \quad A \cap \varnothing = \varnothing.$$

3.7. Prove that for any sets A and B we have[2]

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B.$$

3.8 (Associativity of $\cap$ and $\cup$)**.** For any sets A, B, and C, we have

$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{and} \quad (A \cup B) \cup C = A \cup (B \cup C).$$

Associativity allows us not to care about brackets and sometimes even omit them. We define $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ and $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$.

The intersection and union of an arbitrarily large (in particular, infinite) collection of sets can be defined directly, without reference to intersection or union of two sets.

Let Γ be a collection of sets. The *intersection* of the sets belonging to Γ is the set formed by the elements that belong to *every* set $A \in \Gamma$.

[2]Here, as usual, *iff* stands for "if and only if".

This set is denoted by $\bigcap_{A\in\Gamma} A$. Similarly,

the *union* of the sets belonging to Γ is
the set formed by elements that belong to *at least one* of the sets $A \in \Gamma$.

This set is denoted by $\bigcup_{A\in\Gamma} A$.

If B is a subset of $\bigcup_{A\in\Gamma} A$, then we say that the sets that belong to Γ *cover* B.

3.9. The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for $\Gamma = \{A, B\}$, we have

$$\bigcap_{C\in\Gamma} C = A \cap B \quad \text{and} \quad \bigcup_{C\in\Gamma} C = A \cup B.$$

Riddle 3.10. How are the notions of system of equations and intersection of sets related to each other?

3.G (Two distributivities). *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \tag{1}$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C). \tag{2}$$

3.11. Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.

Riddle 3.12. Generalize Theorem 3.G to the case of arbitrary collections of sets.

3.H (Yet another pair of distributivities). *Let A be a set and let Γ be a set consisting of sets. Then we have*

$$A \cap \bigcup_{B\in\Gamma} B = \bigcup_{B\in\Gamma} (A \cap B) \quad \text{and} \quad A \cup \bigcap_{B\in\Gamma} B = \bigcap_{B\in\Gamma} (A \cup B).$$

The *difference* $A \smallsetminus B$ of sets A and B is
the set of those elements of A which do not belong to B.

3.6 Different differences

Here we do not assume that $A \supset B$.

If $A \supset B$, then the set $A \smallsetminus B$ is also called the *complement* of B in A.

3.13. Prove that for any sets A and B their union $A \cup B$ is the union of the following three sets: $A \smallsetminus B$, $B \smallsetminus A$, and $A \cap B$, which are pairwise disjoint.

3.14. Prove that $A \smallsetminus (A \smallsetminus B) = A \cap B$ for any sets A and B.

3.15. Prove that $A \smallsetminus (B \smallsetminus A) = A \smallsetminus B$ for any sets A and B.

3.16. Prove that $A \subset B$ if and only if $A \smallsetminus B = \varnothing$.

3.17. Prove that $A \cap (B \smallsetminus C) = (A \cap B) \smallsetminus (A \cap C)$ for any sets A, B, and C.

The set $A \bigtriangleup B = (A \smallsetminus B) \cup (B \smallsetminus A)$ is called the *symmetric difference* of the sets A and B.

3.18. Prove that for any sets A and B we have

$$A \bigtriangleup B = (A \cup B) \smallsetminus (A \cap B)$$

Thus, the symmetric difference of sets corresponds to "exclusive or": an element belongs to $A \bigtriangleup B$ iff it belongs to A or to B, but not to both of them.

3.19 (Associativity of symmetric difference)**.** Prove that for any sets A, B, and C we have

$$(A \bigtriangleup B) \bigtriangleup C = A \bigtriangleup (B \bigtriangleup C).$$

Riddle 3.20. Find a symmetric definition of the symmetric difference $(A \bigtriangleup B) \bigtriangleup C$ of three sets and generalize it to arbitrary finite collections of sets.

3.21 (Distributivity)**.** Prove that $(A \bigtriangleup B) \cap C = (A \cap C) \bigtriangleup (B \cap C)$ for any sets A, B, and C.

3.22. Does the following equality hold true for any sets A, B, and C:

$$(A \bigtriangleup B) \cup C = (A \cup C) \bigtriangleup (B \cup C)?$$

3.7 De Morgan formulas

3.I. *Let Γ be an arbitrary collection of subsets of a set X. Then*

$$X \smallsetminus \bigcup_{A \in \Gamma} A = \bigcap_{A \in \Gamma} (X \smallsetminus A), \tag{3}$$

$$X \smallsetminus \bigcap_{A \in \Gamma} A = \bigcup_{A \in \Gamma} (X \smallsetminus A). \tag{4}$$

Formula (4) is deduced from (3) in one step, is not it? These formulas are nonsymmetric cases of a single formulation, which contains in a symmetric way sets and their complements, unions, and intersections.

Riddle 3.23. Find such a formulation.

4 Maps

A *map* f of a set X to a set Y is a triple consisting of X, Y, and a rule,[3] which assigns to every element of X exactly one element of Y.

4.1 The notion of map There are other words with the same meaning: *mapping*, *function*, etc. (Special kinds of maps may have special names like *functional*, *operator*, *sequence*, *family*, *fibration*, etc.)

If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f, or the *f-image* of a. In order to state that $b = f(a)$, one may write also $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$. We also define maps by formulas like $f : X \to Y : a \mapsto b$, where b is explicitly expressed in terms of a.

4.1. Let X and Y be sets consisting of p and q elements, respectively. Find the number of maps $X \to Y$.

4.2 The main classes of maps A map $f : X \to Y$ is a *surjective map*, or just a *surjection* if every element of Y is the image of at least one element of X. (We also say that f is *onto*.) A map $f : X \to Y$ is an *injective map*, *injection*, or *one-to-one map* if every element of Y is the image of at most one element of X. A map is a *bijective map*, *bijection* if it is both surjective and injective.

4.2. Let X and Y be sets consisting of p and q elements, respectively. Find the number of injections $X \to Y$.

4.A. *Let X and Y be sets consisting of p and q elements, respectively. Find the number of surjections $X \to Y$.*

4.3 Image and preimage The *image* of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus, we have

$$f(A) = \{f(x) \mid x \in A\}.$$

The image of the entire set X (i.e., the set $f(X)$) is the *image* of f. It is denoted by $\Im f$.

The *preimage* of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X with images in to B. It is denoted by $f^{-1}(B)$. Thus, we have

$$f^{-1}(B) = \{a \in X \mid f(a) \in B\}.$$

[3]Certainly, the rule (as everything in the set theory) may be thought of as a set. Section 5 below. Namely, the rule can be converted to (or, if you prefer, encoded by) the set Γ_f of the ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that the rule assigns y to x. This is the *graph* of f. It is a subset of $X \times Y$. Recall that $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B. And even if it does not differ, it may happen that the preimage is not the only set with this property. Hence, the preimage *cannot* be defined as a set whose image is the given set.

4.B. *$f\big(f^{-1}(B)\big) \subset B$ for any map $f : X \to Y$ and any $B \subset Y$.*

4.C. *$f\big(f^{-1}(B)\big) = B$ iff $B \subset \Im f$.*

4.D. *Let $f : X \to Y$ be a map, and let $B \subset Y$ be such that $f\big(f^{-1}(B)\big) = B$. Then the following statements are equivalent:*

1. *$f^{-1}(B)$ is the unique subset of X whose image equals B;*
2. *for any $a_1, a_2 \in f^{-1}(B)$, the equality $f(a_1) = f(a_2)$ implies $a_1 = a_2$.*

4.E. *A map $f : X \to Y$ is an injection iff for each $B \subset Y$ such that $f\big(f^{-1}(B)\big) = B$ the preimage $f^{-1}(B)$ is the unique subset of X with image equal to B.*

4.F. *$f^{-1}\big(f(A)\big) \supset A$ for any map $f : X \to Y$ and any $A \subset X$.*

4.G. *$f^{-1}\big(f(A)\big) = A$ iff $f(A) \cap f(X \setminus A) = \varnothing$.*

4.3. Do the following equalities hold true for any $A, B \subset Y$ and $f : X \to Y$?

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \tag{5}$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \tag{6}$$
$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A). \tag{7}$$

4.4. Do the following equalities hold true for any $A, B \subset X$ and $f : X \to Y$?

$$f(A \cup B) = f(A) \cup f(B), \tag{8}$$
$$f(A \cap B) = f(A) \cap f(B), \tag{9}$$
$$f(X \setminus A) = Y \setminus f(A). \tag{10}$$

4.5. Give examples in which two of the above equalities (13)–(15) are false.

4.6. Replace false equalities of 4.4 by correct inclusions.

Riddle 4.7. What simple condition on $f : X \to Y$ should be imposed in order to make correct all equalities of 4.4 for any $A, B \subset X$?

4.8. Prove that for any map $f : X \to Y$ and any subsets $A \subset X$ and $B \subset Y$

$$B \cap f(A) = f\big(f^{-1}(B) \cap A\big).$$

4.4 Identity and inclusion The *identity map* of a set X is the map $\mathrm{id}_X : X \to X : x \mapsto x$. It is denoted by id if X is clear from the context. If A is a subset of X, then the map $\mathrm{in}_A : A \to X : x \mapsto x$ is the *inclusion map*, or just *inclusion*, of A into X. It is denoted just by in when A and X are clear.

4.H. *The preimage of a set B under the inclusion* $\mathrm{in} : A \to X$ *is* $B \cap A$.

4.5 Composition The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z : x \mapsto g\bigl(f(x)\bigr)$.

4.I (Associativity)**.** $h \circ (g \circ f) = (h \circ g) \circ f$ *for any maps* $f : X \to Y$, $g : Y \to Z$, *and* $h : Z \to U$.

4.J. $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ *for any* $f : X \to Y$.

4.K. *A composition of injections is injective.*

4.L. *If the composition $g \circ f$ is injective, then so is f.*

4.M. *A composition of surjections is surjective.*

4.N. *If the composition $g \circ f$ is surjective, then so is g.*

4.O. *A composition of bijections is a bijection.*

4.9. Let a composition $g \circ f$ be bijective. Is then f or g necessarily bijective?

4.6 Inverse and invertible A map $g : Y \to X$ is *inverse* to a map $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. A map having an inverse map is *invertible*.

4.P. *A map is invertible iff it is a bijection.*

4.Q. *If an inverse map exists, then it is unique.*

Let $f : X \to Y$ be a map. For any sets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$, there is a map $A \to B : x \mapsto f(x)$. It is called the *abbreviation* of f to A and B, a *submap*, or a *submapping*.

4.7 Submaps There is no commonly accepted notation for this map. We denote it by $f|_{A,B}$ or even simply by $f|$, when A and B are clear from the context.

If $B = Y$, then $f| : A \to Y$ is denoted by $f|_A$ and called the *restriction* of f to A. This notation and term are commonly accepted.

4.R. *The restriction of a map $f : X \to Y$ to $A \subset X$ is the composition of the inclusion* $\mathrm{in} : A \to X$ *and f. In other words,* $f|_A = f \circ \mathrm{in}$.

4.S. *Any submap (in particular, any restriction) of an injection is injective.*

4.T. *If a map possesses a surjective restriction, then it is surjective.*

5 Constructions of sets

In this section we discuss a few ways of creating new sets from old ones. We have considered above some notions and operations that fit this description. In particular, one can pass from a set to its subsets. A subset can be defined by pointing out properties that distinguish elements of the subset among the elements of the set. Then we considered operations with subsets: intersection, union, subtraction etc. In this section we consider operations for creating sets, that are not subsets of the sets which are the initial material of the construction.

Let X and Y be sets. The set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is called the *direct product*, *Cartesian product*, or just *product* of X and Y and denoted by $X \times Y$.

5.1 Multiplication of sets If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Sets $X \times b$ with $b \in Y$ and $a \times Y$ with $a \in X$ are called *fibers* of the product $X \times Y$.

5.A. *Prove that for any $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$*

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2),$$
$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$
$$(A_1 \times B_1) \smallsetminus (A_2 \times B_2) = \big((A_1 \smallsetminus A_2) \times B_1\big) \cup \big(A_1 \times (B_1 \smallsetminus B_2)\big)$$

The maps

$$\operatorname{pr}_X : X \times Y \to X : (x, y) \mapsto x \quad \text{and} \quad \operatorname{pr}_Y : X \times Y \to Y : (x, y) \mapsto y$$

are called (*natural*) *projections*.

5.B. *Prove that $\operatorname{pr}_X^{-1}(A) = A \times Y$ for each $A \subset X$.*

5.1. Find the corresponding formula for $B \subset Y$.

5.2 Cartesian products of maps Let X, Y, and Z be three sets. A map $f : Z \to X \times Y$ determines the compositions $f_1 = \operatorname{pr}_X \circ f : Z \to X$ and $f_2 = \operatorname{pr}_Y \circ f : Z \to Y$, which are called the *factors* (or *components*) of f. Indeed, f is determined by them as a sort of product.

5.C. *Prove that for any maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique map $f : Z \to X \times Y$ with $\operatorname{pr}_X \circ f = f_1$ and $\operatorname{pr}_Y \circ f = f_2$.*

5.2. Prove that $f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B)$ for any $A \subset X$ and $B \subset Y$.

Any two maps $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ determine a map

$$g_1 \times g_2 : X_1 \times X_2 \to Y_1 \times Y_2 : (x_1, x_2) \mapsto (g_1(x_1), g_2(x_2)),$$

which is their (*Cartesian*) *product*.

5.3. Prove that $(g_1 \times g_2)(A_1 \times A_2) = g_1(A_1) \times g_2(A_2)$ for any $A_1 \subset X_1$ and $A_2 \subset X_2$.

5.4. Prove that $(g_1 \times g_2)^{-1}(B_1 \times B_2) = g_1^{-1}(B_1) \times g_2^{-1}(B_2)$ for any $B_1 \subset Y_1$ and $B_2 \subset Y_2$.

5.5. Prove that the Cartesian product of open maps is open.

5.6. Prove that a metric $\rho : X \times X \to \mathbb{R}$ is continuous with respect to the metric topology.

5.3 Graph of a map A map $f : X \to Y$ determines a subset Γ_f of $X \times Y$ defined by formula $\Gamma_f = \{(x, f(x)) \mid x \in X\}$, it is called the ***graph*** of f.

A map can be recovered from its graph. Recall that a map $f : X \to Y$ is a triple consisting of X, Y, and a rule, which assigns to every element of X exactly one element of Y. The rule is encoded in Γ_f.

5.D. *A set $\Gamma \subset X \times Y$ is the graph of a map $X \to Y$ iff for each $a \in X$ the intersection $\Gamma \cap (a \times Y)$ is a singleton.*

5.7. Prove that for each map $f : X \to Y$ and each set $A \subset X$ we have

$$f(A) = \mathrm{pr}_Y(\Gamma_f \cap (A \times Y)) = \mathrm{pr}_Y(\Gamma_f \cap \mathrm{pr}_X^{-1}(A))$$

and $f^{-1}(B) = \mathrm{pr}_X(\Gamma_f \cap (X \times B))$ for each $B \subset Y$.

The set $\Delta = \{(x, x) \mid x \in X\} = \{(x, y) \in X \times X \mid x = y\}$ (that is the graph of the identity map $\mathrm{id} : X \to X$) is called the ***diagonal*** of $X \times X$.

5.8. Let A and B be subsets of X. Prove that $(A \times B) \cap \Delta = \varnothing$ iff $A \cap B = \varnothing$.

5.9. Let $f : X \to X$ be a map. Prove that the map $\mathrm{pr}_X\big|_{\Gamma_f}$ is bijective.

5.10. Prove that $f : X \to Y$ is injective iff $\mathrm{pr}_Y\big|_{\Gamma_f}$ is injective.

5.11. Denote by T the map $X \times Y \to Y \times X : (x, y) \mapsto (y, x)$. Prove that $\Gamma_{f^{-1}} = T(\Gamma_f)$ for each invertible map $f : X \to Y$.

5.12. Let $f : X \to Y$ be a map. Prove that the graph Γ_f is the preimage of the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$ under the map $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$.

5.4 Relations A statement about an ordered pair of arguments is called a *relation.* In order to emphasize that the number of arguments is two, it is called a *binary* relation. This notion has a straightforward generalization: a statement about an ordered n-tuple of arguments is called an *n-ary relation.* We will deal mostly with binary ones.

Inequality between numbers, $a < b$, is a typical example of relation. Many binary relation is written in a similar way: the arguments surround the symbol of relation. For example, if a relation is denoted by a symbol $\smile$, then the fact that pair (a, b) of arguments satisfies the relation is written by formula $a \smile b$.

A binary relation $\smile$ between elements of sets X and Y determines a set $\{(a, b) \in X \times Y \mid a \smile b\}$. This set is called the *graph* of the relation $\smile$ and denoted by $\Gamma_\smile$.

5.13. Draw a graph for the following relations between elements of the set $[0, 3]$:
(1) the distance between a and b is less than or equal to 1;
(2) the distance between a and b is an integer.

The notion of binary relation generalizes the notion of mapping: any map $f : X \to Y$ can be considered as a relation between elements of X and Y such that its graph coincides with Γ_f. The only difference between the notions of binary relation and mapping is that a mapping $f : X \to Y$ is required to be *univalued*: for each $a \in X$ there is exactly one element $b \in Y$ such that $f(a) = b$. Thus a relation can be considered as a multivalued mapping. In some environments (e.g., in Algebraic Geometry) the word *correspondence* is used as a synonym for binary relation.

5.5 Remarkable classes of relations A relation $\sim$ between elements of a set X is said to be *reflexive* if $a \sim a$ for any $a \in X$.

5.14. A binary relation in X is reflexive iff its graph contains the diagonal $\Delta = \{(a, b) \in X \times X \mid a = b\}$.

A relation $\sim$ between elements of a set X is said to be *irreflexive* if $a \sim a$ for none of $a \in X$.

5.15. A binary relation in X is irreflexive iff its graph and the diagonal $\Delta = \{(a, b) \in X \times X \mid a = b\}$ are disjoint.

A relation $\sim$ is said to be *symmetric* if $a \sim b$ implies $b \sim a$.
A relation $\sim$ is said to be *antisymmetric* if $a \sim b$ and $b \sim a$ imply $a = b$.
A relation $\sim$ is said to be *transitive* if $a \sim b$ and $b \sim c$ imply $a \sim c$.

A relation between elements of a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive.

A relation between elements of a set X is called a *partial order* or just *order* if it is reflexive, antisymmetric and transitive.

5.16. If an equivalence relation is a partial order, then it is equality $a = b$ and its graph is the diagonal.

5.6 Partitions and equivalence relations A collection Σ of subsets of a set X is called a *partition* of X if the elements of Σ are pairwise disjoint (that is $A \cap B = \varnothing$ for any distinct $A, B \in \Sigma$) and cover X (that is $X = \cup_{A \in \Sigma} A$).

5.17. Let $f : X \to Y$ be a map. Then the collection of those of sets $f^{-1}(b)$ which are not empty is a partition of X.

The partition of Theorem 5.17, that is $\{f^{-1}(b) \mid b \in Y, f^{-1}(b) \neq \varnothing\}$, is denoted by $S(f)$.

Riddle 5.18. Can Theorem 5.17 be generalized to an arbitrary relation?

Each partition Σ of a set X determines a relation between elements of X: two elements satisfy this relation if they belong to the same element of the partition Σ. We will denote this relation by symbol $\sim_\Sigma$. It can be defined as follows: $a \sim_\Sigma b$ if there exists $S \in \Sigma$ such that $a, b \in S$.

5.19. For any partition Σ of a set X the relation $\sim_\Sigma$ is an equivalence relation in X.

Vice versa, any equivalence relation gives rise to a partition. Indeed, let $\sim$ be an equivalence relation in a set X. For $a \in X$ denote by $a(\text{mod} \sim)$ the set $\{b \in X \mid b \sim a\}$ of elements related by $\sim$ to a.

5.E. *For any equivalence relation $\sim$ in a set X, the sets $a(\text{mod} \sim)$ form a partition of X.*

5.F. *For any partition Σ of a set X into non-empty subsets, sets $a(\text{mod} \sim_\Sigma)$ coincide with elements of Σ.*

5.G. *Any equivalence relation $\sim$ in a set X coincides with the equivalence relation defined by the partition of X to sets $a(\text{mod} \sim)$.*

Thus, partitions of a set into nonempty subsets and equivalence relations on the set are essentially the same. More precisely, they are two ways for describing the same phenomenon.

5.7 Quotient sets Let X be a set, S a partition of X. The set whose elements are members of the partition S (which are subsets of X) is called the *quotient set* or *factor set* of X by S. It is denoted by $X/_S$.

Riddle 5.20. How does this operation relate to division of numbers? Why is there a similarity in terminology and notation?

The set $X/_S$ is also called the *set of equivalence classes* for the equivalence relation corresponding to the partition S.

At first glance, the definition of quotient set contradicts one of the very profound principles of the set theory, which states that a set is determined by its elements, see Section 1.2. Indeed, according to this principle, $X/_S = S$ since the sets S and $X/_S$ are composed of the same elements. Hence, there seems to be no need to introduce $X/_S$.

The real sense of the notion of quotient set lies not in its literal set-theoretic meaning, but in our way of thinking about elements of partitions. If we remember that they are subsets of the original set and want to keep track of their internal structure (or, at least, of their elements), then we speak of a partition. If we think of them as atoms, getting rid of their possible internal structure, then we speak about the quotient set.

The map $\mathrm{pr} : X \to X/_S$ that sends $x \in X$ to the element of S containing x is the (*canonical*) *projection* or *factorization map*. A subset of X which is a union of elements of a partition is said to be *saturated*. The smallest saturated set containing a subset A of X is called the *saturation* of A.

5.21. Prove that $A \subset X$ is an element of a partition S of X iff A is the preimage of an element of $X/_S$ under the natural projection $\mathrm{pr} : X \to X/_S$.

5.H. *Prove that the saturation of a set A equals* $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

5.I. *Prove that a set is saturated iff it is equal to its saturation.*

5.8 Quotients and maps Let S be a partition of a set X into nonempty subsets. Let $f : X \to Y$ be a map which is constant on each element of S. Then there is a map $X/_S \to Y$ which sends each element A of S to the element $f(a)$, where $a \in A$. This map is denoted by $f/_S$ and called the *quotient map* or *factor map* of f (by the partition S).

5.J. *1) Prove that a map $f : X \to Y$ is constant on each element of a partition S of X iff there exists a map $g : X/_S \to Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \mathrm{pr}}\big\downarrow & \nearrow g & \\ X/_S & & \end{array}$$

2) Prove that such a map g coincides with $f/_S$.

More generally, let S and T be partitions of sets X and Y. Then every map $f : X \to Y$ that maps each subset in S to a subset in T determines a map $X/_S \to Y/_T$ which sends an element A of the partition S to the element of the partition T containing $f(A)$. This map is denoted by $f/_{(S,T)}$ and called the *quotient map* or *factor map* of f (*with respect to* S and T).

5.K. *Formulate and prove for $f/_{S,T}$ a statement generalizing 5.J.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{pr}_X}\downarrow & & {\scriptstyle \mathrm{pr}_Y}\downarrow \\ X/_S & \xrightarrow{g} & Y/_T \end{array}$$

A map $f : X \to Y$ determines the partition of the set X into nonempty preimages of the elements of Y. This partition is denoted by $S(f)$.

5.L. *The map $f/_{S(f)} : X/_{S(f)} \to Y$ is injective.*

This map is the *injective factor* (or *injective quotient*) of f.

6 Ordered Sets

This section is devoted to orders. They are structures on sets and occupy in Mathematics a position almost as profound as topological structures. After a short general introduction, we focus on relations between structures of these two types. Like metric spaces, partially ordered sets possess natural topological structures. This is a source of interesting and important examples of topological spaces. All finite topological spaces come from this source.

6.1 Strict orders A *binary relation* on a set X is a set of ordered pairs of elements of X, i.e., a subset $R \subset X \times X$. Many relations are denoted by special symbols, like $\prec$, $\vdash$, $\equiv$, or $\sim$. When such notation is used, there is a tradition to write xRy instead of writing $(x, y) \in R$. So, we write $x \vdash y$, or $x \sim y$, or $x \prec y$, etc. This generalizes the usual notation for the classical binary relations $=$, $<$, $>$, $\le$, $\subset$, etc.

A binary relation $\prec$ on a set X is called a *strict partial order*, or just a *strict order* if it satisfies the following two conditions:

- *Irreflexivity*: There is no $a \in X$ such that $a \prec a$.
- *Transitivity*: For any $a, b, c \in X$, if $a \prec b$ and $b \prec c$ then $a \prec c$.

6.1 (Antisymmetry)**.** Let $\prec$ be a strict partial order on a set X. Then there are no $x, y \in X$ such that $x \prec y$ and $y \prec x$ simultaneously.

6.2. Relation $<$ in the set $\mathbb{R}$ of real numbers is a strict order.

The formula $a \prec b$ is sometimes read as "a is less than b" or "b is greater than a", but it is often read as "b follows a" or "a precedes b". The advantage of the latter two ways of reading is that then the relation $\prec$ is not associated too closely with the inequality between real numbers.

A binary relation $\preceq$ on a set X is called a *nonstrict partial order*, or just *nonstrict order*, if it satisfies the following three conditions:

- *Transitivity*: For any $a, b, c \in X$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.
- *Antisymmetry*: For any $a, b \in X$, if $a \preceq b$ and $b \preceq a$, then $a = b$.
- *Reflexivity*: $a \preceq a$ for any $a \in X$.

6.2 Nonstrict orders

6.3. The relation $\leq$ on $\mathbb{R}$ is a nonstrict order.

6.4. In the set $\mathbb{N}$ of positive integers, the relation $a \mid b$ (a *divides* b) is a nonstrict partial order.

6.5. Is the relation $a \mid b$ a nonstrict partial order on the set $\mathbb{Z}$ of integers?

6.6. Inclusion determines a nonstrict partial order on the set of subsets of any set X.

6.3 Relation between strict and nonstrict orders

6.7. For each strict order $\prec$, there is a relation $\preceq$ defined on the same set as follows: $a \preceq b$ if either $a \prec b$, or $a = b$. This relation is a nonstrict order.

6.8. For each nonstrict order $\preceq$, there is a relation $\prec$ defined on the same set as follows: $a \prec b$ if $a \preceq b$ and $a \neq b$. This relation is a strict order.

6.9. The constructions of Problems 6.7 and 6.8 are mutually inverse: applied one after another in any order, they give the initial relation.

Thus, strict and nonstrict orders determine each other via constructions of Problems 6.7 and 6.8. They are just different incarnations of the same structure. We have already met a similar phenomenon in topology: open and closed sets in a topological space determine each other and provide different ways for describing a topological structure.

A set equipped with a partial order (either strict or nonstrict) is called a *partially ordered set* or, briefly, a *poset*. More formally speaking, a partially ordered set is a pair $(X, \prec)$ formed by a set X and a strict partial order $\prec$ on X. Certainly, instead of a strict partial order $\prec$ we can use the corresponding nonstrict order $\preceq$.

Which of the orders, strict or nonstrict, prevails in each specific case is a matter of convenience, taste, and tradition. Although it would be handy to keep both of them available, nonstrict orders conquer situation by situation. For instance, nobody introduces special notation for strict divisibility. Another example: the symbol $\subseteq$, which is used to denote nonstrict inclusion, is replaced by the symbol $\subset$, which is almost never understood as a designation solely for strict inclusion.

In abstract considerations, we use both kinds of orders: strict partial orders are denoted by the symbol $\prec$, nonstrict ones by the symbol $\preceq$.

6.4 Cones Let $(X, \prec)$ be a poset and $a \in X$. The set $\{x \in X \mid a \prec x\}$ is called the *upper cone* of a, and the set $\{x \in X \mid x \prec a\}$ the *lower cone* of a. The element a does not belong to its cones. Adding a to them, we obtain *completed* cones: the *upper completed cone* or *star* $C^+_X(a) = \{x \in X \mid a \preceq x\}$ and the *lower completed cone* $C^-_X(a) = \{x \in X \mid x \preceq a\}$.

6.A (Properties of Cones). *Let $(X, \prec)$ be a poset. Then we have:*

1. $C^+_X(b) \subset C^+_X(a)$, *provided that* $b \in C^+_X(a)$;
2. $a \in C^+_X(a)$ *for each* $a \in X$;
3. $C^+_X(a) = C^+_X(b)$ *implies* $a = b$.

6.B (Cones Determine an Order). *Let X be an arbitrary set. Suppose for each $a \in X$ we fix a subset $C_a \subset X$ so that*

1. $b \in C_a$ *implies* $C_b \subset C_a$,
2. $a \in C_a$ *for each* $a \in X$, *and*
3. $C_a = C_b$ *implies* $a = b$.

We write $a \prec b$ if $b \in C_a$. Then the relation $\prec$ is a nonstrict order on X, and for this order we have $C^+_X(a) = C_a$.

6.10. Let $C \subset \mathbb{R}^3$ be a set. Consider the relation $\lhd_C$ on $\mathbb{R}^3$ defined as follows: $a \lhd_C b$ if $b - a \in C$. What properties of C imply that $\lhd_C$ is a partial order on $\mathbb{R}^3$? What are the upper and lower cones in the poset $(\mathbb{R}^3, \lhd_C)$?

6.11. Prove that each convex cone C in $\mathbb{R}^3$ with vertex $(0,0,0)$ and such that $P \cap C = \{(0,0,0)\}$ for some plane P satisfies the conditions found in the solution to Problem 6.10.

6.12. Consider the space-time $\mathbb{R}^4$ of special relativity theory, where points represent moment-point events and the first three coordinates x_1, x_2, x_3 are the spatial coordinates, while the fourth one, t, is the time. This space carries a relation "the *event* (x_1, x_2, x_3, t) *precedes* (and may influence) the *event* $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{t})$. The relation is defined by the inequality

$$c(\tilde{t} - t) \geq \sqrt{(\widetilde{x}_1 - x_1)^2 + (\widetilde{x}_2 - x_2)^2 + (\widetilde{x}_3 - x_3)^2}.$$

Is this a partial order? If yes, then what are the upper and lower cones of an event?

6.13. Answer the versions of questions of the preceding problem in the case of two- and three-dimensional analogs of this space, where the number of spatial coordinates is 1 and 2, respectively.

6.5 Position of an element with respect to a set Let $(X, \prec)$ be a poset, $A \subset X$ a subset. Then b is the ***greatest element*** of A if $b \in A$ and $c \preceq b$ for every $c \in A$. Similarly, b is the ***smallest element*** of A if $b \in A$ and $b \preceq c$ for every $c \in A$.

6.14. An element $b \in A$ is the smallest element of A iff $A \subset C_X^+(b)$; an element $b \in A$ is the greatest element of A iff $A \subset C_X^-(b)$.

6.15. Each set has at most one greatest and at most one smallest element.

An element b of a set A is a ***maximal*** element of A if A contains no element c such that $b \prec c$. An element b is a ***minimal*** element of A if A contains no element c such that $c \prec b$.

6.16. An element b of A is maximal iff $A \cap C_X^-(b) = b$; an element b of A is minimal iff $A \cap C_X^+(b) = b$.

Riddle 6.17. 1) How are the notions of maximal and greatest elements related? 2) What can you say about a poset in which these notions coincide for each subset?

6.6 Linear orders Please, notice: the definition of a strict order does not require that for any $a, b \in X$ we have either $a \prec b$, or $b \prec a$, or $a = b$. The latter condition is called a ***trichotomy***. In terms of the corresponding nonstrict order, it is reformulated as follows: any two elements $a, b \in X$ are ***comparable***: either $a \preceq b$, or $b \preceq a$.

A strict order satisfying trichotomy is said to be ***linear*** (or ***total***). The corresponding poset is said to be ***linearly*** ordered (or ***totally*** ordered). It is also called just an ***ordered set***.[4] Some orders do satisfy trichotomy.

6.18. The order $<$ on the set $\mathbb{R}$ of real numbers is linear.

[4]Quite a bit of confusion was brought into the terminology by Bourbaki. Then linear orders were called orders, nonlinear orders were called partial orders, and in occasions when it was not known if the order under consideration was linear, the fact that this was unknown was explicitly stated. Bourbaki suggested to drop the word *partial*. Their motivation for this was that a partial order is a phenomenon more general than a linear order, and hence deserves a shorter and simpler name. This suggestion was commonly accepted in the French literature, but in English one it would imply abolishing a nice short word *poset*, which seems to be an absolutely impossible thing to do.

This is the most important example of a linearly ordered set. The words and images rooted in it are often extended to all linearly ordered sets. For example, cones are called *rays*, upper cones become *right rays*, while lower cones become *left rays*.

6.19. A poset $(X, \prec)$ is linearly ordered iff $X = C_X^+(a) \cup C_X^-(a)$ for each $a \in X$.

6.20. The order $a \mid b$ on the set $\mathbb{N}$ of positive integers is not linear.

6.21. For which X is the relation of inclusion on the set of all subsets of X a linear order?

6.7 Splitting a transitive relation into equivalence and partial order In the definitions of equivalence and partial order relations, the condition of transitivity seems to be the most important. Below, we supply a formal justification of this feeling by showing that the other conditions are natural companions of transitivity, although they are not its consequences.

6.C. *Let $\prec$ be a transitive relation on a set X. Then the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \prec b \text{ or } a = b$$

is also transitive (and, furthermore, it is certainly reflexive, i.e., $a \precsim a$ for each $a \in X$).

A binary relation $\precsim$ on a set X is called a *preorder* if it is transitive and reflective, i.e., satisfies the following conditions:

- *Transitivity*. If $a \precsim b$ and $b \precsim c$, then $a \precsim c$.
- *Reflexivity*. We have $a \precsim a$ for any a.

A set X equipped with a preorder is *preordered*.

If a preorder is antisymmetric, then this is a nonstrict order.

6.22. Is the relation $a \mid b$ a preorder on the set $\mathbb{Z}$ of integers?

6.D. *If $(X, \precsim)$ is a preordered set, then the relation $\sim$ defined by*

$$a \sim b \text{ if } a \precsim b \text{ and } b \precsim a$$

is an equivalence relation (i.e., it is symmetric, reflexive, and transitive) on X.

6.23. What equivalence relation is defined on $\mathbb{Z}$ by the preorder $a \mid b$?

6.E. *Let $(X, \precsim)$ be a preordered set, and let $\sim$ be an equivalence relation defined on X by $\precsim$ according to 6.D. Then $a' \sim a$, $a \precsim b$, and $b \sim b'$ imply $a' \precsim b'$ and in this way $\precsim$ determines a relation on the set of equivalence classes $X/_\sim$. This relation is a nonstrict partial order.*

Thus, any transitive relation generates an equivalence relation and a partial order on the set of equivalence classes.

6.F. *How this chain of constructions would degenerate if the original relation was*

1. *an equivalence relation, or*
2. *nonstrict partial order?*

Proofs and Comments

3.I
(a)

$$x \in \bigcap_{A\in\Gamma}(X \smallsetminus A) \iff \forall\, A \in \Gamma :\ x \in X \smallsetminus A$$
$$\iff \forall\, A \in \Gamma :\ x \notin A \iff x \notin \bigcup_{A\in\Gamma} A \iff x \in X \smallsetminus \bigcup_{A\in\Gamma} A.$$

(b) Replace both sides of the formula by their complements in X and put $B = X \smallsetminus A$.

4.B If $x \in f^{-1}(B)$, then $f(x) \in B$.

4.C $\boxed{\Longrightarrow}$ Obvious. $\boxed{\Longleftarrow}$ For each $y \in B$, there exists an element x such that $f(x) = y$. By the definition of the preimage, $x \in f^{-1}(B)$, whence $y \in f(f^{-1}(B))$. Thus, $B \subset f(f^{-1}(B))$. The opposite inclusion holds true for any set, see 4.B.

4.D (a) $\Longrightarrow$ (b) Assume that $f(C) = B$ implies $C = f^{-1}(B)$. If there exist distinct $a_1, a_2 \in f^{-1}(B)$ such that $f(a_1) = f(a_2)$, then also $f(f^{-1}(B) \smallsetminus a_2) = B$, which contradicts the assumption.
(b) $\Longrightarrow$ (a) Assume now that there exists $C \neq f^{-1}(B)$ such that $f(C) = B$. Clearly, $C \subset f^{-1}(B)$. Therefore, C can differ from $f^{-1}(B)$ only if $f^{-1}(B) \smallsetminus C \neq \varnothing$. Take $a_1 \in f^{-1}(B) \smallsetminus C$, and let $b = f(a_1)$. Since $f(C) = B$, there exists $a_2 \in C$ with $f(a_2) = f(a_1)$, but $a_2 \neq a_1$ because $a_2 \in C$, while $a_1 \notin C$.

4.E This follows from 4.D.

4.F Let $x \in A$. Then $f(x) = y \in f(A)$, whence $x \in f^{-1}(f(A))$.

4.G Both equalities are obviously equivalent to the following statement: $f(x) \notin f(A)$ for each $x \notin A$.

4.H $\mathrm{in}^{-1}(B) = \{x \in A \mid x \in B\} = A \cap B$.

4.I Let $x \in X$. Then

$$h \circ (g \circ f)(x) = h(g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = (h \circ g) \circ f(x).$$

4.K Let $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ because f is injective, and $g(f(x_1)) \neq g(f(x_2))$ because g is injective.

4.L If f is not injective, then there exist $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. However, then $(g \circ f)(x_1) = (g \circ f)(x_2)$, which contradicts the injectivity of $g \circ f$.

4.M Let $f : X \to Y$ and $g : Y \to Z$ be surjective. Then we have $f(X) = Y$, whence $g(f(X)) = g(Y) = Z$.

4.N This follows from the obvious inclusion $\Im(g \circ f) \subset \Im g$.

4.O This follows from 4.K and 4.M.

4.P $\boxed{\Longrightarrow}$ Use 4.L and 4.N. $\boxed{\Longleftarrow}$ Let $f : X \to Y$ be a bijection. Then, by the surjectivity, for each $y \in Y$ there exists $x \in X$ such that $y = f(x)$, and, by the injectivity, such an element of X is unique. Putting $g(y) = x$, we obtain a map $g : Y \to X$. It is easy to check (please, do it!) that g is inverse to f.

4.Q This is actually obvious. On the other hand, it is interesting to look at a "mechanical" proof. Let two maps $g, h : Y \to X$ be inverse to a map $f : X \to Y$. Consider the composition $g \circ f \circ h : Y \to X$. On the one hand, we have $g \circ f \circ h = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$. On the other hand, we have $g \circ f \circ h = g \circ (f \circ h) = g \circ \mathrm{id}_Y = g$.

5.A For example, let us prove the second relation:

$$(x, y) \in (A_1 \times B_1) \cap (A_2 \times B_2) \iff x \in A_1,\ y \in B_1,\ x \in A_2,\ y \in B_2$$
$$\iff x \in A_1 \cap A_2,\ y \in B_1 \cap B_2 \iff (x, y) \in (A_1 \cap A_2) \times (B_1 \cap B_2).$$

5.B Indeed,

$$\mathrm{pr}_X^{-1}(A) = \{z \in X \times Y \mid \mathrm{pr}_X(z) \in A\} = \{(x, y) \in X \times Y \mid x \in A\} = A \times Y.$$

5.C Set $f(z) = (f_1(z), f_2(z))$. If $f(z) = (x, y) \in X \times Y$, then $x = (\mathrm{pr}_X \circ f)(z) = f_1(z)$. We similarly have $y = f_2(z)$.

5.D $\boxed{\Longrightarrow}$ Indeed, $\Gamma_f \cap (x \times Y) = \{(x, f(x))\}$ is a singleton.
$\boxed{\Longleftarrow}$ If $\Gamma \cap (x \times Y)$ is a singleton $\{(x, y)\}$, then we can put $f(x) = y$.

5.19 Reflexivity: $a \sim_\Sigma a$ is a tautology (a belongs to the same element of Σ as a). Symmetry is also a tautology, because a and b are involved symmetrically in the definition of $a \sim_\Sigma b$.

Transitivity: Let $a \sim_\Sigma b$ and $b \sim_\Sigma c$. Then a and b belongs to $U \in \Sigma$ and b and c belong to $V \in \Sigma$. Hence $U \cap V \neq \varnothing$. Since distinct elements of a partition are disjoint while U and V are not, it follows that $U = V$. Thus a and c belong to the same element of Σ. Therefore $a \sim_\Sigma c$.

5.E By reflexivity of $\sim$, any $a \in X$ belongs to $a(\mathrm{mod} \sim)$. Therefore sets $a(\mathrm{mod} \sim)$ with $a \in X$ cover X.

Then observe that if $a \sim b$ then $a(\mathrm{mod} \sim) = b(\mathrm{mod} \sim)$. Indeed, let $c \in a(\mathrm{mod} \sim)$. By the definition of $a(\mathrm{mod} \sim)$ this means that $c \sim a$. By transitivity, $c \sim a$ and $a \sim b$ imply $c \sim b$, that is $c \in b(\mathrm{mod} \sim)$. Thus we proved that $a(\mathrm{mod} \sim) \subset b(\mathrm{mod} \sim)$. By symmetry, $a \sim b$ implies $b \sim a$ and the arguments above prove the opposite inclusion $a(\mathrm{mod} \sim) \supset b(\mathrm{mod} \sim)$.

Now let us prove that if $a \sim b$ does not hold true, then $a(\mathrm{mod} \sim) \cap b(\mathrm{mod} \sim) = \varnothing$. Assume the contrary: $a \sim b$ does not hold true, but there exists $c \in X$ which belongs to both $a(\mathrm{mod} \sim)$ and $b(\mathrm{mod} \sim)$. Then $c \sim a$ and $c \sim b$. For symmetry of $\sim$, $c \sim a$ implies $a \sim c$. Further, by transitivity, $a \sim c$ and $c \sim b$ implies $a \sim b$. This contradicts to the assumption above.

Thus we have proved that either $a(\text{mod} \sim) = b(\text{mod} \sim)$ (when $a \sim b$), or $a(\text{mod} \sim) \cap b(\text{mod} \sim) = \varnothing$ (otherwise).

5.H First, the preimage $\text{pr}^{-1}\big(\text{pr}(A)\big)$ is saturated, secondly, it is the least because if $B \supset A$ is a saturated set, then $B = \text{pr}^{-1}\big(\text{pr}(B)\big) \supset \text{pr}^{-1}\big(\text{pr}(A)\big)$.

5.J 1) $\boxed{\Longrightarrow}$ Put $g = f/_S$. $\boxed{\Longleftarrow}$ The set $f^{-1}(y) = p^{-1}(g^{-1}(y))$ is saturated, i.e., it consists of elements of the partition S. Therefore, f is constant at each of the elements of the partition. 2) If A is an element of S, a is the point of the quotient set corresponding to A, and $x \in A$, then $f/_S(a) = f(A) = g(p(x)) = g(a)$.

5.K The map f maps elements of S to those of T iff there exists a map $g : X/_S \to Y/_T$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \text{pr}_X}\big\downarrow & & {\scriptstyle \text{pr}_Y}\big\downarrow \\ X/_S & \xrightarrow{\;g\;} & Y/_T \end{array}$$

is commutative. Then we have $f/_{(S,T)} = g$.

5.L This is so because distinct elements of the partition $S(f)$ are preimages of distinct points in Y.

6.7 We need to check that the relation "$a \prec b$ or $a = b$" satisfies the three conditions from the definition of a nonstrict order. Doing this, we can use only the fact that $\prec$ satisfies the conditions from the definition of a strict order. Let us check the transitivity. Suppose that $a \preceq b$ and $b \preceq c$. This means that either 1) $a \prec b \prec c$, or 2) $a = b \prec c$, or 3) $a \prec b = c$, or 4) $a = b = c$.
1) In this case, $a \prec c$ by transitivity of $\prec$, and so $a \preceq c$. 2) We have $a \prec c$, whence $a \preceq c$. 3) We have $a \prec c$, whence $a \preceq c$. 4) Finally, $a = c$, whence $a \preceq c$. Other conditions are checked similarly.

6.A Assertion (a) follows from transitivity of the order. Indeed, consider an arbitrary an $c \in C^+_X(b)$. By the definition of a cone, we have $b \preceq c$, while the condition $b \in C^+_X(a)$ means that $a \preceq b$. By transitivity, this implies that $a \preceq c$, i.e., $c \in C^+_X(a)$. We have thus proved that each element of $C^+_X(b)$ belongs to $C^+_X(a)$. Hence, $C^+_X(b) \subset C^+_X(a)$, as required.
Assertion (b) follows from the definition of a cone and the reflexivity of order. Indeed, by definition, $C^+_X(a)$ consists of all b such that $a \preceq b$, and, by reflexivity of order, $a \preceq a$.
Assertion (c) follows similarly from antisymmetry: the assumption $C^+_X(a) = C^+_X(b)$ together with assertion (b) implies that $a \preceq b$ and $b \preceq a$, which together with antisymmetry implies that $a = b$.

6.B By Theorem 6.A, cones in a poset have the properties that form the hypothesis of the theorem to be proved. When proving Theorem 6.A, we showed that these properties follow from the corresponding conditions in the definition of a partial nonstrict order. In fact, they are equivalent to these conditions.

Permuting words in the proof of Theorem 6.A, we to obtain a proof of Theorem 6.B.

Hints, Comments, Advises, Solutions, and Answers

2.1 No, because $\varnothing$ has no elements, while $\varnothing \in \{\varnothing\}$.

2.2 The set $\{\varnothing\}$ consists of one element, which is the empty set $\varnothing$. Certainly, this element itself is the empty set and contains no elements, but the set $\{\varnothing\}$ consists of a single element $\varnothing$.

2.3 1) and 2) are correct, while 3) is not.

2.4 Yes, the set $\{\{\varnothing\}\}$ is a singleton: its single element is the set $\{\varnothing\}$.

2.5 Yes, because these sets contains the same elements, namely the numbers 1, 2, and 3.

2.6 2, 3, 1, 2, 2, 2, 1, 2 for $x \neq 1/2$ and 1 if $x = 1/2$.

3.1 (a) $\{1, 2, 3, 4\}$; (b) $\{\ \}$; (c) $\{-1, -2, -3, -4, -5, -6, \dots\}$

3.10 The set of solutions for a system of equations is equal to the intersection of the sets of solutions of individual equations in the system.

4.3 Yes, they do. Let us prove, for example, the latter equality. Let $x \in f^{-1}(Y \smallsetminus A)$. Then $f(x) \in Y \smallsetminus A$, whence $f(x) \notin A$. Therefore, $x \notin f^{-1}(A)$ and $x \in X \smallsetminus f^{-1}(A)$. We have thus proved that $f^{-1}(Y \smallsetminus A) \subset X \smallsetminus f^{-1}(A)$. Each step in this argument is reversible. The reversing gives rise to the opposite inclusion.

4.4 Let us prove (8). If $y \in f(A \cup B)$, then we can find $x \in A \cup B$ such that $f(x) = y$. If $x \in A$, then $y \in f(A)$, while if $x \in B$, then $y \in f(B)$. In both cases, we have $y \in f(A) \cup f(B)$. The opposite inclusion admits an even simpler proof. The inclusion $A \subset A \cup B$ implies $f(A) \subset f(A \cup B)$. Similarly, $f(B) \subset f(A \cup B)$. Thus, $f(A) \cup f(B) \subset f(A \cup B)$. The other two equalities may happen to be wrong, see 4.5 and 4.6.

4.5 Consider the constant map $f : \{0, 1\} \to \{0\}$. Let $A = \{0\}$ and $B = \{1\}$. Then $f(A) \cap f(B) = \{0\}$, while $f(A \cap B) = f(\varnothing) = \varnothing$. Similarly, $f(X \smallsetminus A) = f(B) = \{0\} \neq \varnothing$, although $Y \smallsetminus f(A) = \varnothing$.

4.6 We have $f(A \cap B) \subset f(A) \cap f(B)$. (Prove this!) However, there is no natural inclusion between $f(X \smallsetminus A)$ and $Y \smallsetminus f(A)$. Indeed, we can arbitrarily change a map on $X \smallsetminus A$ without changing it on A, and hence without changing $Y \smallsetminus f(A)$.

4.7 The bijectivity of f suffices for any equality of this kind. The injectivity is necessary and sufficient for (9), while the surjectivity is necessary for (10). Thus, the bijectivity of f is necessary to make correct all equalities of 4.4.

4.8 $\boxed{\subset}$ Let $y \in B \cap f(A)$. Then $y = f(x)$, where $x \in A$. On the other hand, $x \in f^{-1}(B)$, whence $x \in f^{-1}(B) \cap A$, and therefore $y \in f(f^{-1}(B) \cap A)$. $\boxed{\supset}$ Prove the opposite inclusion on your own.

4.9 No, not necessarily. Example: $f : \{0\} \to \{0, 1\}$, $g : \{0, 1\} \to \{0\}$. Surely, f must be injective (see 4.L), and g surjective (see 4.N).

5.1 $\mathrm{pr}_Y^{-1}(B) = X \times B$.

5.2 Plug in the definitions.

5.3 This is rather straightforward.

5.4 This is also quite straightforward.

5.5 Recall the definition of the product topology and use 5.3.

5.6 Let us check that ρ is continuous at each point $(x_1, x_2) \in X \times X$. Indeed, let $d = \rho(x_1, x_2)$, $\varepsilon > 0$. Then, using the triangle inequality, we easily see that $\rho\big(B_{\varepsilon/2}(x_1) \times B_{\varepsilon/2}(x_2)\big) \subset (d - \varepsilon, d + \varepsilon)$.

5.7 We have:

$$\mathrm{pr}_Y(\Gamma_f \cap (A \times Y)) = \mathrm{pr}_Y(\{(x, f(x)) \mid x \in A\}) = \{f(x) \mid x \in A\} = f(A).$$

Prove the second identity on your own.

5.8 Indeed, we have

$$(A \times B) \cap \Delta = \{(x, y) \mid x \in A,\ y \in B,\ x = y\} = \{(x, x) \mid x \in A \cap B\}.$$

5.9 $\mathrm{pr}_X |_{\Gamma_f} : (x, f(x)) \leftrightarrow x$.

5.10 Indeed, $f(x_1) = f(x_2)$ iff $\mathrm{pr}_Y(x_1, f(x_1)) = \mathrm{pr}_Y(x_2, f(x_2))$.

5.11 This obviously follows from the relation $T(x, f(x)) = (f(x), x) = (y, f^{-1}(y))$.

5.12 This is quite straightforward.

5.21 The map pr sends each element a of X to the element A of the partition (regarded as an element of the quotient set) containing a, and so the preimage $\mathrm{pr}^{-1}(A) = \mathrm{pr}^{-1}(\mathrm{pr}(x))$ is also the element A of the partition containing the point $a \in X$.

6.5 No, because it is not antisymmetric. Indeed, $-1 \mid 1$ and $1 \mid -1$, but $-1 \neq 1$.

6.10 The hypotheses of Theorem 6.B turn into the following restrictions on C: C is closed with respect to addition, contains the zero, and no nontrivial translation bijectively maps C onto C.

6.17 1) Obviously, the greatest element is maximal and the smallest one is minimal, but the converse statements are not true. 2) These notions coincide for any subset of a poset iff any two elements of the poset are comparable (i.e., one of them is greater than the other). $\boxed{\Longrightarrow}$ Indeed, consider, e.g., a two-element subset. If the two elements were incomparable, then each of them would be maximal, and hence, by assumption, the greatest one. However, the greatest element is unique. A contradiction. $\boxed{\Longleftarrow}$ Comparability of any two elements obviously implies that in any subset any maximal element is the greatest one, and any minimal element is the smallest one.

6.21 The relation of inclusion on the set of all subsets of X is a linear order iff X is either empty or a singleton.

6.C If $a \precsim b \precsim c$, then we have $a \prec b \prec c$, $a = b = c$, $a \prec b = c$, or $a = b \prec c$. In all four cases, we have $a \precsim c$.

6.22 Yes, it is. A number a always divides a (formally speaking, even 0 divides 0). Further, if a divides b and b divides c, then a divides c.

6.D The relation $\sim$ is obviously reflexive, symmetric, and also transitive.

6.23 $a \sim b$ iff $a = \pm b$.

6.E Indeed, if $a' \sim a$, $a \precsim b$, and $b \sim b'$, then $a' \precsim a \precsim b \precsim b'$, whence $a' \precsim b'$. Clearly, the relation defined on the equivalence classes is transitive and reflexive. Now, if two equivalence classes $[a]$ and $[b]$ satisfy both $a \precsim b$ and $b \precsim a$, then $[a] = [b]$, i.e., the relation is antisymmetric, and, hence, it is a nonstrict order.

6.F (a) In this case, we obtain the trivial nonstrict order on a singleton; (b) In this case, we obtain the same nonstrict order on the same set.