Metric, neighborhoods, topology

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In modern mathematics there is a large part related to notions of continuity. In this part, a background environment and commonly accepted language are built around the notion of topological space and continuous map. These notions are not complicated.

Below a short motivation for the notion of topological space is given. It starts with the definition of metric space, as usual. The topological notions that are introduced in a metric setup, are limited to the notion of neighborhood.

Then a few natural properties of neighborhoods are collected and proclaimed to be axioms, which define the structure in an abstract set. Secretly, this is a definition of topological structure. It comes motivated. It looks more complicated than the standard definition of topological structure in terms of open sets. The standard definition is given immediately after that, and then equivalence of these two approaches is carefully stated and proved.

1 Metric Spaces

1.1 Distances

In many mathematical contexts, we meet sets, in which it makes sense to speak about distances between their elements. For example, in the set of real numbers $\mathbb{R}$ for a distance between real numbers a and b it is natural to take $|a - b|$.

In all such situations we expect that distances have some properties. Namely, a distance is expected to be a non-negative real number, the distance between a and b is zero if and only if $a = b$; the distance from a to

b is the same as the distance from b to a. Finally, the distance from a to b does not exceed the sum of distances from a to c and from c to b. These properties are called *axioms of metric.* Here are there usual formulations.

Let S be a set. A function $d : S \times S \to \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ is called a *metric* (or *distance function*) on S if

AxM1 $d(a, b) = 0$ iff $a = b$;

AxM2 $d(a, b) = d(b, a)$ for any $a, b \in S$;

AxM3 $d(a, b) \leq d(a, c) + d(c, b)$ for any $a, b, c \in S$.

The pair (S, d), where d is a metric on S, is called a *metric space*. Elements of the set S are called *points.* The third axiom of metric is called the *triangle inequality*.

1.2 Examples of metric spaces

The line $\mathbb{R}$ with $d(a, b) = |a - b|$ mentioned above is a metric space.

The usual distance between points on the plane can be calculated in terms of coordinates of these points using the Pythagoras theorem. This gives the following example of a metric space: the plane $\mathbb{R}^2$ with $d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

Generalization: the function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (a, b) \mapsto \sqrt{\sum_{i=1}^n (a_i - b_i)^2}$ is a metric on $\mathbb{R}^n$. It is called the *Euclidean metric.* I skip a proof that this is a metric. Verification of the first two axioms of metric is straightforward. Verification for the triangle inequality is more complicated, but can be found easily.

The first two examples are special cases of the third one with $n = 1$ and 2, respectively. These metrics are always meant when $\mathbb{R}$ or $\mathbb{R}^n$ are considered as metric spaces, unless other metric is specified explicitly.

Of course, there are many metrics on the same set. For example, given a metric d, one can build another metric just by multiplying d by any positive constant.

Any set S with the function

$$d : S \times S \to \mathbb{R}_+ : \ (x, y) \mapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric space.

1.3 Balls and spheres

Let (S, d) be a metric space, $a \in S$ a point, r a positive real number. Then the sets

$$B_r(a) = \{\, x \in S \mid d(a, x) < r \,\}, \tag{1}$$
$$D_r(a) = \{\, x \in S \mid d(a, x) \leq r \,\}, \tag{2}$$
$$S_r(a) = \{\, x \in S \mid d(a, x) = r \,\} \tag{3}$$

are called, respectively, the *open ball*, *closed ball* (or *disk*), and *sphere* of the space (S, d) with center a and radius r.

In $\mathbb{R}$, an open ball is an open interval, in $\mathbb{R}^2$ an open ball is an open disk (bounded by a circle), in $\mathbb{R}^3$ an open ball is what we usually call open ball.

1.4 Neighborhoods

Let S be a metric space with metric d, let $p \in S$. A set $N \subset S$ is called a *neighborhood* of p if it contains some open ball with center p. In other words, N is a neighborhood of p if there exists $r \in \mathbb{R}_+$ such that $B_r(p) \subset N$.

1.5 Digression: an easy definition of continuous functions

The notion of neighborhood is very handy. For example, the famous epsilon-delta definition for continuity of a function becomes very simple: *a function f is continuous at a point p if for any neighborhood N of $f(p)$ the preimage $f^{-1}(N) = \{x \mid f(x) \in N\}$ is a neighborhood of p.*

This definition for a function $f : \mathbb{R} \to \mathbb{R}$ is really equivalent to the epsilon-delta definition, but it looks more conceptual. It contains only one quantifier, while the epsilon-delta definition contains three. As we will see, it works far beyond the environment of functions $\mathbb{R} \to \mathbb{R}$.

Here is how it works for maps between metric spaces. The definition stays the same: *a map $f : X \to Y$ between metric spaces X and Y is continuous at a* *point p if the preimage $f^{-1}(N)$ of any neighborhood N of $f(p)$ is a neighborhood of p.*

Theorem 1.1. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff each ball centered at $f(a)$ contains the image of some ball centered at a.*

Proof. Let as assume that f is continuous at a point $a \in X$. Let $B_\varepsilon(f(a))$ be a ball centered at $f(a)$. As an open set, it is a neighborhood of $f(a)$ in Y. By local continuity of f at a, its preimage $f^{-1}B_\varepsilon(f(a))$ is a neighborhood of a. Hence, there exists a ball $B_\delta(a) \subset f^{-1}B_\varepsilon(f(a))$. Applying f to both sides of this inclusion formula, we obtain $f(B_\delta(a)) \subset B_\varepsilon(f(a))$.

Let us assume that each ball $B_\varepsilon(f(a))$ contains the image of a ball $B_\delta(a)$ and prove that then f is continuous at a. Let N be a neighborhood of $f(a)$. In a metric space Y this means that there exists a ball $B_\varepsilon(f(a)) \subset N$. By our assumption, $B_\varepsilon(f(a))$ contains the image of some ball $B_\delta(a)$. Therefore, $f^{-1}(N) \supset B_\delta(a)$ is a neighborhood of a. $\square$

Theorem 1.2. *Let X and Y be metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Proof. The condition "for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\left(f(x), f(a)\right) < \varepsilon$" means that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. Now, apply the preceding Theorem 1.1. $\square$

1.6 Properties of neighborhoods

A neighborhood of p is a set which includes all points that are sufficiently close to p. Therefore the following two properties seem natural:

1. *The intersection of any two neighborhoods of p is a neighborhood of p.*
2. *Any set which contains a neighborhood of p is a neighborhood of p.*

It would be natural also to expect that *a neighborhood is shared by neighbors.* In other words, *a neighborhood N of a point p is a neighborhood for points sufficiently close to p.* "Sufficiently close" means "belonging to a neighborhood". This time it is probably a new neighborhood, more narrow neighborhood, because the original one could contain quite remote points. Therefore the statement: "a neighborhood is shared by neighbors" is transformed to a more complicated and formal statement:

3. *For any neighborhood N of p there is a neighborhood $M \subset N$ such that N is a neighborhood of each $q \in M$.*

If N contains an open ball B with center p, then B is a neighborhood of p and N is a neighborhood for each point of B.

Most arguments in which neighborhoods appear are based on the properties 1-3 above. A reasonable notion of neighborhood can be extended to situations without distances. The three properties stated above hold true in all these situations and inspire to use the same intuition and terminology.

2 Neighborhood spaces

2.1 Axioms for neighborhoods

Let S be a set, let for each $p \in S$ a collection $\mathcal{N}_p$ of subsets of S is fixed, and these collections satisfy the following requirements:

AxN1 $\mathcal{N}_p$ is not empty for any $p \in S$;

AxN2 $p \in U$ for each $U \in \mathcal{N}_p$;

AxN3 if $U \in N_p$ and $U \subset V$, then $V \in \mathcal{N}_p$;

AxN4 if $U, V \in N_p$, then $U \cap V \in \mathcal{N}_p$;

AxN5 for any $U \in \mathcal{N}_p$ there exists $V \in \mathcal{N}_p$ such that $U \in \mathcal{N}_q$ for any $q \in V$.

Observe that in the requirement 5 we have $V \subset U$, since $U \in \mathcal{N}_b$ implies $b \in U$ by the first requirement, hence any $b \in V$ belongs to U.

Denote by $\mathcal{N}$ the family $\{\mathcal{N}_p \mid p \in S\}$ of all $\mathcal{N}_p$. Then

- the pair $(S, \mathcal{N})$ (i.e., the set S equipped with the collections $\mathcal{N}_p$ for each $p \in S$) is called a *neighborhood space*,
- elements of S are called its *points*,
- each $U \in \mathcal{N}_p$ is called a *neighborhood* of p in this neighborhood space,
- $\mathcal{N}$ is called a *neighborhood structure* in S.
- The requirements AxN1 – AxN5 are called the *axioms of neighborhood structure*.

Let us reformulate the axioms of neighborhood structure using these new words wherever possible.

AxN1 Each point has a neighborhood.

AxN2 A point is contained in each of its neighborhoods.

AxN3 Any set which contains a neighborhood of a point is also a neighborhood of the point.

AxN4 The intersection of two neighborhoods of a point is a neighborhood of the point.

AxN5 Each neighborhood U of a point p contains a neighborhood V of p such that U is a neighborhood for each point of V.

AxN1 and AxN2 imply that the whole set S is a neighborhood for each point.

AxN5 can be restated also as follows:

Each neighborhood U of a point p is shared by all the points of some neighborhood V of p.

2.2 Relation to metric spaces

Any metric space provides an example of a neighborhood space, because neighborhoods of points in a metric space satisfy AxN1 – AxN5.

Different metrics on the same set may define the same neighborhoods. This is what happens, for example, if the balls are the same although the metrics differ. This is the case if one of the metrics is obtained from the other one as a result of multiplication by a positive real number.

Metrics may differ more substantially, but still define the same neighborhood structure. If for any point p any ball defined by one of the metrics contains a ball defined by the other metric and vice versa, then the neighborhood structures coincide.

For example, on the plain $\mathbb{R}^2$ the Euclidean metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2},$$

the "Manhattan metric"

$$d_{Manhattan}((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|$$

and

$$d_{max}((a_1, a_2), (b_1, b_2)) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

define the same neighborhood structure. To realize how different these metrics are and why their neighborhood structures coincide, I recommend to figure out what spheres and balls they define.

Some neighborhood structures cannot be obtained from any metric. For example, if S consists of two points, a and b, then $\mathcal{N}_a = \mathcal{N}_b = S$ satisfy all the requirements, but cannot be obtained from any metric on S, because for any metric d on S the ball $B_{d(a,b)}(a) = \{a\}$ is a neighborhood of a, that is $\{a\} \in \mathcal{N}_a$, and $\mathcal{N}_a \neq S = \{a, b\}$.

Thus, under the transition from metric spaces to neighborhood spaces some information is forgotten, and, on the other hand, new spaces appeared (since not all neighborhood spaces can be obtained from metric ones). We ignore some information encoded in a metric (which is unessential for a wide range of problems) and consider a larger scope of objects.

2.3 Intermediate recapitulation

This is a typical success story about a development of an abstract mathematical notion. The notion of neighborhood structure that we came to is equivalent to the notion of *topological space* - one of the most profound notions of modern mathematics.

At this level of knowledge, we could not yet see and appreciate the most important achievement of the transition. The new objects (neighborhood spaces) are more flexible than the initial ones (metric spaces). With the new ones one can do many operations (e.g., so called "cut and paste" operations), which are impossible, or, at least, unnatural and difficult to perform, with the old ones.

Of course, this is not an account of the real story. The transition was not performed in a single stroke. I just presented a plausible motivation for the notion of topological space. Usually this notion appears dogmatically in a final form, which emerges mysteriously from a thin air.

In the story above, an important final stage is still missing. Although the notion of neighborhood structure is quite intuitive and motivated, it is too cumbersome to be convenient. In order to describe a specific neighborhood structure literally, one has to care about the set of all neighborhoods for each point. The structure can be easily recovered from other equivalent, but more concise structures. Limits of point sequences, closures of sets, open sets, etc. are defined in terms of neighborhoods and can be used for recovering neighborhoods. Hence, the neighborhood structure can be defined in terms of any of these notions. Which of them are better suited for defining the neighborhood structure is a highly non-trivial question. The choice was not straightforward and took a long time. But nowadays it is commonly accepted to define the structure of topological space on basis of the notion of open set.

In the next section the modern definition of topological space, which is based on open sets, is presented. Then we will prove that a topological space is nothing but a neighborhood space.

3 Topological spaces

3.1 Topological structure

Let S be a set. Let Ω be a collection of its subsets such that:

AxT1 The union of any collection of elements of Ω belongs to Ω.

AxT2 The intersection of any finite set of elements of Ω belongs to Ω.

AxT3 The empty set $\varnothing$ and the whole S belong to Ω.

Then

- Ω is called a *topological structure* or just a *topology* on S;
- the pair (S, Ω) is called a *topological space*;
- elements of S are called *points* of this topological space;
- elements of Ω are called *open sets* of the topological space (S, Ω);
- the conditions in the definition above are called the *axioms of topological structure*.

Let us reformulate the axioms of topological structure using the words *open set* wherever possible.

AxT1 The union of any collection of open sets is open.

AxT2 The intersection of any finite collection of open sets is open.

AxT3 The empty set and the whole space are open.

3.2 Neighborhoods in a topological space

In a topological space (S, Ω), a neighborhood of a point p is defined as a set which contains the point p together with some open set that contains p. In formulas: N is a neighborhood of p if there exists $U \in \Omega$ such that $p \in U \subset N$.

Observe that, according to this definition, an open set is a neighborhood of each of its points.

Theorem 3.1. *The set of neighborhoods in a topological space satisfies all the axioms for neighborhood structure.*

Proof. AxN1 (each point has a neighborhood). Indeed, the whole S is open by AxT3. Therefore S is a neighborhood for every point.

AxN2 (a point is contained in each of its neighborhoods). It follows immediately from the definition of neighborhood.

AxN3 (any set containing a neighborhood of a point is also its neighborhood). Also, it follows immediately from the definition: if N is a neighborhood of p, then there exists an open set U such that $p \in U \subset N$, and if $M \supset N$, then $p \in U \subset M$.

AxN4 (the intersection of two neighborhoods of a point is a neighborhood of the point). Let N and M be neighborhoods of p. Then there exist open sets U and V such that $p \in U \subset N$ and $p \in V \subset M$. Then $U \cap V$ is open by AxT2, and $p \in U \cap V \subset N \cap M$.

AxN5 (each neighborhood U of a point p contains a neighborhood V of p such that U is a neighborhood for each point of V). Since U is a neighborhood of p, there exists an open set W such that $p \in W \subset U$. This set is a neighborhood of each of its points. Hence W can be taken as V. $\square$

Denote the neighborhood structure defined by a topological structure Ω by $\mathcal{N}(\Omega)$. Thus each topological space (S, Ω) converts into a neighborhood space $(S, \mathcal{N}(\Omega))$.

3.3 Open sets in a neighborhood space

Let us return to neighborhood spaces. In a neighborhood space, a set which is a neighborhood of any of its points is called *open*. In other words, U is open if $U \in \mathcal{N}_p$ for each $p \in U$.

In a metric space this definition is applicable, too, as any metric space is a neighborhood space. In a metric space this definition looks as follows: *a set is open in a metric space if, together with any of its points, it contains an open ball centered at this point.*

Theorem 3.2. *In a neighborhood space the set of all open sets (defined by the neighborhood structure as above) satisfies the axioms of topological structure.*

Proof. Let us begin with AxT3. As it was observed above, the set of all the points of a neighborhood space is a neighborhood of each point. Thus, this set is open. On the other hand, the empty set is open: it has no point, hence it is a neighborhood for any of them.

AxT1 (the union of any collection of open sets is open). This follows from AxN3. Indeed, each point of a union of open sets belongs to one of the open sets that are united, this open set is its neighborhood, and hence the whole union, as a larger set, is its neighborhood by AxN3.

AxT2 (the intersection of any two open sets is open). This follows obviously from the AxN4. □

Denote the topological structure defined by a neighborhood structure $\mathcal{N}$ by $\Omega(\mathcal{N})$. Thus each neighborhood space $(S, \mathcal{N})$ turns into a topological space $(S, \Omega(\mathcal{N}))$.

Notice that the last axiom of neighborhood structure AxN5 was not used in the proof above. It is also needed, but its role is more delicate.

Theorem 3.3. *In the topological space that is built as above out of a neighborhood space, the neighborhoods coincide with the neighborhoods of the initial neighborhood space. In formula:* $\mathcal{N}(\Omega(\mathcal{N})) = \mathcal{N}$.

Proof. Let $(S, \mathcal{N})$ be a neighborhood space. Let us prove first the inclusion $\mathcal{N}(\Omega(\mathcal{N})) \subset \mathcal{N}$. Let $V \in \mathcal{N}_p(\Omega(\mathcal{N}))$, in words: let V be a neighborhood of $p \in S$ in the sense of definition for neighborhoods in a topological space

$(X, \Omega(\mathcal{N}))$. By this definition, there exists an open set $U \in \Omega(\mathcal{N}))$ such that $p \in U \subset V$. By the definition of topological structure $\Omega(\mathcal{N})$, the set U is a neighborhood of each of its points (in the original neighborhood space). In formula: $U \in \mathcal{N}_q$ for any $q \in U$. Since $V \supset U$, $V \in \mathcal{N}_q$ by AxN3 for each $q \in U$. In particular, $V \in \mathcal{N}_p$, that is V a neighborhood of p which belongs to the original neighborhood structure $\mathcal{N}$.

Conversely, let $V \in \mathcal{N}_p$. Let U is the set of points q such that $V \in \mathcal{N}_q$. Observe that $p \in U$ because V is a neighborhood of p. Further, $U \subset V$ since V is a neighborhood for each point of U. Therefore by AxN5 for each $q \in U$ there exists a neighborhood W of q such that V is a neighborhood for each point of W. Hence $W \subset U$. Thus, each point of U has a neighborhood contained in U. Hence U is open, that is $U \in \Omega(\mathcal{N})$ and hence V is a neighborhood in the sense of definition for neighborhoods in the topological space $(S, \Omega(\mathcal{N}))$. In formula, $V \in \mathcal{N}(\Omega(\mathcal{N}))$. □

Theorem 3.4. $\Omega(\mathcal{N}(\Omega)) = \Omega$.

Proof. Let (S, Ω) be a topological space, $U \in \Omega$. Then U is a neighborhood of each of its point, i.e., $U \in \mathcal{N}_p(\Omega)$ for each $p \in U$. Therefore, U is open in the topological structure defined by the neighborhood structure $\mathcal{N}(\Omega)$, that is $U \in \Omega(\mathcal{N}(\Omega))$. Thus $\Omega \subset \Omega(\mathcal{N}(\Omega))$.

Conversely, let $U \in \Omega(\mathcal{N}(\Omega))$. Then $U \in \mathcal{N}_p(\Omega)$ for each $p \in U$. This means that there exists $V_p \in \Omega$ such that $p \in V_p \subset U$. Then $U = \cup_{p \in U} V_p$. By AxT1, U belongs to Ω as the union of sets $V_p \in \Omega$. Thus $\Omega(\mathcal{N}(\Omega)) \subset \Omega$. □

Theorems 3.3 and 3.4 imply that the maps $\mathcal{N} \mapsto \Omega(\mathcal{N})$ and $\Omega \mapsto \mathcal{N}(\Omega)$ are inverse to each other. So, neighborhood structures and topological structures on the same set are in bijective correspondence to each other.

4 Continuous maps

4.1 Definition and basic properties

Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if the preimage of each open subset of Y is an open subset of X.

Recall again that the *preimage* of a subset $B \subset Y$ under a map $f : X \to Y$ is $\{a \in X \mid f(a) \in B\}$ (in words: this is the set of all the elements of X which are mapped by f to elements of B). The preimage of B under f is denoted by $f^{-1}(B)$.

This is the global version of the notion of continuity at a point considered above. We will study the relation between the global and local continuities later.

Theorem 4.1. *A map is continuous iff the preimage of each closed set is closed.*

Proof. Let $f : X \to Y$ be a map. If $f : X \to Y$ is continuous, then, for each closed set $F \subset Y$, the set $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$ is open, and therefore $f^{-1}(F)$ is closed. To prove the converse statement, exchange the words *open* and *closed* in the above argument. $\square$

Theorem 4.2. *The identity map of any topological space is continuous.*

Theorem 4.3. *Any constant map (i.e., a map with one-point image) is continuous.*

Proof. The preimage of any set under a constant map either is empty or coincides with the whole space. $\square$

Exercise 4.1. Let Ω_1 and Ω_2 be two topological structures in a space X. Prove that the identity map

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous iff $\Omega_2 \subset \Omega_1$.

Exercise 4.2. Let $f : X \to Y$ be a continuous map. Find out whether or not it is continuous with respect to

1. a larger topology on X and the same topology on Y,
2. a smaller topology on X and the same topology on Y,
3. a larger topology on Y and the same topology on X,
4. a smaller topology on Y and the same topology on X.

Exercise 4.3. Let X be a discrete space, Y an arbitrary space.
1) Which maps $X \to Y$ are continuous?
2) Which maps $Y \to X$ are continuous for each topology on Y?

Exercise 4.4. Let X be an indiscrete space, Y an arbitrary space.
2) Which maps $Y \to X$ are continuous?
1) Which maps $X \to Y$ are continuous for each topology on Y?

Theorem 4.4. *Let A be a subspace of X. Then the inclusion* $\mathrm{in} : A \to X$ *is continuous.*

Proof. If a set U is open in X, then its preimage $\mathrm{in}^{-1}(U) = U \cap A$ is open in A by the definition of the induced topology. $\square$

Exercise 4.5. The topology Ω_A induced on $A \subset X$ by the topology of X is the smallest topology on A with respect to which the inclusion $\mathrm{in} : A \to X$ is continuous.

Theorem 4.5. *A composition of continuous maps is continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. We must show that for every $U \subset Z$ that is open in Z its preimage $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. The set $g^{-1}(U)$ is open in Y by continuity of g. In turn, its preimage $f^{-1}(g^{-1}(U))$ is open in X by the continuity of f. $\square$

Recall that the restriction of a map $f : X \to Y$ to $A \subset X$ is the map $f|_A : A \to Y$ defined by formula $(f|A)(x) = x$ for $x \in A$.

Theorem 4.6. *A restriction of a continuous map is continuous.*

Proof. Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $A \subset X$. Then $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$. $\square$

4.2 Local Continuity

A map f from a topological space X to a topological space Y is said to be *continuous at a point* $a \in X$ if the preimage of every neighborhood of $f(a)$ is a neighborhood of a.

Thus, we come back to the definition which we discussed above in the environment of metric spaces.

Theorem 4.7. *A map $f : X \to Y$ is continuous iff it is continuous at each point of X.*

Proof. Assume that f is continuous. Let us prove that f is continuous at every $a \in X$. Let N be a neighborhood of $f(a)$. By the definition of neighborhood, it contains an open neighborhood: there exists an open set U such that $f(a) \in U \subset N$. Then $a \in f^{-1}(U) \subset f^{-1}(N)$. The set $f^{-1}(U)$ is open in X, because U is open in Y and f is continuous. Thus $f^{-1}(N)$ contains open set $f^{-1}(U)$ which contains a. Therefore $f^{-1}(N)$ is a neighborhood of a.

Now let us assume that f is continuous at every point $a \in X$ and prove that f is continuous. We must check that the preimage of each open set is open. Let $V \subset Y$ be an open set in Y. Take $a \in f^{-1}(V)$. By continuity of f at a, the set $f^{-1}(V)$ is a neighborhood of a. Hence, there exists an open set U_a such that $a \in U_a \subset f^{-1}(V)$. Take such U_a for each $a \in f^{-1}(V)$ and unite all of them. The union $U = \cup_{a \in f^{-1}(V)} U_a$ is an open set (as a union of open sets), it is contained in $f^{-1}(V)$ as each U_a is contained, and it contains $f^{-1}(V)$, as each $a \in f^{-1}(V)$ belongs to its U_a. Thus $f^{-1}(V) = U$ and is an open set. $\square$