# Advanced Linear Algebra MAT 315

Oleg Viro

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## Solutions for midterm 1. Problem 4

**Problem 4.** Let $F$ be a field and $S$ be a subset of $F$.

**(a)** Prove that among subfields $K \subset F$ such that $S \subset K$, there exists the smallest one, $K_0$.

**Proof.** In Lecture 2, in the proof of existence of a prime subfield in any field,
there is Lemma according to which
the intersection of any collection of subfields in a field $F$ is a subfield of $F$.
Hence, the intersection of all subfields $K \subset F$ such that $S \subset K$ is a subfield of $F$.
This subfield is contained in any subfield $K \subset F$ such that $S \subset K$.
Thus it is the smallest of those $K$'s. □

**(b)** Find a necessary condition for finiteness of this minimal subfield $K_0$.
**Solution.** Here are two necessary conditions.
(1) If $K_0$ is finite, then $S \subset K_0$ must be finite.

(2) If $K_0$ is finite, then the characteristic of $F$ is not 0.
Indeed, the prime subfield of $F$ must be finite, and this happens iff the characteristic of $F$ is 0.

2 / 24

## Problem 5

**Problem 5.** Let $\mathbb{F}$ be a field and $\varphi\colon \mathbb{F} \to \mathbb{F}$ be a field homomorphism.
**(a)** Is $\varphi$ a linear map $\mathbb{F}^1 \to \mathbb{F}^1$? Justify your answer.
**Solution.** No, unless $\varphi = \mathrm{id}$. Indeed, if $\varphi \neq \mathrm{id}$, then there exists $\alpha$ such that $\varphi(\alpha) \neq \alpha$. Since $\varphi$ is a field homomorphism, then

$$\varphi((\alpha)) = \varphi((\alpha \cdot 1)) = (\varphi(\alpha \cdot 1)) = (\varphi(\alpha) \cdot \varphi(1)) = (\varphi(\alpha) \cdot 1) = (\varphi(\alpha))$$

On the other hand, if $\varphi$ was a linear map, we would have

$$\varphi((\alpha)) = \varphi((\alpha \cdot 1)) = \varphi(\alpha(1)) = \alpha(\varphi(1)) = \alpha(1) = (\alpha).$$

Therefore $\varphi(\alpha) = \alpha$, but this contradicts to the assumption that $\varphi(\alpha) \neq \alpha$.

**(b)** Give an example of a field homomorphism $\varphi\colon \mathbb{F} \to \mathbb{F}$ such that $\varphi \neq \mathrm{id}_{\mathbb{F}}$ for some field $\mathbb{F}$.
**Solution.** $\mathbb{F} = \mathbb{C}$, and $\varphi$ is a complex conjugation $x + iy \mapsto x - iy$, which is a field homomorphism, see handout of Lecture 2.

3 / 24

## Linear maps



### Linear maps

Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$.

**Definition** A map $T : V \to W$ is said to be **linear** if:

$T(u+v) = Tu + Tv$ for all $u, v \in V$ ($T$ is **additive**);

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$ ($T$ is **homogeneous**).

Linear maps or linear transformations? $Tv$ or $T(v)$?

**Notation** $\mathcal{L}(V, W) = \{\text{all the linear maps } V \to W\}$

Other notations: $\mathrm{Hom}_{\mathbb{F}}(V, W)$ or $\mathrm{Hom}(V, W)$.



### Examples of linear maps

**Zero** $0 \in \mathcal{L}(V, W) : x \mapsto 0$

**Identity** $I \in \mathcal{L}(V, V) : x \mapsto x$ Other notations: $\mathrm{id}$, or $\mathrm{id}_V$, or $1$.

**Inclusion** $\mathrm{in} \in \mathcal{L}(V, W) : x \mapsto x$ if $V \subset W$

## Examples of linear maps

Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.

Integration $\mathbb{R}[x] \to \mathbb{R} : p(x) \mapsto \int_0^1 p(x)dx$.

Multiplication by a polynomial $q(x)$ $T : \mathbb{F}[x] \to \mathbb{F}[x] : Tp(x) = q(x)p(x)$.

Backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

Forward shift $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$

7 / 24

## A linear map takes 0 to 0

**Theorem.** Let $T : V \to W$ be a linear map. Then $T(0) = 0$.

**Proof.** $T(0) = T(0 + 0) = T(0) + T(0)$.

So, $T(0) = T(0) + T(0)$.

Add $-T(0)$ to both sides.

$0 = T(0)$. □

8 / 24

## Linear operations in $\mathcal{L}(V,W)$

**Definition** Let $S,T : V \to W$ be maps and $\lambda \in \mathbb{F}$.

The **sum** $S+T$ and the **product** $\lambda T$ are maps $V \to W$ defined by

$(S+T)(v) = Sv + Tv$ and $(\lambda T)(v) = \lambda(Tv)$ for all $v \in V$.

**Theorem.** If $S,T$ are linear maps, then $S+T$ and $\lambda T$ are linear maps.

**Proof. Exercise!** It's easy! □

**Theorem** With the operations of addition and scalar multiplication, $\mathcal{L}(V,W)$ **is a vector space.**

**Proof. Exercise!** It's easy! □

Special case: $W = \mathbb{F}$. Then $\mathcal{L}(V,W) = \mathcal{L}(V,\mathbb{F})$ is called the **dual space** and is denoted by $V'$.
Elements of $V'$ are linear maps $V \to \mathbb{F}$. They are called **linear functionals** or **covectors**.



## Composition

**Definition (should be well known).** Let $T : U \to V$ and $S : V \to W$ be maps.
The **composition** $S \circ T$ is a map $U \to W$ defined by formula
$(S \circ T)(u) = S(T(u))$ for all $u \in U$.

Diagramatic presentation: $U \xrightarrow{T} V \xrightarrow{S} W$, with a curved arrow $U \to W$ labelled $S \circ T$

Composition is also called a **product**. (Say, in Axler's textbook.)

Often $S \circ T$ is denoted by $ST$, like a product.

**Theorem.** If $S$ and $T$ are linear maps, then $S \circ T$ is a linear map.

**Proof. Exercise!** It's easy! □

**Properties of composition.**

**associativity** $(T_1T_2)T_3 = T_1(T_2T_3)$.
**identity** $T\,\mathrm{id}_V = T = \mathrm{id}_W\,T$.
**distributivity** $(S_1+S_2)T = S_1T + S_2T$ and $(T_1+T_2)S = T_1S + T_2S$.
**homogeneity** $(\lambda S)T = \lambda(ST) = S(\lambda T)$.

## Categories

A category provides a convenient **language** to speak about **objects of unspecified nature**, but **related** to each other **in a very specific way**. A **category** consists of:

**objects** and
**morphisms**: for any two objects $X, Y$ morphisms $X \to Y$, and
**compositions** of morphisms: $X \xrightarrow{f} Y \xrightarrow{g} Z$

The composition is **associative**: $h \circ (g \circ f) = (h \circ g) \circ f$ $\quad A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

$= \quad A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

With any object $X$, the **identity morphism** $\mathrm{id}_X : X \to X$ is associated:

for $A \xrightarrow{f} X \xrightarrow{\mathrm{id}_X} X$ we have $\mathrm{id}_X \circ f = f$

and for $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{g} B$ we have $g \circ \mathrm{id}_X = g$.

12 / 24

## Examples of categories

**Example 1. The category of sets.**
Objects are sets, morphisms are maps, compositions are compositions of maps.

**Example 2. The category of vector spaces over a field $\mathbb{F}$.**
Objects are vector spaces over $\mathbb{F}$, morphisms are linear maps,
compositions are compositions of linear maps.

**Example 3. The category of linear maps.** Let $\mathbb{F}$ be a field.
Objects are linear maps $V \to W$, where $V$ and $W$ are vector spaces over $\mathbb{F}$.
A morphism $(V \xrightarrow{T} W) \to (X \xrightarrow{S} Y)$ is a pair $(V \xrightarrow{L} X, W \xrightarrow{M} Y)$ of linear maps such that $M \circ T = S \circ L$.

It is presented by a diagram: $\begin{array}{ccc} V & \xrightarrow{L} & X \\ \downarrow T & & \downarrow S \\ W & \xrightarrow{M} & Y \end{array}$ which is **commutative**: $M \circ T = S \circ L$. Composition:

$$\begin{pmatrix} A & \xleftarrow{N} & X \\ \downarrow U & & \downarrow S \\ B & \xleftarrow{R} & Y \end{pmatrix} \circ \begin{pmatrix} X & \xleftarrow{L} & V \\ \downarrow S & & \downarrow T \\ Y & \xleftarrow{M} & W \end{pmatrix} = \begin{pmatrix} A & \xleftarrow{N \circ L} & V \\ \downarrow U & & \downarrow T \\ B & \xleftarrow{R \circ M} & W \end{pmatrix}$$

13 / 24

## Operators

**Definition** A linear map from a vector space to itself is called an **operator**.

**Notation** $\mathcal{L}(V) = \{\text{all linear maps } V \to V\} = \mathcal{L}(V, V)$.

**Category of operators in vectors spaces over a field $\mathbb{F}$**

**objects** are operators $T : V \to V$,

a **morphism** $(V \xrightarrow{T} V) \to (W \xrightarrow{S} W)$

is a linear map $V \xrightarrow{L} W$ such that $S \circ L = L \circ T$.

or, rather, a commutative diagram
$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \downarrow{\scriptstyle T} & & {\scriptstyle S}\downarrow \\ V & \xrightarrow{L} & W \end{array},$$

a **composition** of morphisms is the composition of the linear maps.

**Axler**: "The deepest and most important parts of linear algebra ... deal with operators."

14 / 24

## Inverses and invertibles

In any category:

**Definition**

Morphisms $T : V \to W$ and $S : W \to V$ are said to be **inverse** to each other

if $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$.

A morphism $T : V \to W$ is called **invertible** if there exists a morphism inverse to $T$.

**Uniqueness of Inverse.** An morphism inverse to an invertible morphism is unique.

**Proof** Let $S_1$ and $S_2$ be inverse to $T : V \to W$. Then
$$S_1 = S_1 \,\mathrm{id}_W = S_1(TS_2) = (S_1T)S_2 = \mathrm{id}_V \, S_2 = S_2 \qquad \square$$

**Notation** If $T$ is invertible, then its inverse is denoted by $T^{-1}$.

For a morphism $T : V \to W$, the inverse morphism $T^{-1}$ is defined by two properties:
$$TT^{-1} = \mathrm{id}_W \quad \text{and} \quad T^{-1}T = \mathrm{id}_V.$$

15 / 24

## Isomorphism in a category

**Definition.** An invertible morphism is called an **isomorphism**.
Objects $V$ and $W$ are called **isomorphic** if $\exists$ an isomorphism $V \to W$ .

**Properties of isomorphisms**

- An identity morphism is an isomorphism.
- The composition of isomorphisms is an isomorphism.
- The map inverse to an isomorphism is an isomorphism.

**Relation of being isomorphic is equivalence.**
It is reflexive, symmetric and transitive.

A category does not recognize any difference between its isomorphic objects,
although the objects may be not identically the same.

16 / 24

## Invertible map $=$ bijection

Which sets are isomorphic in the category of sets and maps?

**Theorem. Invertibility is equivalent to bijectivity.**

You should know this. If not, see the textbook, page 81.

17 / 24

## Null space

**Definition (reminder)** For $T \in \mathcal{L}(V, W)$, the null space of $T$ is

$$\operatorname{null} T = T^{-1}\{0\} = \{v \in V \mid Tv = 0\}.$$

Another name: **kernel**. Notation: $\operatorname{Ker} T$.

**Examples**

- For $T : V \to W : v \mapsto 0$, $\operatorname{null} T = V$
- For differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $\operatorname{null} D = \{\text{constants}\}$
- For multiplication by $x^3$ $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F}) : Tp = x^3 p(x)$, $\operatorname{null} T = 0$
- For backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, F^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$
  $\operatorname{null} T = \{(a, 0, 0, \dots) \mid a \in \mathbb{F}\}$



## Null space is a subspace

**Theorem.** For $T \in \mathcal{L}(V, W)$, $\operatorname{null} T$ is a subspace of $V$.

**Proof.** As we know $T(0) = 0$. Hence $0 \in \operatorname{null} T$.

$u, v \in \operatorname{null} T \implies T(u + v) = T(u) + T(v) = 0 + 0 = 0 \implies u + v \in \operatorname{null} T$.

$u \in \operatorname{null} T, \lambda \in \mathbb{F} \implies T(\lambda u) = \lambda T u = \lambda 0 = 0 \implies \lambda u \in \operatorname{null} T$. □

## Injectivity and the null space

**Definition (reminder).**
A map $T : V \to W$ is called **injective** if $Tu = Tv \implies u = v$.

A map $T : V \to W$ is injective $\iff$ $u \neq v \implies Tu \neq Tv$.

$T$ is injective $\iff$ $\operatorname{null} T = \{0\}$.

**Proof**

$\implies$ Recall $0 \in \operatorname{null} T$. If $\operatorname{null} T \neq \{0\}$, then $\exists v \in \operatorname{null} T$, $v \neq 0$.
So, $Tv = T0 = 0$ and $T$ is not injective. □

$\impliedby$ Let $u, v \in V$, $Tu = Tv$. Then $0 = Tu - Tv = T(u - v)$.
Hence $u - v \in \operatorname{null} T = \{0\}$ $\implies u = v$. □



## Range

**Definition.**
For a map $T : V \to W$, the **range** of $T$ is $\operatorname{range} T = T(V) = \{Tv \mid v \in V\}$.

Another name: **image**. Notation: $\operatorname{Im} T$.

**Examples**

- For $T : V \to W : v \mapsto 0$, $\operatorname{range} T = \{0\}$.
- For differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $\operatorname{range} D = \mathcal{P}(\mathbb{R})$.
- For multiplication by $x^3$ $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F}) : Tp = x^3 p(x)$,
  $\operatorname{range} T =$ polynomials without monomials of degree $< 3$.

## Surjectivity and range

**Definition (reminder).**
A map $T : V \to W$ is called **surjective** if $\operatorname{range} T = W$.

**The range of a linear map is a subspace.**
For $T \in \mathcal{L}(V, W)$, $\operatorname{range} T$ is a subspace of $W$.

**Proof** $0 \in \operatorname{range} T$, since $T(0) = 0$.

If $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$, then $\exists v \in V : w = Tv$, $T(\lambda v) = \lambda T v = \lambda w \in \operatorname{range} T$.

$$w_1, w_2 \in \operatorname{range} T \implies \exists v_1, v_2 \in V : w_1 = Tv_1, w_2 = Tv_2$$
$$\implies w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2) \in \operatorname{range} T.$$ □



## Inverse to a linear map is linear

**Theorem** If $V$ and $W$ are vector spaces and a linear map $T : V \to W$ is invertible, then $T^{-1}$ is linear.

This means that a morphism in the category vector spaces is isomorphism
$\iff$ it is an isomorphism in the category of sets.

**Proof. Additivity.** Let $w_1, w_2 \in W$. Then

$$T^{-1}(w_1 + w_2) = T^{-1}(\mathrm{id}_W\, w_1 + \mathrm{id}_W\, w_2) = T^{-1}(TT^{-1}w_1 + TT^{-1}w_2)$$
$$= T^{-1}T(T^{-1}w_1 + T^{-1}w_2) = \mathrm{id}_V(T^{-1}w_1 + T^{-1}w_2) = T^{-1}w_1 + T^{-1}w_2\,.$$

**Proof. Homogeneity.**

$$T^{-1}(\lambda w) = T^{-1}(\lambda\, \mathrm{id}_W\, w) = T^{-1}(\lambda TT^{-1}w) = T^{-1}(\lambda T(T^{-1}w))$$
$$= T^{-1}T(\lambda T^{-1}w) = \mathrm{id}_V(\lambda T^{-1}w) = \lambda T^{-1}w\,.$$ □

**Corollary 1** A linear map $T : V \to W$ is an isomorphism in the category of vector spaces, if and only if it is bijective. □

**Corollary 2** A linear map $T : V \to W$ is an isomorphism in the category of vector spaces, if and only if $\operatorname{null} T = 0$ and $\operatorname{range} T = W$. □