Linear Algebra MAT 310 / Advanced Linear Algebra MAT 315

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Welcome to MAT 310 /MAT 315

This is the first lecture for courses MAT 310 and MAT 315.
My name is Oleg Viro, I will teach MAT 315, when it will split out on February 23.

Dr. Alena Erchenko will teach MAT 310. Her email is alena.erchenko@stonybrook.edu. To access her office hours, you need to go to Zoom meetings on the
main course page on Blackboard.
I will give lectures on February 4 and 9,
Dr. Erchenko will give lectures on February 11 and 16.

On February 18 we will have the **first midterm exam**. The results will be available on February 22. Some students will be given an option to switch to MAT 315.

Recitation R02 will become recitation R01 for MAT 315. The students in section R02, who will stay in MAT 310, will need to move either to R01 or R03 (and the opposite for students moving to MAT 315).

Recitation R01 for MAT 310 will **become hybrid** starting from February 23rd.

Blackboard and Gradescope

Everyone has to have access to Blackboard (both main course page and recitation)
and Gradescope.

Everyone should have received an email from Gradescope with an invitation
to join MAT 310. If someone has not, please contact Dr. Erchenko.

There is a **practice test on Gradescope** you **have to complete by Feb 3rd**. This will be used to determine **if you are attending** the course for our
official report to the university.

Homework and exam submissions are to be done **through Gradescope**.

Exams will be proctored on Zoom. For exams you will need a **webcam**,
stable internet connection and an ability to **scan and submit documents in a short timeframe**.

Homework is due **every Wednesday** starting the second week.
Control questions: 1. Where will you get homework?
2. How to submit a homework?

Two courses of Linear Algebra

There is a tradition to study Linear Algebra in a **couple of courses**.
At Mathematics Department in Stony Brook,
the **first** course is either MAT 211 or MAT 307/308,
the **second** course is either MAT 310 or MAT 315.

What is Linear Algebra about? What are its **main objects?**

In courses like MAT 211, the main objects are
systems of linear equations and **matrices**.
In MAT 310 and MAT 315, the main objects are different:
vector spaces, linear maps and **linear operators**.
The same Linear Algebra, but, **first**, we teach to **calculate**, and only **second, to think**.

This pattern echos the **Calculus path:**
first, a number of **Calculus courses** teach to calculate, later, **Analysis courses** explain what was calculated.

In MAT 310 and MAT 315, our main emphasize is on notions, theorems and proofs.

Today I will try to outline the final picture you will get at the end.
I am going to speak **about** mathematics, rather than to **speak mathematics**.

Structure of Linear Algebra

- **Bare Linear Algebra.** The first major part of Linear Algebra can be described as
study of bare vector spaces, their linear maps and operators in them.
The words **bare vector spaces** mean vector spaces without any extra structure,
like a quadratic, bilinear or Hermitian form.
The main **results** of this part are classifications of
vector spaces, linear maps and operators up to appropriate equivalence relations.

- **Tensor algebra.** This part is devoted to the study of
the **hierarchy of vector spaces** which is generated by a single vector space.
It starts with the **dual space** which is made of linear functionals on a vector space.
The hierarchy consists of spaces of polylinear functionals and tensors.
A linear map induces linear maps on ingredients of the hierarchy. **Determinants**, **traces** and other
characteristics of a linear map come naturally from the maps in hierarchy.

- **Inner product spaces.** This part repeats the first one, but for
vector spaces equipped with extra structure, like quadratic, bilinear or Hermitian forms.
This includes any orthogonality issues, self-adjoined and unitary operators, etc.
Again, the main results are **classification theorems**.
On the level of spaces, the extra structures make classification more complicated.
On the level of operators, the extra structures which are respected by the operators
prohibit the most complicated types of operators from the bare Linear Algebra.

Personal Linear Algebra

Almost all mathematicians use Linear Algebra in their research.
In different areas of Mathematics, different parts of Linear Algebra are used.
Every mathematician has his/her personal Linear Algebra.

Sheldon Axler, the author of our textbook "Linear Algebra done right", works in Functional Analysis.
I wholeheartedly recommend reading his nice elegant textbook.
Take a look at the movies in YouTube with Axler's lectures based on the textbook:
https://www.youtube.com/playlist?list=PLGAnmvB9m7zOBVCZBUUmSinFV0wEir2Vw

Axler used to work with infinite-dimensional spaces, and his research influenced his way of teaching Linear Algebra.
For operators in infinite-dimensional spaces, determinants cannot be defined.
Axler **postponed determinants** to the very end of the book and did not mention tensors.
In Functional Analysis all vector spaces are over $\mathbb{C}$ or $\mathbb{R}$.
In the textbook, ground fields are restricted to $\mathbb{C}$ and $\mathbb{R}$.
Terminology follows the choices that are traditional in Functional Analysis.

I am a topologist. My personal Linear Algebra includes a homological algebra over fields.
For many years I taught Differential Geometry.
Differential Geometry uses quite a lot of Tensor Algebra.
My personal Linear Algebra includes Tensor Algebra.

The rôle of Linear Algebra

Linear Algebra courses occupy a key place in the undergraduate program.
In particular, MAT 310 and MAT 315, together with Logic and Analysis courses, are the first proof-based undergraduate math courses.
To a great extent, this special place is due to the special rôle that Linear Algebra plays in Mathematics and Science.

Almost any process is linear in a sufficiently small neighborhood almost everywhere.
This principle lies at the foundation of all Mathematical Analysis and its applications.
This is why it is that important to know linear processes.
Linear Algebra is a careful study of the mathematical language for linearity.

The twentieth-century Physics has expanded the range of application of Linear Algebra, by adding the superposition principle, according to which the state space of any quantum system is a vector space over $\mathbb{C}$.
Constructions of Linear Algebra have given rise to the fundamental laws of nature.
For example, vector duality explaines Bohr's quantum complementarity principle.
Linear representation group theory underlied the periodic table, the zoology of elementary particles, and even the space-time structure.

Linear Algebra always provided a geometric view on solutions of computational problems.

Linear Algebra Archipelago

In Linear Algebra, the most fundamental objects are **vector spaces**.
Explicitly or implicitly, a vector space is present in any tiny piece of Linear Algebra.

A vector space comes with a **field**. Each vector space is **over** some field.
Hence in any piece of Linear Algebra, some field is present.

However, in most situations, all vector spaces are over the same field.
If this happens, the field is called the **ground field.** We will denote it by $\mathbb{F}$.

Vector spaces are related mainly via linear maps.
A linear map may relate only vector spaces over the same ground field.

So, Linear Algebra looks like an **archipelago of islands**. Each island accommodates vectors spaces over the same ground field, linear maps between them, and all the relevant knowledge.

In basic Linear Algebra courses, like MAT 211, fields do not show up explicitly.
In fact, in MAT 211 the only ground field is $\mathbb{R}$, the field of real numbers.
MAT 211 is confined in $\mathbb{R}$-island.

In the texbook by Axler *Linear Algebra Done Right* and in MAT 310,
a field is either $\mathbb{C}$ or $\mathbb{R}$.

In MAT 315, by default, a field will be arbitrary.

Fields which you know

- The field $\mathbb{Q}$ of rational numbers.

Recall: $\mathbb{Q}$ consists of fractions $\dfrac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$.

- The field $\mathbb{R}$ of real numbers.

Recall: $\mathbb{R}$ consists of numbers which can be presented by decimals.

- The field $\mathbb{C}$ of complex numbers.

Recall: $\mathbb{C}$ consists of numbers $a + ib$, with $a, b \in \mathbb{R}$, $i^2 = -1$.

Neither $\mathbb{R}$ nor $\mathbb{C}$ is studied in the prerequisite courses for MAT 310 or MAT 315.

$\mathbb{R}$ is studied in MAT 319/320 (Foundations of / Introduction to Analysis),
$\mathbb{C}$ is studied in MAT 342 (Complex Analysis),
arbitrary fields are studied in MAT 312/313 (Applied/Abstract Algebra).

None of these courses is among the prerequisite courses for MAT 310 or MAT 315.

Understanding of real numbers as decimals suffices for our Linear Algebra courses.
For either MAT 310 or MAT 315,
we need to learn more about complex numbers than you know from high school.
In the next lecture we will study the basics about fields.
In particular, we will study complex numbers from scratch.

Why so many fields

Linear Algebra over $\mathbb{C}$ is easier.

The reason for this is

Fundamental theorem of algebra.
Any polynomial equation with complex coefficients has a root.

> If any polynomial equation with coefficients in a field F has a root,
> then F is said to be **alebraically closed.**

When solving Linear Algebra problems, one has to solve polynomial equations. Therefore it's nice if the ground field is algebraically closed.

However, it is impossible to stay with $\mathbb{C}$.
In numerous applications of Linear Algebra, a field comes being built into a problem.
Often, this field is $\mathbb{R}$, but not necessarily.

It may happen that the field does not contain an element
which would be crucial for solving a specific problem.
Then one can replace the field with a more useful one,
so that the desired element would be added.

Vector spaces over the original field will be automatically extended, either.
A problem will be solved in the extended vector space,
and then the solution has to be translated back to the original setup.

Archipelago or multiverse?

Only comparatevely sophisticated problems lead to equations of high degree.
Most equations in Linear Algebra are **linear**. They are easy to solve in any field.

Therefore islands in the Linear Algebra archipelago are very similar to each other: most profound structure theorems (laying down the major laws) are the same on every island,
although one can find a substantial diversity in sophisticated details.

The islands are similar so much that **another metaphor** comes to mind:
Linear Algebra is a **multiverse** of parallel universes, which are listed by ground fields.

If needed, one can jump from one universe to another. Usually it is not needed.

A vector space V over $\mathbb{C}$ can be considered as a vector space $V_{\mathbb{R}}$ over $\mathbb{R}$.

Any vector space V over $\mathbb{R}$ can be **complexified** to a vector space $V \otimes \mathbb{C}$ over $\mathbb{C}$.

Applying both operations,
$$V \mapsto V_{\mathbb{R}} \mapsto V_{\mathbb{R}} \otimes \mathbb{C} \quad \text{or} \quad V \mapsto V \otimes \mathbb{C} \mapsto (V \otimes \mathbb{C})_{\mathbb{R}},$$
doubles the (dimension of) vectors space.

We will come back to this, but now it's time to talk about the main objects.

Vector spaces

Traditionally Linear Algebra studies only finite-dimensional vector spaces.
Infinite-dimensional vector spaces are studied in Functional Analysis.
A vector space is a set with two operations: **addition** and **scalar multiplication.**
Elements of the set are called **vectors**, elements of the ground field are called **scalars**.
The addition adds a **vector** to a **vector** and the result is a **vector**,
while the scalar multiplication multiplies a **vector** by a **scalar** and the result is a **vector**. The operations satisfy a number of natural requirements.
Out of our respect to vector spaces, let us look at their **formal complete definition:**

A **vector space** over a field $\mathbb{F}$ is a set V equipped with addition $V \times V \to V : (v, w) \mapsto v + w$ and scalar multiplication $\mathbb{F} \times V \to V : (\lambda, v) \mapsto \lambda v$ such that
the addition is **associative** and **commutative**:

$$(u + v) + w = u + (v + w) \text{ and } u + v = v + u \text{ for } \forall u, v, w \in V,$$

has **zero** $0 \in V$ such that $0 + v = v$ for $\forall v \in V$,
each element $v \in V$ has **additive inverse** $-v$ such that $v + (-v) = 0$,
the multiplication is **associative**: $\lambda(\mu v) = (\lambda\mu)v$ for $\forall \lambda, \mu \in \mathbb{F}$ and $\forall v \in V$,
1 is the **multiplicative identity**: $1v = v$ for $\forall v \in V$,
two **distributivity properties** hold true:

$$\alpha(v + w) = \alpha v + \alpha w \text{ for } \forall \alpha \in \mathbb{F} \text{ and } \forall v, w \in V \text{ and}$$
$$(\alpha + \beta)v = \alpha v + \beta v \text{ for } \forall \alpha, \beta \in \mathbb{F} \text{ and } \forall v \in V.$$

Coordinate spaces

Despite of its long cumbersome definition, a vector space per se is simple and boring,
like an empty stage. It can be identified with some of $\mathbb{F}^n$'s, which are straightforward generalizations of $\mathbb{R}^n$, obtained by replacing $\mathbb{R}$ with an arbitrary ground field $\mathbb{F}$.

The set $\mathbb{F}^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{F}\}$ is called the **coordinate** n**-space** over $\mathbb{F}$,
its element $(x_1, x_2, \dots, x_n)$ is called a **vector** or n**-vector**
and x_j is called the j th **coordinate** of that vector.

The addition of $\mathbb{F}$ gives rise to **addition in** $\mathbb{F}^n$:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

The scalar multiplication $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is generalized to a **scalar multiplication**
$\mathbb{F} \times \mathbb{F}^n \to \mathbb{F}^n : (\lambda, (x_1, \dots, x_n)) \mapsto (\lambda x_1, \dots, \lambda x_n)$.
All the requirements from the definition of vector space hold true here.
Thus, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.
In a strong sense, this is an archetypical example:
Any finite-dimensional vector space over $\mathbb{F}$ is a copy of (is isomorphic to) some $\mathbb{F}^n$.

In order to learn how vector spaces can be compared, we have to learn **linear maps.**

But first, let us consider **more examples** of vector spaces.

Spaces of functions

Let X, Y be sets. Then Y^X denotes the set of all maps $X \to Y$.

Let $\mathbb{F}$ be a field, S be a set. Consider $V = \mathbb{F}^S$, the set of all maps $S \to \mathbb{F}$.

Addition in V:

for $f, g \in \mathbb{F}^S$, define $f + g$ by $(f+g)(x) = f(x) + g(x)$ for $\forall x \in S$.

Scalar multiplication:

for $f \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$, define λf by $(\lambda f)(x) = \lambda f(x)$ for $\forall x \in S$.

$\mathbb{F}^S$ with these addition and scalar multiplication is a vector space over $\mathbb{F}$.

Exercise. Prove this.

$\mathbb{F}^S$ generalizes $\mathbb{F}^n$:

if $S = \{1, 2, \ldots, n\}$, then $f \in \mathbb{F}^S$ can be identified with $(f(1), f(2), \ldots, f(n)) \in \mathbb{F}^n$.

Another interesting special case of $\mathbb{F}^S$ arises when $S = \mathbb{N}$.
It can be described as $\mathbb{F}^\infty = \{(x_1, x_2, x_3, \ldots) \mid x_j \in \mathbb{F})\}$,
the set of infinite sequences of elements of $\mathbb{F}$.

If S is infinite, then $\mathbb{F}^S$ is an infinite-dimensional vector space over $\mathbb{F}$.
It falls out of the world of Linear Algebra!

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More examples

$\mathbb{C}$ is a vector space over $\mathbb{R}$. We say that $\mathbb{C}$ is a **real vector space**.

$\mathbb{C} = \mathbb{C}^1$ is also a **complex vector space.**, i.e., a vector space over $\mathbb{C}$.

Each vector space over $\mathbb{C}$ can be considered as a vector space over $\mathbb{R}$.

Generalization: If K is a subfield of a field L,
then each vector space over L is a vector space over K.

What is the smallest vector space over $\mathbb{F}$? $0 = \{0\}$.

Is the set of all real 2×3-matrices a vector space?

A **polynomial in a variable** x over a field $\mathbb{F}$ is an expression
$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, where $a_k \in \mathbb{F}$.

Polynomials in a variable X over a field $\mathbb{F}$ form a vector space over $\mathbb{F}$.

Notation $\mathbb{F}[x]$. In Axler's book $\mathcal{P}(\mathbb{F})$.

The vector space $\mathbb{F}[x]$ is infinite-dimensional,
but it contains lots of finite-dimensional subspaces.

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Linear maps

Vector spaces **socialize** to each other via linear maps.

Let V and W be vector spaces over $\mathbb{F}$.
Vaguely speaking, a map $T : V \to W$ is **linear** if it respects
the linear operations (i.e., the addition and scalar multiplication) in V and W.
Here is what this means:

A map $T : V \to W$ is said to be **linear**, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (T is **additive**)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (T is **homogeneous**).

Terminology: linear **maps** or linear **transformations**? **Notation:** Tv or $T(v)$?

Notation $\{\text{all linear maps } V \to W\} = \mathcal{L}(V, W)$ or $\mathrm{Hom}_{\mathbb{F}}(V, W)$ or $\mathrm{Hom}(V, W)$.

Examples of linear maps

$0 \in \mathcal{L}(V, W)$. $v \mapsto 0$ for any $v \in V$.

Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.

Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.

Integration $\mathbb{R}[x] \to \mathbb{R} : p(x) \mapsto \int_0^1 p(x)dx$.

Multiplication by a polynomial $q(x)$ $T : \mathbb{F}[x] \to \mathbb{F}[x] : Tp(x) = q(x)p(x)$.

Backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

Forward shift $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$

A linear map takes 0 to 0

Theorem Let $T : V \to W$ be a linear map. Then $T(0) = 0$.

Proof. $T(0) = T(0+0) = T(0) + T(0)$.

So, $T(0) = T(0) + T(0)$.

Add $-T(0)$ to both sides.

$0 = T(0)$. □

Composition

Definition (should be well known). Let $T : U \to V$ and $S : V \to W$ be maps.
The **composition** $S \circ T$ is a map $U \to W$ defined by formula
$(S \circ T)(u) = S(T(u))$ for all $u \in U$.

Diagramatic presentation:
$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

Composition is also called a **product**. (Say, in Axler's textbook.)

Often $S \circ T$ is denoted by ST, like a product.

Theorem. If S and T are linear maps, then $S \circ T$ is a linear map.

Proof. Exercise! It's easy! □

Algebraic properties of composition.

associativity $(T_1T_2)T_3 = T_1(T_2T_3)$

identity $T\,\mathrm{id}_V = T = \mathrm{id}_W\,T$

Inverse to a linear map is linear

Theorem If V and W are vector spaces and a linear map $T : V \to W$ is invertible, then T^{-1} is linear.

Proof. Additivity. Let $w_1, w_2 \in W$. Then

$$T^{-1}(w_1 + w_2) = T^{-1}(\mathrm{id}_W\, w_1 + \mathrm{id}_W\, w_2) = T^{-1}(TT^{-1}w_1 + TT^{-1}w_2)$$
$$= T^{-1}T(T^{-1}w_1 + T^{-1}w_2) = \mathrm{id}_V(T^{-1}w_1 + T^{-1}w_2) = T^{-1}w_1 + T^{-1}w_2 .$$

Proof. Homogeneity.

$$T^{-1}(\lambda w) = T^{-1}(\lambda\, \mathrm{id}_W\, w) = T^{-1}(\lambda TT^{-1}w) = T^{-1}(\lambda T(T^{-1}w))$$
$$= T^{-1}T(\lambda T^{-1}w) = \mathrm{id}_V(\lambda T^{-1}w) = \lambda T^{-1}w .$$

□

Isomorphisms

Invertible linear map $T : V \to W$ is called **linear isomorphism.**

Vector spaces V, W are said to be **isomorphic** if there exists a linear isomorphism $V \to W$.

Properties of isomorphisms

- An identity linear map is a linear isomorphism.
- The composition of linear isomorphisms is a linear isomorphism.
- The map inverse to a linear isomorphism is a linear isomorphism.

Relation of being isomorphic is equivalence.
It is reflexive, symmetric and transitive.

Linear Algebra do not recognize any difference between isomorphic vector spaces,
although the vector spaces may be not identically the same.

What vector spaces are isomorphic?

Finite-dimensional vector spaces over the same field are isomorphic
iff they have the same **dimension.**

Dimension is a non-negative integer. It solves the classification problem for finite-dimensional vector spaces.

Maps between linear maps

What is a classification of linear maps
similar to isomorphism classification of vector spaces?
First, let us understand, what is a linear map of one linear map to another linear map?

There is a natural answer to this question.

Given linear maps $T : V \to W$ and $S : X \to Y$, a linear map $T \to S$ is a pair $(V \xrightarrow{L} X\ ,\ W \xrightarrow{M} Y)$ of linear maps such that $M \circ T = S \circ L$.

It is presented by a diagram: $\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$ which is **commutative**: $M \circ T = S \circ L$.

Composition: $\left(\begin{tikzcd} A & X \arrow[l, "N"'] \arrow[d, "S"'] \\ B \arrow[from=1-1, "U"] & Y \arrow[l, "R"] \end{tikzcd}\right) \circ \left(\begin{tikzcd} X \arrow[d, "S"] & V \arrow[l, "L"'] \arrow[d, "T"'] \\ Y & W \arrow[l, "M"] \end{tikzcd}\right) = \left(\begin{tikzcd} A \arrow[d, "U"] & V \arrow[l, "N \circ L"'] \arrow[d, "T"'] \\ B & W \arrow[l, "R \circ M"] \end{tikzcd}\right)$

Isomorphisms of linear maps

Let $T : V \to W$ and $S : X \to Y$ be linear maps,

and $\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$ be a linear map $T \to S$.

Recall that this means commutativity of the diagram, i.e., $M \circ T = S \circ L$.
If L and M are linear isomorphisms, we say that they form an isomorphism $T \to S$

This gives rise to an equivalence relation
on the collection of linear maps between vector spaces over the same ground field.

Sometimes this equivalence relation is called **left-right equivalence** or **LR-equivalence**.
This name is motivated by the relation $M \circ T = S \circ L$ re-written as $S = M \circ T \circ L^{-1}$.

The classification of linear maps up to this isomorphism is simple
and will be presented soon in both courses.
The result can be formulated as follows:
Up to isomorphism,
a linear map $S : V \to W$ is characterized by three non-negative integers:
the dimensions of V and W and the **rank** of S.
The rank is also a dimension. Namely, the dimension of subspace $S(V) \subset W$.

Operators

A linear map from a vector space to itself is called a **(linear) operator**.
Notation $\mathcal{L}(V) = \{\text{all linear maps } V \to V\} = \mathcal{L}(V,V)$.

Category of operators in vectors spaces over a field $\mathbb{F}$

objects are operators $T : V \to V$,

a **morphism** $(V \xrightarrow{T} V) \to (W \xrightarrow{S} W)$

is a linear map $V \xrightarrow{L} W$ such that $S \circ L = L \circ T$.

or, rather, a commutative diagram
$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "T"] & W \arrow[d, "S"'] \\ V \arrow[r, "L"] & W \end{tikzcd},$$

a **composition** of morphisms is the composition of the linear maps.

Axler: "The deepest and most important parts of linear algebra ... deal with operators."

The answer depends on the ground field. It will be done over $\mathbb{C}$.

The answer is formulated in terms of Jordan form.

Linear operations in $\mathcal{L}(V,W)$

Definition Let $S, T : V \to W$ be maps and $\lambda \in \mathbb{F}$.

The **sum** $S + T$ and the **product** λT are maps $V \to W$ defined by

$(S+T)(v) = Sv + Tv$ and $(\lambda T)(v) = \lambda(Tv)$ for all $v \in V$.

Theorem. If S, T are linear maps, then $S + T$ and λT are linear maps.

Proof. Exercise! It's easy! □

Theorem With the operations of addition and scalar multiplication, $\mathcal{L}(V,W)$ **is a vector space.**

Proof. Exercise! It's easy! □

Special case: $W = \mathbb{F}$. Then $\mathcal{L}(V,W) = \mathcal{L}(V,\mathbb{F})$ is called the **dual space** and is denoted by V'.

Elements of V' are linear maps $V \to \mathbb{F}$. They are called **linear functionals** or **covectors**.