MAT 315, Advanced Linear Algebra Name ______________________________
Homework 5
due by March 12 Score____________

Problem 1. Let X be a set and $\mathbb{F}$ a field. Denote by $\mathbb{F}^X$ the set of all maps $X \to \mathbb{F}$. As it was shown in the first lectures, $\mathbb{F}^X$ has a natural structure of vector space over $\mathbb{F}$.

(a) For which X the dimension of this vector space is finite?
(b) Find a natural basis of $\mathbb{F}^X$.
(c) For any finite-dimensional vector space V over $\mathbb{F}$, find X such that V is isomorphic to $\mathbb{F}^X$.

Problem 2. For finite sets X and Y find a natural isomorphism between $\mathbb{F}^{X\times Y}$ and $\mathcal{L}(V^\checkmark, W^\checkmark; \mathbb{F})$, where $V = \mathbb{F}^X$ and $W = \mathbb{F}^Y$.

Problem 3. Let U, V, W be finite-dimensional vector spaces over a field $\mathbb{F}$. Find a bilinear map $\mathcal{M} : V \times W \to \mathcal{L}(V^\checkmark, W^\checkmark; \mathbb{F})$ such that for any bilinear map $F : V \times W \to U$ there exists a unique linear map $G : \mathcal{L}(V^\checkmark, W^\checkmark; \mathbb{F})$ such that $G \circ \mathcal{M} = F$. The latter condition means that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[rr, "F"] \arrow[dr, "\mathcal{M}"'] & & U \\ & \mathcal{L}(V^\checkmark, W^\checkmark; \mathbb{F}) \arrow[ur, "G"'] & \end{tikzcd}$$

Remark. Problem 1 is for warming up. Its purpose to remind definitions.

The vector space $\mathcal{L}(V^\checkmark, W^\checkmark; \mathbb{F})$, which appears in Problem 3, is the tensor product $V \otimes W$ of V and W. It was not defined in the lectures. Problem 3 is about is a characteristic property of $V \otimes W$, the property which defines $V \otimes W$.

The statement of Problem 2 means that $\mathbb{F}^{X\times Y}$ is isomorphic to $\mathbb{F}^X \otimes \mathbb{F}^Y$.