

**Homework 5**

due by March 12

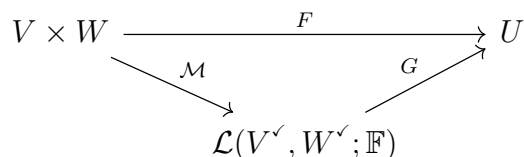
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**Problem 1.** Let  $X$  be a set and  $\mathbb{F}$  a field. Denote by  $\mathbb{F}^X$  the set of all maps  $X \rightarrow \mathbb{F}$ . As it was shown in the first lectures,  $\mathbb{F}^X$  has a natural structure of vector space over  $\mathbb{F}$ .

- (a) For which  $X$  the dimension of this vector space is finite?
- (b) Find a natural basis of  $\mathbb{F}^X$ .
- (c) For any finite-dimensional vector space  $V$  over  $\mathbb{F}$ , find  $X$  such that  $V$  is isomorphic to  $\mathbb{F}^X$ .

**Problem 2.** For finite sets  $X$  and  $Y$  find a natural isomorphism between  $\mathbb{F}^{X \times Y}$  and  $\mathcal{L}(V', W'; \mathbb{F})$ , where  $V = \mathbb{F}^X$  and  $W = \mathbb{F}^Y$ .

**Problem 3.** Let  $U, V, W$  be finite-dimensional vector spaces over a field  $\mathbb{F}$ . Find a bilinear map  $\mathcal{M} : V \times W \rightarrow \mathcal{L}(V', W'; \mathbb{F})$  such that for any bilinear map  $F : V \times W \rightarrow U$  there exists a unique linear map  $G : \mathcal{L}(V', W'; \mathbb{F}) \rightarrow U$  such that  $G \circ \mathcal{M} = F$ . The latter condition means that the following diagram commutes.



**Remark.** Problem 1 is for warming up. Its purpose to remind definitions.

The vector space  $\mathcal{L}(V', W'; \mathbb{F})$ , which appears in Problem 3, is the tensor product  $V \otimes W$  of  $V$  and  $W$ . It was not defined in the lectures. Problem 3 is about is a characteristic property of  $V \otimes W$ , the property which defines  $V \otimes W$ .

The statement of Problem 2 means that  $\mathbb{F}^{X \times Y}$  is isomorphic to  $\mathbb{F}^X \otimes \mathbb{F}^Y$ .