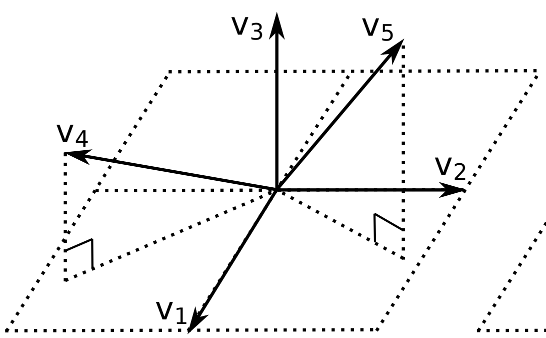
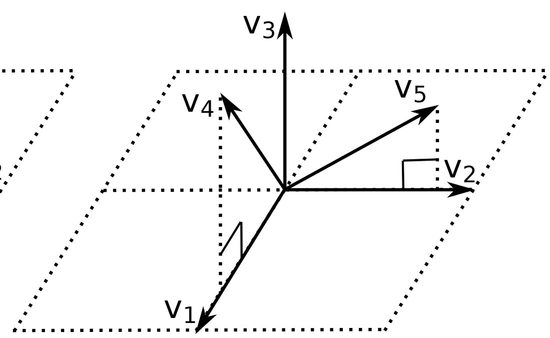
MAT310 SPRING 2020 - HOMEWORK 2

DUE: FEBRUARY 20TH

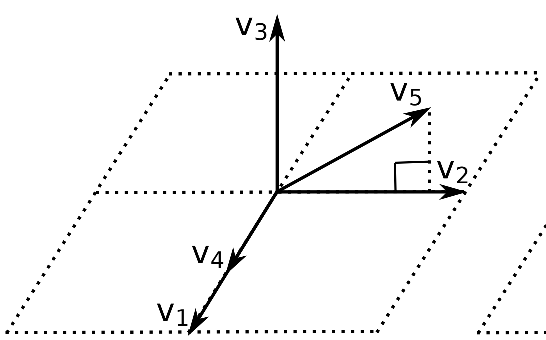
Problem 1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_5$ be vectors in $\mathbb{R}^3$, as shown in the four figures below. In each figure, find *all* linearly dependent sets consisting of three of these five vectors, or else state that there are none if this is the case. *No justification needed.* (Note that in each of these figures, $\mathbf{v}_1$ and $\mathbf{v}_2$ span the displayed plane, $\mathbf{v}_3$ points "up" and is perpendicular to this plane, and for any other vector *not* in the plane, we draw a dotted vertical line indicating its position above the plane.)



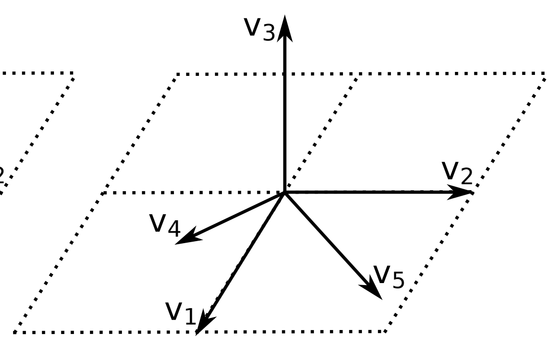
(a)



(b)



(c)



(d)

Problem 2. Let

$$U = \left\{ p \in \mathcal{P}_4(\mathbb{R}) \mid \int_{-1}^{1} p(x)dx = 0 \right\}.$$

(a) Find a basis for U. Justify that your answer is a spanning set and is linearly independent.

(b) What is the dimension of U?
(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Problem 3. A matrix $A \in \mathbb{R}^{n,n}$ is said to be symmetric if $A^T = A$. Let $\mathrm{Sym}_n \subset \mathbb{R}^{n,n}$ denote the set of all symmetric $n \times n$ real matrices.

(a) Find a basis for Sym_2. Justify that your answer is a spanning set and is linearly independent.
(b) What is the dimension of Sym_2?
(c) Find a subspace W of $\mathbb{R}^{2,2}$ such that $\mathbb{R}^{2,2} = \mathrm{Sym}_2 \oplus W$.

Problem 4. Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ be finite subsets of a vector space V. Prove that if $n > m$, and $X \subset \mathrm{span}(Y)$, then X is linearly dependent. [Hint: For $1 \leq i \leq n$, write

$$\mathbf{x}_i = \sum_{j=1}^{m} a_{ij}\mathbf{y}_j,$$

and consider the $m \times n$ matrix $A := [a_{ij}]$. What can you say about the rank of A?]

Problem 5.

(a) Let U, W be subspaces of a vector space V. Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

[Hint: Pick a basis $\mathfrak{B}_0 := \{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ for $U \cap W$. Complete $\mathfrak{B}_0$ into a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $\{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ for U and W respectively. Show that $\mathfrak{B} := \{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is a basis for $U + W$.]
(b) Prove that $\dim(U \cap W) \geq \dim(U) + \dim(W) - \dim(V)$.
(c) Is it possible for two planes in $\mathbb{R}^3$ intersect at a single point, say at the origin? What about in $\mathbb{R}^4$? In each case, either provide an example, or justify that it is not possible.

Problem 6. Let $U_1, \ldots, U_k$ be subspaces of a vector space V. Prove that

$$U_1 + \ldots + U_k = U_1 \oplus \ldots \oplus U_k$$

if and only if for any collection of linearly independent sets $X_i \subset U_i$ with $1 \leq i \leq k$, the union $X_1 \cup \ldots \cup X_k$ is also linearly independent.