Advanced Linear Algebra MAT 315

Oleg Viro

03/03/2020, Lecture 5

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In the proof of Lemma, we used the assumption that V is finite-dimensional. Lemma holds true without this assumption. For infinite-dimensional space it requires tools like transfinite induction. But Theorem holds true only for a finite-dimensional V, anyway.

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