Advanced Linear Algebra MAT 315

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Affine subsets

3.79-81 Definition affine subset
Let V be a vector space, $v \in V$ and U be a subspace of V. Then
$v + U = \{v + u \mid u \in U\}$ is called an **affine subset** of V **parallel** to U.

3.85 Two affine subsets parallel to U are equal or disjoint
$(v + U) \cap (w + U) \neq \varnothing \implies v + U = w + U$.

Proof
$(v + U) \cap (w + U) \neq \varnothing$ means $\exists u_1, u_2 \in U$ such that $v + u_1 = w + u_2$.
Hence $v - w = u_2 - u_1 \in U$. Then $v + u = w + ((v - w) + u) \in w + U$ for $\forall u \in U$.
Hence $v + U \subset w + U$. Similarly, $w + U \subset v + U$, hence $v + U = w + U$. ∎

A **partition** of a set X is a collection Γ of its subsets which
are **disoint** i.e., $A \cap B = \varnothing$ for any $A, B \in \Gamma$
and **cover** X i.e., $X = \cup_{A \in \Gamma} A$.

3.85 means that affine sets parallel to U form a partition of V.

The set of elements of a partition of X is called the **quotient set** of X by the partition.

Quotient space

3.83 Definition quotient space V/U Let U be a subspace of a vector space V.
The set of affine subsets of V parallel to U is called the **quotient space** V/U.
In formula: $V/U = \{v + U \mid v \in V\}$.

3.86 Definition addition and scalar multiplication on V/U
$(v + U) + (w + U) = (v + w) + U$, $\lambda(v + U) = (\lambda v) + U$ for $v, w \in V$, $\lambda \in \mathbb{F}$.

Is this definition good? $v + U = \hat{v} + U \implies (v + w) + U = (\hat{v} + w) + U$?
Yes, $v + U = \hat{v} + U \implies v - \hat{v} \in U \implies (v + w) - (\hat{v} + w) = v - \hat{v} \in U$
$\implies (v + w) + U = (\hat{v} + w) + U$ ∎.

Similarly with scalar multiplication.

3.87 Quotient space is a vector space
V/U with these operations is a vector space.

Verify!