Advanced Linear Algebra MAT 315

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02/27/2020, Lecture 4

Quotients of Vector Spaces

Affine subsets

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The set of elements of a partition of X is called the **quotient set** of X by the partition.

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Verify!