Advanced Linear Algebra MAT 315

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Isomorphisms of linear maps

Recall that there is a **category of linear maps**, in which
an **object** is a linear map $T : V \to W$,

and a **morphism** $\begin{array}{c} V \\ \downarrow{\scriptstyle T} \\ W \end{array} \to \begin{array}{c} X \\ \downarrow{\scriptstyle S} \\ Y \end{array}$ is a commutative diagram $\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$

composition: $\left(\begin{tikzcd} A & X \arrow[l, "N"] \arrow[d, "S"'] \\ B \arrow[u, leftarrow, "U"] & Y \arrow[l, "R"'] \end{tikzcd}\right) \circ \left(\begin{tikzcd} X \arrow[d, "S"] & V \arrow[l, "L"] \arrow[d, "T"'] \\ Y & W \arrow[l, "M"'] \end{tikzcd}\right) = \left(\begin{tikzcd} A \arrow[d, "U"] & V \arrow[l, "N\circ L"] \arrow[d, "T"'] \\ B & W \arrow[l, "R\circ M"'] \end{tikzcd}\right)$

identity: $\begin{tikzcd} V \arrow[r, "\mathrm{id}_V"'] \arrow[d, "T"] & V \arrow[d, "T"'] \\ W \arrow[r, "\mathrm{id}_W"] & W \end{tikzcd}$, **inverse**: $\left(\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}\right)^{-1} = \begin{tikzcd} X \arrow[d, "S"] & V \arrow[l, "L^{-1}"'] \arrow[d, "T"'] \\ Y & W \arrow[l, "M^{-1}"] \end{tikzcd}$

In this category, an **isomorphism** is a commutative diagram $\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$
in which L and M are isomorphisms of vector spaces.

Right-left equivalence

In other words, an isomorphism $\begin{array}{c} V \\ \downarrow{\scriptstyle T} \\ W \end{array} \to \begin{array}{c} X \\ \downarrow{\scriptstyle S} \\ Y \end{array} = \begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$

is a pair of isomorphisms $V \xrightarrow{L} X$ and $W \xrightarrow{M} Y$ such that $S = MTL^{-1}$.

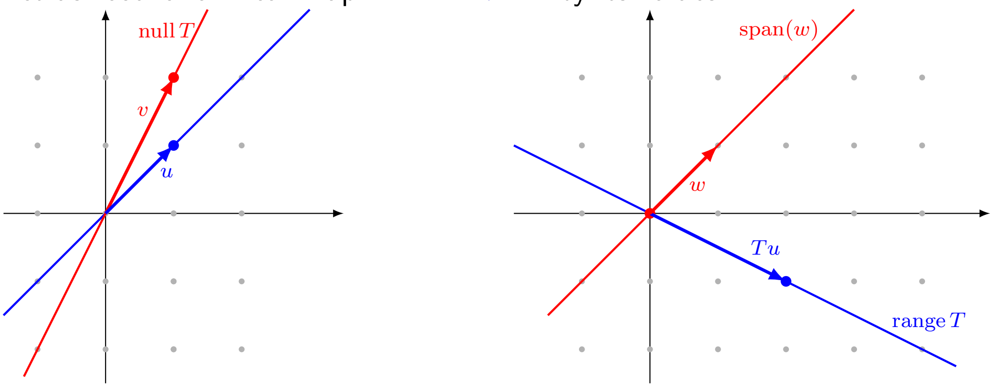
S is isomorphic to T $\iff$
S can be obtained from T by multiplication from right and left by linear isomorphisms.

Another name: **right-left equivalence** or **R-L-equivalence**.

Riddle What is **left-equivalence**?
right-eqivalence?

Example of linear map

Let us recover a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by its values.



Take $(1,1)$. It's image is $T(1,1) = (2,-1)$. What have we learned about T?
$T(\operatorname{span}(1,1)) = \operatorname{span}(2,-1)$.

Take $(1,2)$. It's image is $T(1,2) = (0,0)$. $T(\operatorname{span}(1,2)) = \{(0,0)\}$. $\operatorname{span}(1,2) \subset \operatorname{null} T$.
In fact $\operatorname{span}(1,2) = \operatorname{null} T$. Why?

Choose a basis v of $\operatorname{null} T$. Extend it to a basis (v,u) of $\mathbb{R}^2$. Tu is basis of $\operatorname{range} T$.
Extend Tu to a basis (Tu, w) of $\mathbb{R}^2$.

$$\mathbb{R}^2 = \operatorname{span}(v) \oplus \operatorname{span}(u) \xrightarrow{0 \oplus T} \operatorname{span}(w) \oplus \operatorname{span}(Tu) = \mathbb{R}^2 .$$

Isomorphism classification of linear maps

Theorem Let V and W be vector spaces over $\mathbb{F}$ of finite dimensions $p = \dim V$ and $q = \dim W$. Let $V \xrightarrow{T} W$ be a linear map with $\operatorname{rk} T = r$ and $\dim T = n$.

Then T is R-L-equivalent to $0 \oplus \mathrm{id} : \mathbb{F}^n \oplus \mathbb{F}^r \to \mathbb{F}^{q-r} \oplus \mathbb{F}^r$.

Proof. Let $v = (v_1, \dots, v_n)$ be a basis of $\operatorname{null} T$. Extend it to a basis
$v_1, \dots, v_n, u_1, \dots, u_{p-n}$ of V. Denote $\operatorname{span}(u_1, \dots, u_{p-n})$ by U.
Clearly, $V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{span}(u_1, \dots, u_{p-n}) = \operatorname{null} T \oplus U$.
The restriction $T|_U : U \to W$ is injective, because $U \cap \operatorname{null} T = 0$, and
$\phi = T_{(v_1,\dots,v_n)} \oplus T_{(u_1,\dots,u_{p-n})} : \mathbb{F}^n \oplus \mathbb{F}^{p-n} \to \operatorname{null} T \oplus U$ is an isomorphism.
$(Tu_1, \dots, Tu_{p-n})$ is a basis of $\operatorname{range} T$. Hence, $p - n = r$.
Extend $Tu_1, \dots, Tu_r$ to a basis $w_1, \dots, w_{q-r}, Tu_1, \dots, Tu_r$ of W.
Denote $\operatorname{span}(w_1, \dots, w_{q-r})$ by C.
$\psi = T_{w_1,\dots,w_{q-r}} \oplus T_{Tu_1,\dots,Tu_r} : \mathbb{F}^{q-r} \oplus \mathbb{F}^r \to C \oplus \operatorname{range} T$ is an isomorphism.
Isomorphisms ϕ and ψ form an isomorphism $(0 \oplus \mathrm{id}) \to T$:

$$\begin{tikzcd}
\mathbb{F}^n \oplus \mathbb{F}^r \arrow[r, "\phi"] \arrow[d, "0\oplus\mathrm{id}"] & \operatorname{null} T \oplus U \arrow[r, "="] \arrow[d, "0\oplus T"] & V \arrow[d, "T"] \\
\mathbb{F}^{q-r} \oplus \mathbb{F}^r \arrow[r, "\psi"] & C \oplus \operatorname{range} T \arrow[r, "="] & W
\end{tikzcd} \quad \blacksquare$$

Numerical invariants of a linear map

3.22 Corollary. Fundamental Theorem of Linear Maps.
Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V,W)$.
Then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \operatorname{rk} T$.

Proof. By Theorem applied to $T| : V \to \operatorname{range} T$
there exists an isomorphism $\mathbb{F}^{\dim \operatorname{null} T} \oplus \mathbb{F}^{\dim \operatorname{range} T} \to V$. ∎

$\operatorname{rk} T \leq \dim W$ for any linear map $T : V \to W$.

Proof. $\operatorname{range} T$ is a subspace of W. ∎

A linear map $T : V \to W$ with $\dim V = p$, $\dim W = q$ and $\operatorname{rk} T = r$ exists
$\iff$ $r \leq p$ and $r \leq q$.

Linear maps $T : V \to W$ and $T' : V' \to W'$ are isomorphic
$\iff$ $\dim V = \dim V'$, $\dim W = \dim W'$ and $\operatorname{rk} T = \operatorname{rk} T'$.

Two more corollaries

3.23 Theorem. A map to a smaller dimensional space is not injective
If $\dim V > \dim W$ then any linear map $V \to W$ is not injective.

Proof. Let $T \in \mathcal{L}(V,W)$. Then $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T$
$\geq \dim V - \dim W > 0.$ Hence $\operatorname{null} T > 0$. ∎

3.24 Theorem. A map to a larger dimensional space is not surjective
If $\dim V < \dim W$, then any linear map $V \to W$ is not surjective.

Proof. Let $T \in \mathcal{L}(V,W)$. Then $\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T$
$\leq \dim V < \dim W.$ Hence $\operatorname{range} T \neq W$. ∎

Bijectivity of an operator

Recall: 3.67 A linear map from a vector space to itself is called an **operator**.

3.69 For an operator in a finite dimensional vector space injectivity is equivalent to surjectivity

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Then
T is bijective $\iff$ T is injective $\iff$ T is surjective .

Proof. bijective $\implies$ injective. By definition.

injective $\implies$ surjective. T is injective $\implies$ $\operatorname{rk} T = \dim V - \dim \operatorname{null} T = \dim V$ ∎

surjective $\implies$ injective. T is surjective $\implies$ $\dim \operatorname{null} T = \dim V - \operatorname{rk} T = 0$ ∎

injective or surjective $\implies$ surjective and injective $\implies$ bijective ∎

In infinite-dimensional space

surjectivity $\not\Rightarrow$ injectivity

Example: $\mathbb{F}^\infty \to \mathbb{F}^\infty : (x_1, x_2, \dots, x_n \dots) \mapsto (x_2, \dots, x_n \dots)$

injectivity $\not\Rightarrow$ surjectivity

Example: $\mathbb{F}^\infty \to \mathbb{F}^\infty : (x_1, x_2, \dots, x_n \dots) \mapsto (0, x_1, x_2, \dots, x_n \dots)$

Back to matrices

Encoding a linear map by its values on a basis

3.5 Linear maps and basis of domain

Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_n \in W$. Then
$\exists$ a unique linear map $T : V \to W$ such that $Tv_j = w_j$ for $j = 1, \dots, n$.

Proof. Existense.
Consider linear maps $T_v : \mathbb{F}^n \to V$ and $T_w : \mathbb{F}^n \to W$,
where $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$.
T_v is invertible, because v is a basis of V.

The map $V \xrightarrow{T_v^{-1}} \mathbb{F}^n \xrightarrow{T_w} W$ maps $v_j \mapsto e_j \mapsto w_j$. ∎

Uniqueness. Let $T : V \to W$ be any linear map with $Tv_j = w_j$ for $j = 1, \dots, n$.
Then $\mathbb{F}^n \xrightarrow{T_v} V \xrightarrow{T} W$ maps $e_j \mapsto v_j \mapsto w_j$. Hence $T_v \circ T = T_w$
and $T = T_v^{-1} \circ T_w$. ∎

Reformulation. Any map $\{v_1, \dots, v_n\} \to W$ of a basis of V to a vector space W extends uniquely to a linear map $V \to W$.

Coordinate systems

We have seen that:

- Any finite-dimensional vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with $n = \dim V$.
- Any basis $u = (u_1, \dots, u_n)$ of V determines an isomorphism
$$T_u : \mathbb{F}^n \to V : (x_1 \dots, x_n) \mapsto x_1u_1 + \dots x_nu_n.$$
- Any isomorphism $T : \mathbb{F}^n \to V$ is T_u, where $u = (Te_1, \dots, Te_n)$.

Definition. An isomorphism $T_u : \mathbb{F}^n \to V$ is called the **coordinate system** in V determined by a basis $u = (u_1, \dots, u_n)$.
For a vector $v \in V$, the coordinates $x_1, \dots, x_n$ of $T_u^{-1}(v)$ are called the **coordinates** of v in the basis u.

The coordinates $x_1, \dots, x_n$ of v in a basis $u_1, \dots, u_n$ are determined by the equality
$$v = x_1u_1 + \dots + x_nu_n.$$

The equality $v = x_1u_1 + \dots + x_nu_n$ is called a **decomposition** of v in the basis $u_1, \dots, u_n$.

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From a linear map $\mathbb{F}^p \to \mathbb{F}^q$ to its matrix

Any linear map $T : \mathbb{F}^p \to \mathbb{F}^q$ is defined by the list $u = (u_1, \dots, u_p) = (Te_1, \dots, Te_p)$ according to the formula $T(x_1, \dots, x_p) = x_1u_1 + \dots + x_pu_p = x_1Te_1 + \dots + x_pTe_p$.

Recall that $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_p = (0, \dots, 0, 1)$.

Let $Te_i = (A_{1,i}, \dots, A_{q,i})$ for each $i = 1, \dots, p$. Then

$$\begin{aligned}T(x_1, \dots, x_p) &= x_1Te_1 + \dots + x_pTe_p \\ &= x_1(A_{1,1}, \dots, A_{q,1}) + x_2(A_{1,2}, ..., A_{q,2}) + \dots + x_p(A_{1,p}, \dots, A_{q,p}) \\ &= (A_{1,1}x_1 + A_{1,2}x_2 + \dots + A_{1,p}x_p, \quad \dots \ , \quad A_{q,1}x_1 + A_{q,2}x_2 + \dots + A_{q,p}x_p).\end{aligned}$$

Let us think of elements of a coordinate space $\mathbb{F}^m$ as columns of m numbers. Then

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} A_{1,1}x_1 + A_{1,2}x_2 + \dots + A_{1,p}x_p \\ A_{2,1}x_1 + A_{2,2}x_2 + \dots + A_{2,p}x_p \\ \vdots \\ A_{q,1}x_1 + A_{q,2}x_2 + \dots + A_{q,p}x_p \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,p} \\ A_{2,1} & A_{2,2} & \dots & A_{2,p} \\ \vdots & \vdots & \dots & \vdots \\ A_{q,1} & A_{q,2} & \dots & A_{q,p} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$

Conclusion: any linear map $\mathbb{F}^p \to \mathbb{F}^q$ is a multiplication by a $q \times p$-matrix.

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Matrices

3.30 Definition. Let q and p denote positive integers.
A $q \times p$ **matrix** A is a rectangular array of elements of $\mathbb{F}$ with q rows and p columns:

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & & \vdots \\ A_{q,1} & \dots & A_{q,p} \end{pmatrix}.$$

$A_{j,k}$ is the entry in row j and column k.

3.32 Definition. Let $T \in \mathcal{L}(V,W)$, $v = (v_1, \dots, v_p)$ a basis in V, $w = (w_1, \dots, w_q)$ a basis in W. The matrix of T with respect to these bases is the q-by-p matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defned by $Tv_k = A_{1,k}w_1 + \dots + A_{q,k}w_q$.

$$(Tv_1, \dots, Tv_p) = (w_1, \dots, w_q) \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & & \vdots \\ A_{q,1} & \dots & A_{q,p} \end{pmatrix}$$

The kth column of $\mathcal{M}(T)$ is formed of the coordinates of the kth basis vector v_k.

The matrix of composition

3.43 The matrix of a composition of linear maps
If $T : U \to V$ and $S : V \to W$ are linear maps, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Proof. Let u, v, w be bases of U, V, W, respectively, and $\mathcal{M}(S) = A$, $\mathcal{M}(T) = B$.

$$\begin{aligned}(ST)u_k = S(B_{1,k}v_1 + B_{2,k}v_2 + \dots + B_{n,k}v_n) &= B_{1,k}Sv_1 + B_{2,k}Sv_2 + \dots + B_{n,k}Sv_n \\ &= B_{1,k}(A_{1,1}w_1 + A_{2,1}w_2 + \dots + A_{m,1}w_m) \\ &+ B_{2,k}(A_{1,2}w_1 + A_{2,2}w_2 + \dots + A_{m,2}w_m) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &+ B_{n,k}(A_{1,n}w_1 + A_{2,n}w_2 + \dots + A_{m,n}w_m) \\ &= \sum_{r=1}^{n} A_{1,r}B_{r,k}w_1 + \sum_{r=1}^{n} A_{2,r}B_{r,k}w_2 + \dots + \sum_{r=1}^{n} A_{m,r}B_{r,k}w_m\end{aligned}$$

$$= (w_1 \dots w_m) \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{pmatrix}$$

Hence $(STu_1 \dots STu_p) = (w_1 \dots w_m) \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,1} & \dots & B_{1,p} \\ \vdots & & \vdots \\ B_{n,1} & \dots & B_{n,p} \end{pmatrix}$

Systems of linear equations vs. linear maps

Any system of q linear equations with p unknowns looks as follows:

$$\begin{cases} A_{1,1}x_1 + \cdots + A_{1,p}x_p = b_1 \\ A_{2,1}x_1 + \cdots + A_{2,p}x_p = b_2 \\ \dots\dots\dots\dots\dots\dots\dots \\ A_{q,1}x_1 + \cdots + A_{q,p}x_p = b_q \end{cases}$$

It can be re-written as a matrix equation $AX = B$,
where

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & \vdots & \vdots \\ A_{q,1} & \dots & A_{q,p} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}.$$

Each solution of $AX = B$ is a vector from $T^{-1}(B)$,
where $T : \mathbb{F}^p \to \mathbb{F}^q$ defined by matrix A, namely $T : X \mapsto AX$.

This allows to convert results about linear maps
into results about systems of linear equations.

Corollaries about systems of linear equations

Recall: 3.23 $\dim V > \dim W \implies$ any linear map $V \to W$ is not injective.

3.26 (Corollary of 3.23) **Homogeneous system of linear equations.**
A homogeneous system of linear equations with more variables than equations
has nonzero solutions.

For a system
$$\begin{cases} \sum_{k=1}^{p} A_{1,k}x_k = 0 \\ \vdots \\ \sum_{k=1}^{p} A_{q,k}x_k = 0 \end{cases}$$

Define $T : \mathbb{F}^p \to \mathbb{F}^q$ by
$T(x_1, \dots, x_p) = \left(\sum_{k=1}^{p} A_{1,k}x_k, \dots, \sum_{k=1}^{p} A_{q,k}x_k\right)$
If $p > q$, then T is not injective by 3.23 and $\operatorname{null} T \neq 0$. ∎

Recall: 3.24 $\dim V < \dim W \implies$ any linear map $V \to W$ is not surjective.

3.29 (Corollary of 2.34) **Inhomogeneous system of linear equations**
An inhomogeneous system of linear equations with more equations than variables
has no solution for some choice of the constant terms.