# Advanced Linear Algebra MAT 315

Oleg Viro

02/25/2020, Lecture 3

# Classification of linear maps, revisited and continued

# Isomorphisms of linear maps

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, **inverse**:
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In this category, an **isomorphism** is a commutative diagram
$$\begin{tikzcd} V \arrow[r, "L"'] \arrow[d, "T"] & X \arrow[d, "S"'] \\ W \arrow[r, "M"] & Y \end{tikzcd}$$
in which $L$ and $M$ are isomorphisms of vector spaces.

# Right-left equivalence

In other words, an isomorphism
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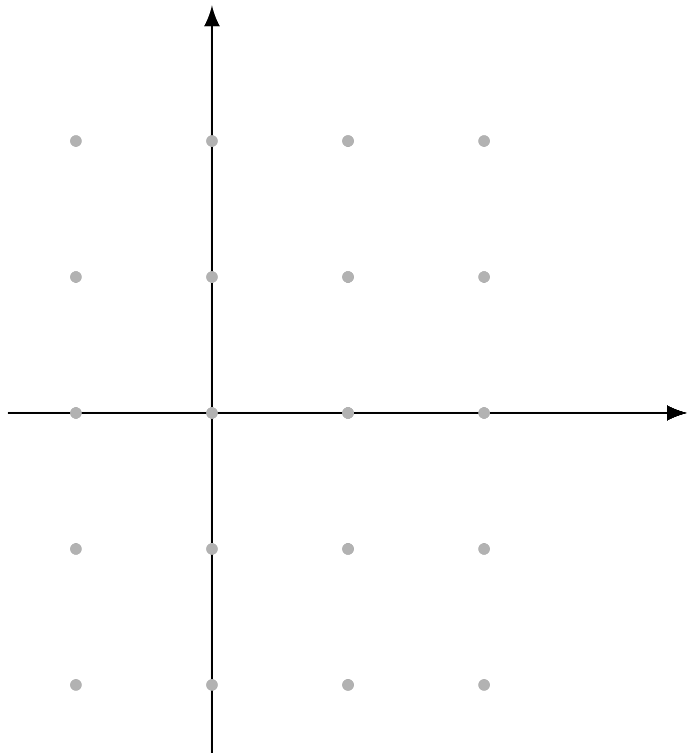

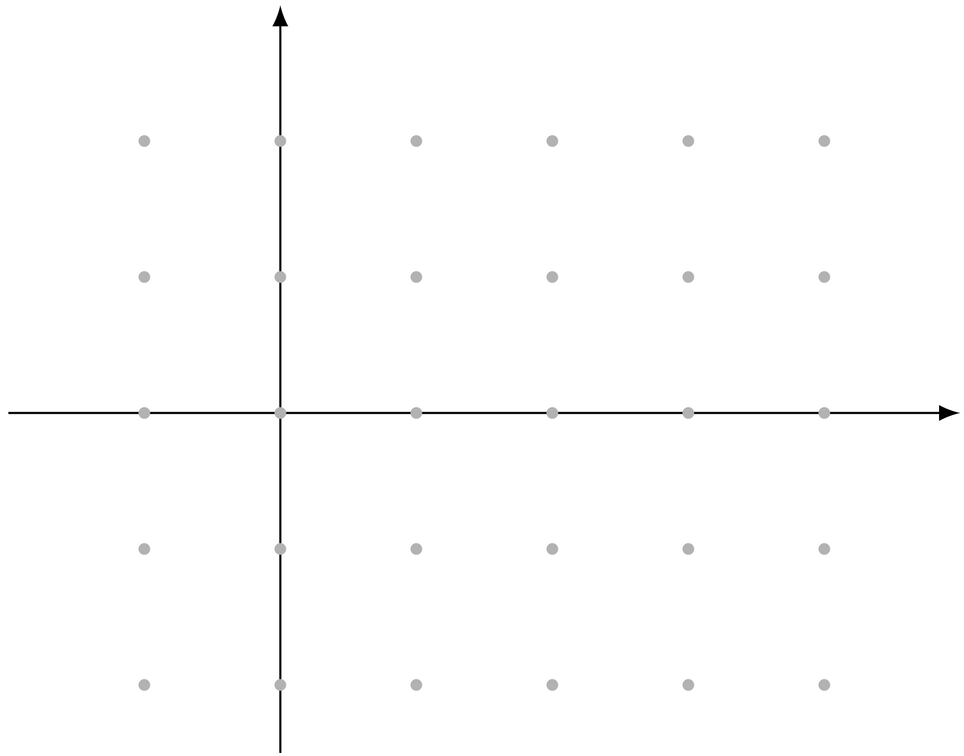

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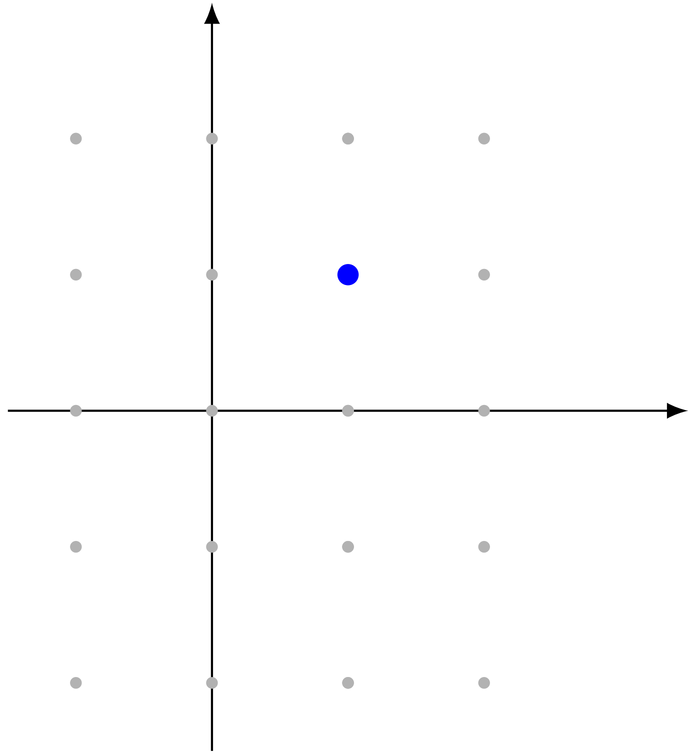

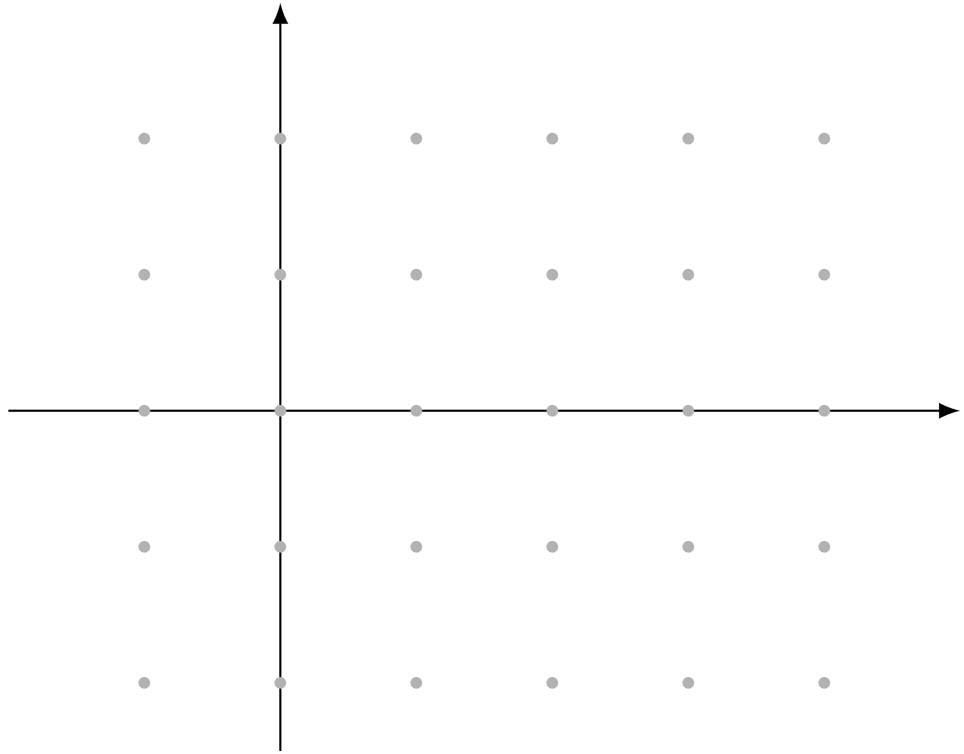

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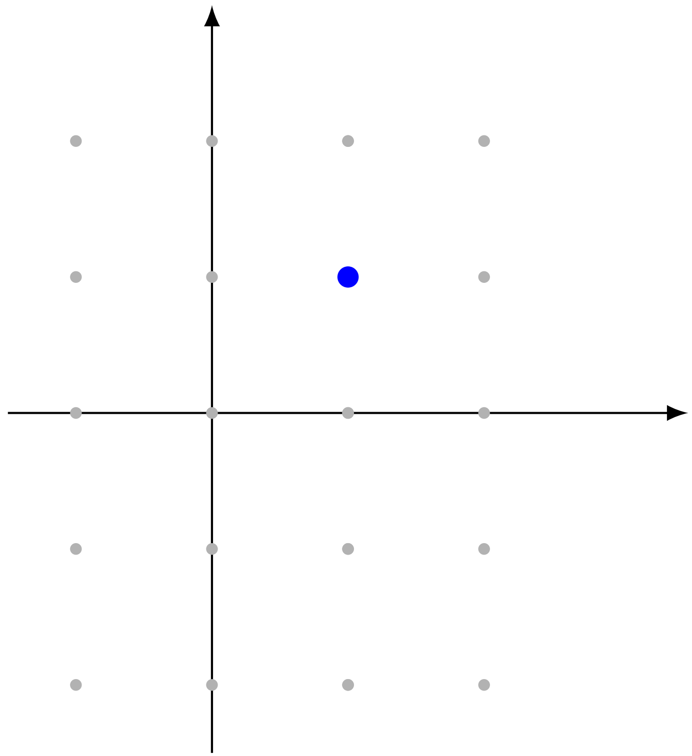

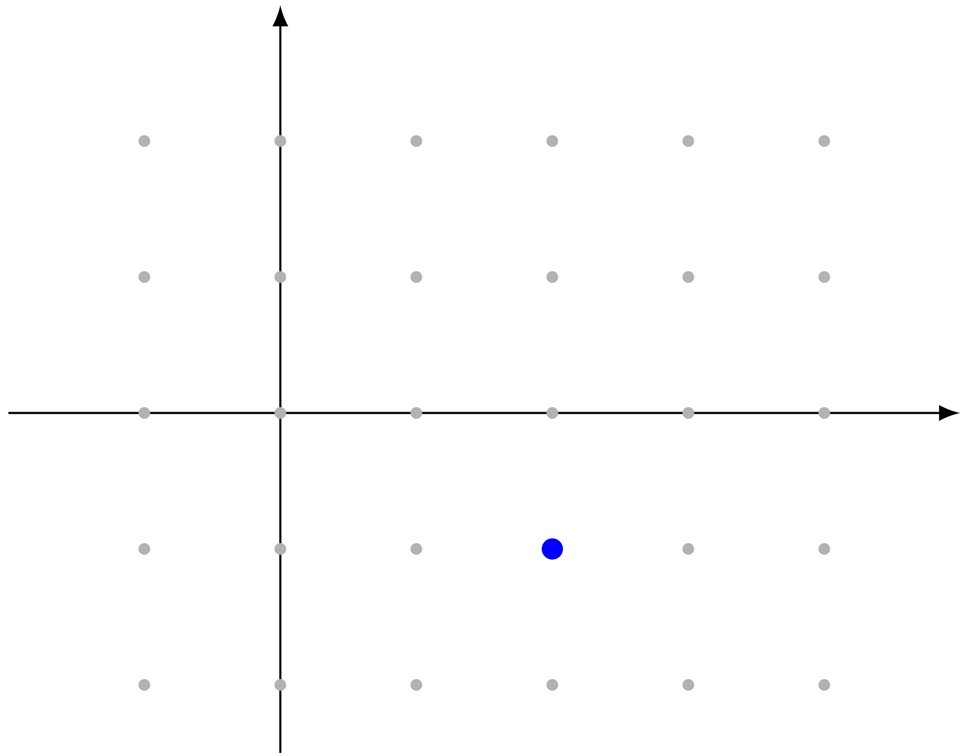

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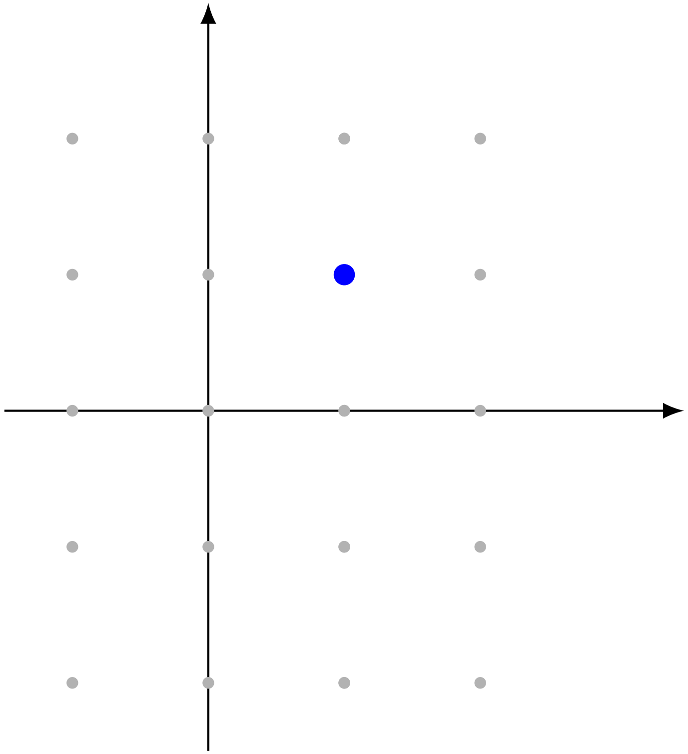

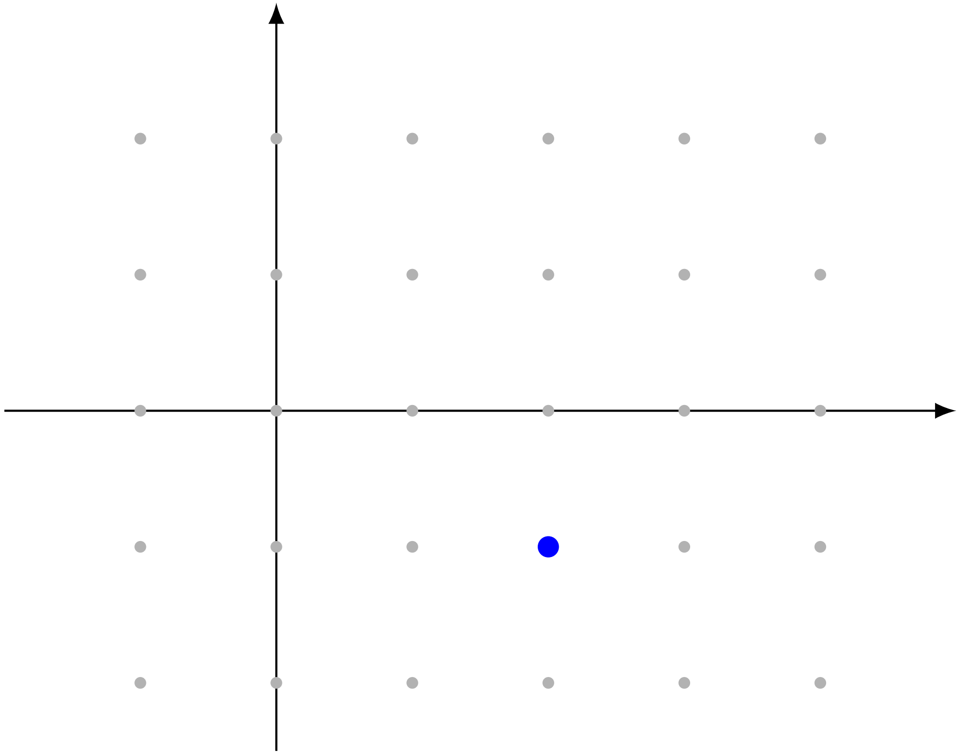

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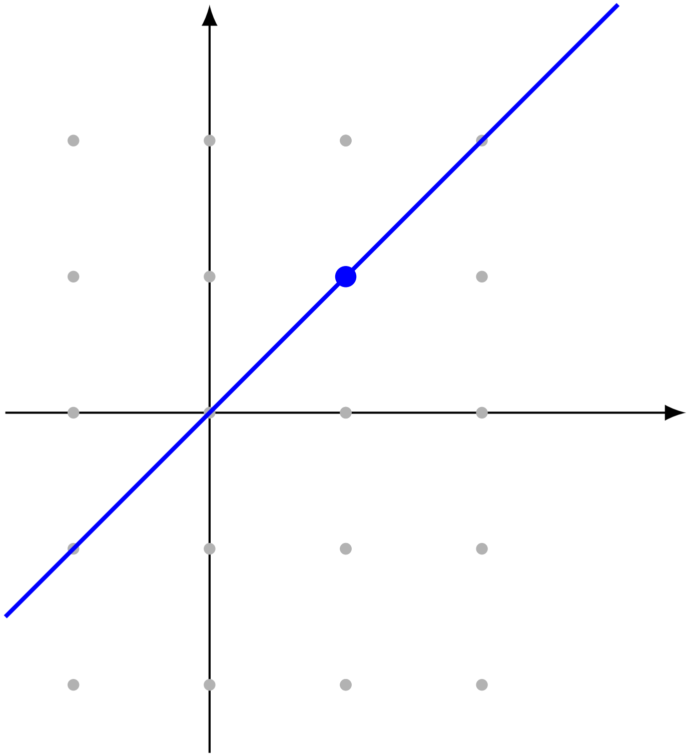

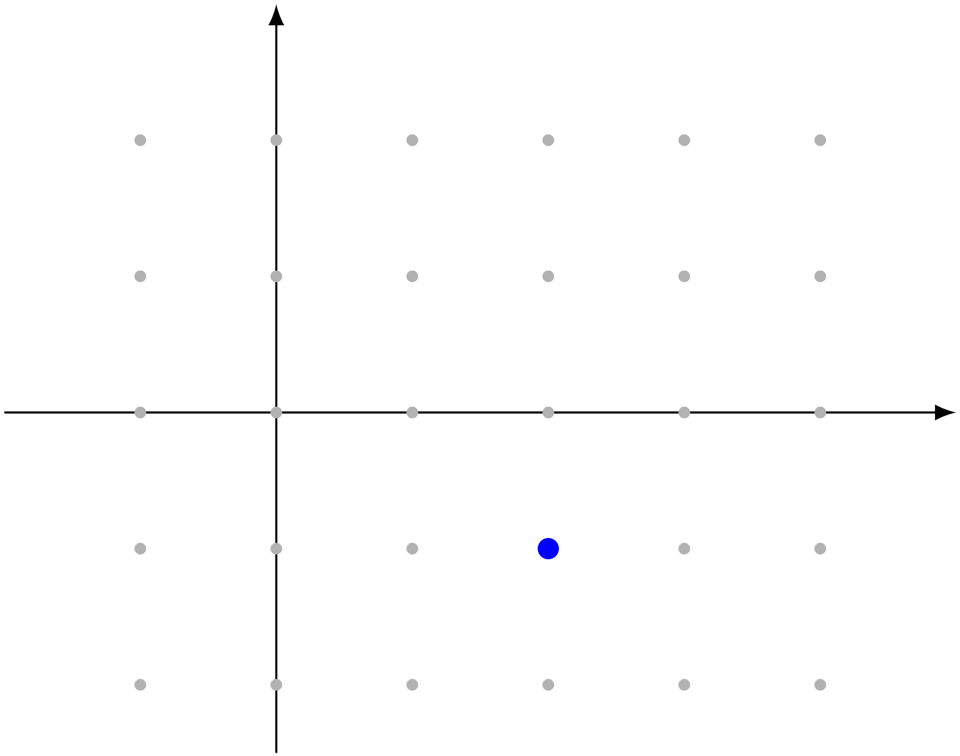

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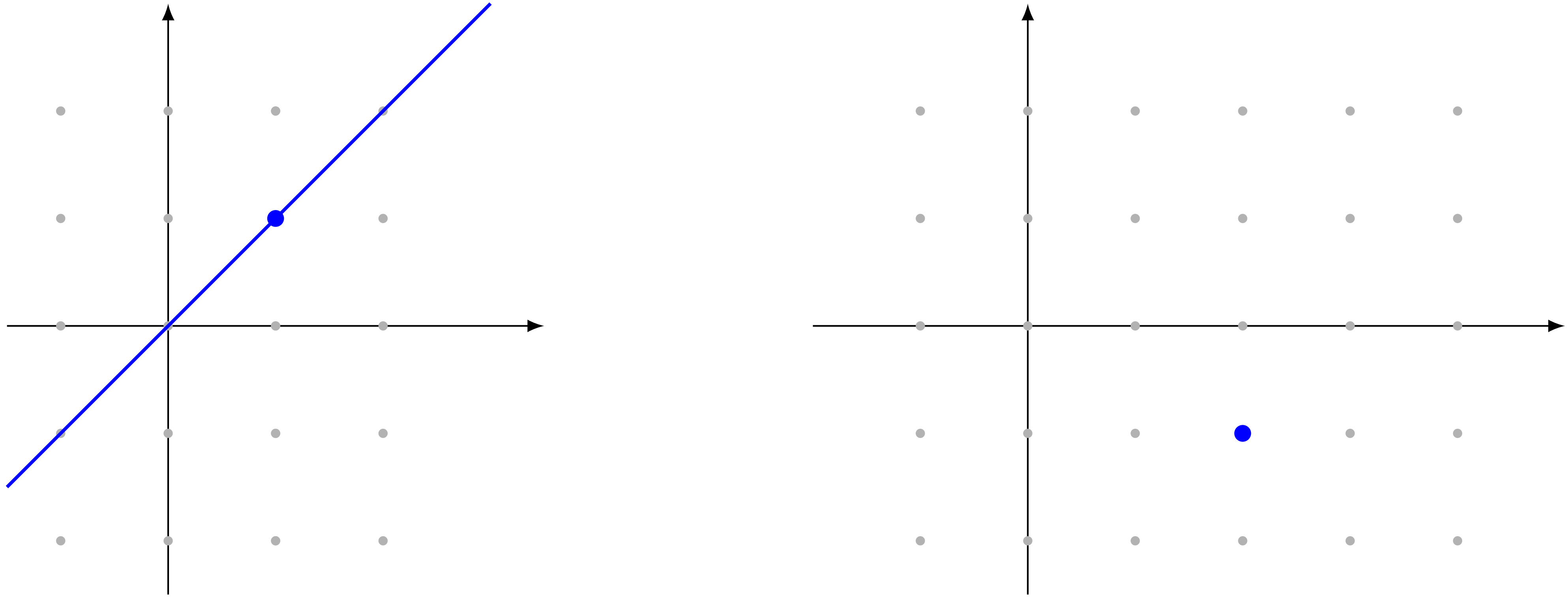

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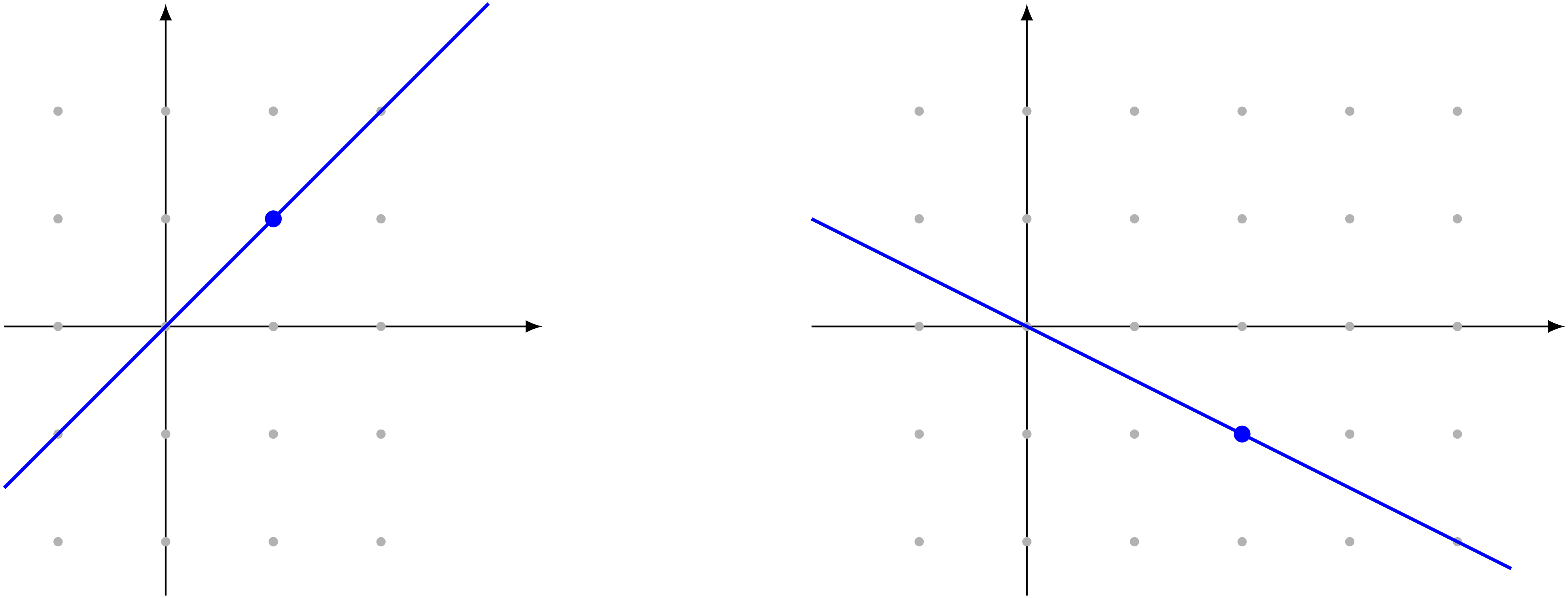

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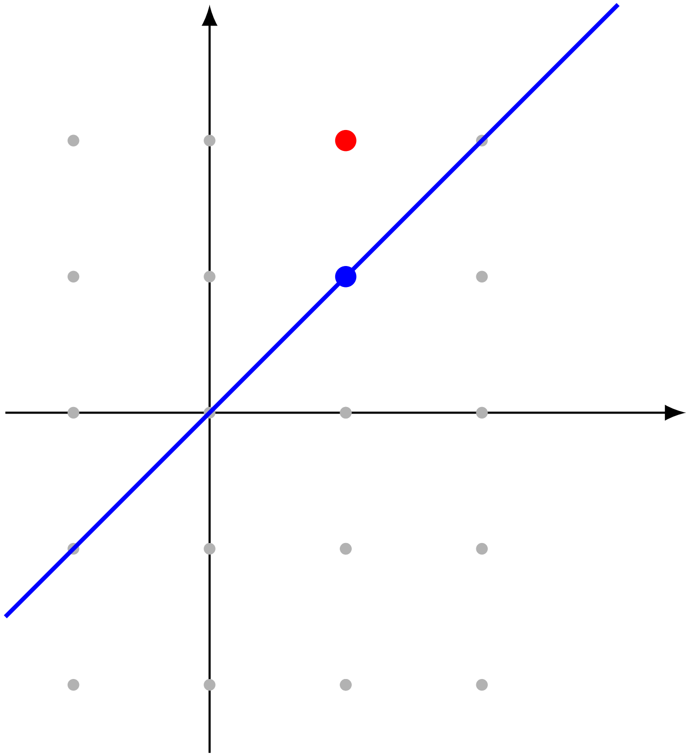

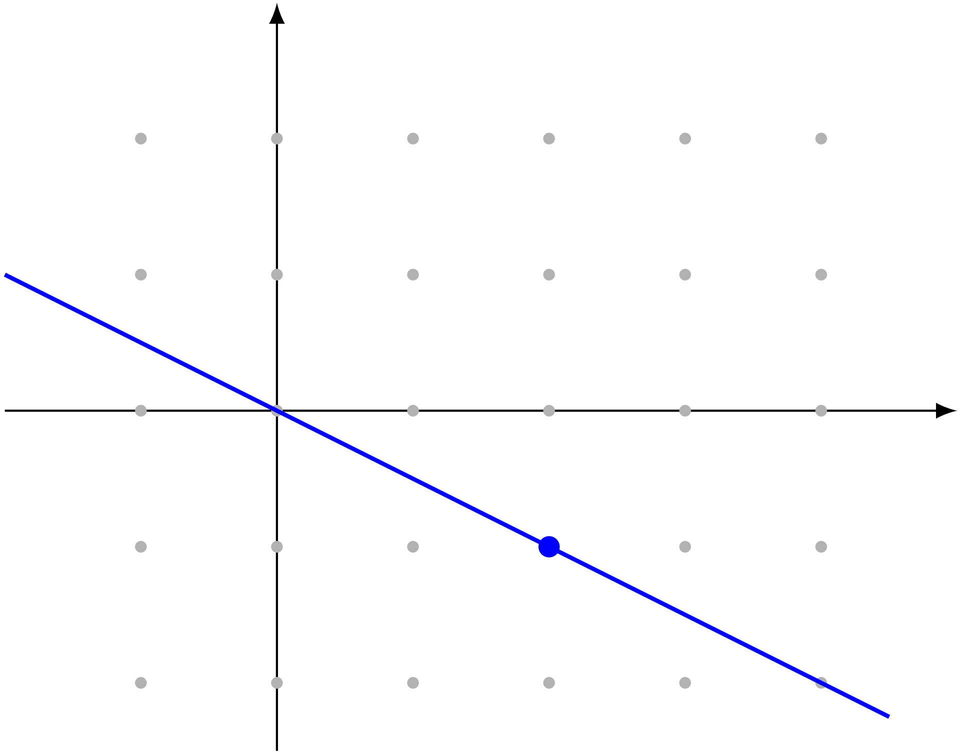

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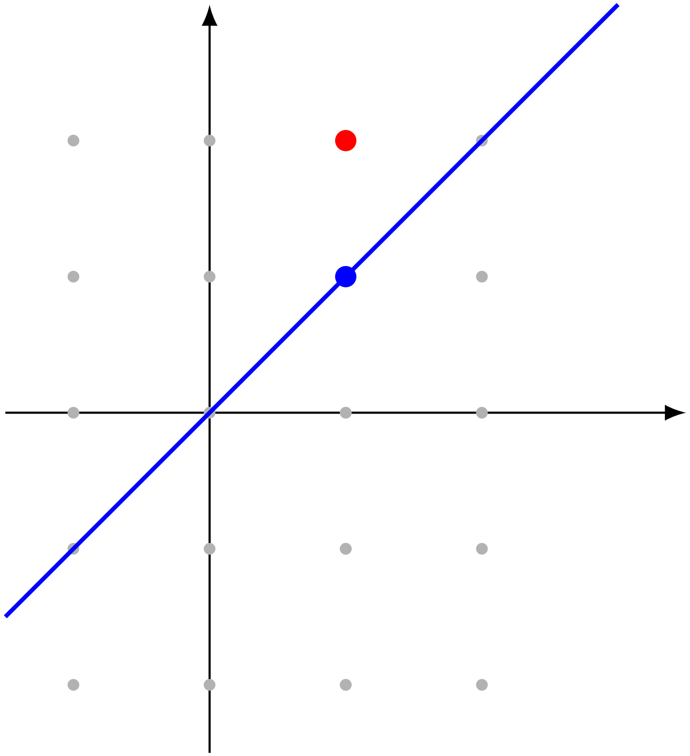

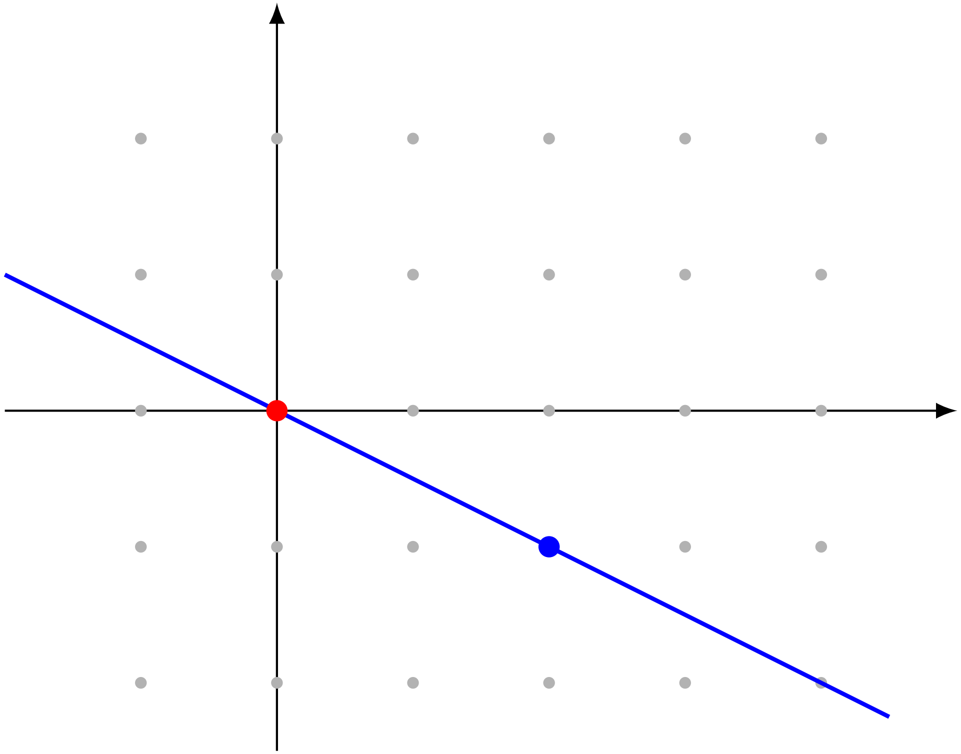

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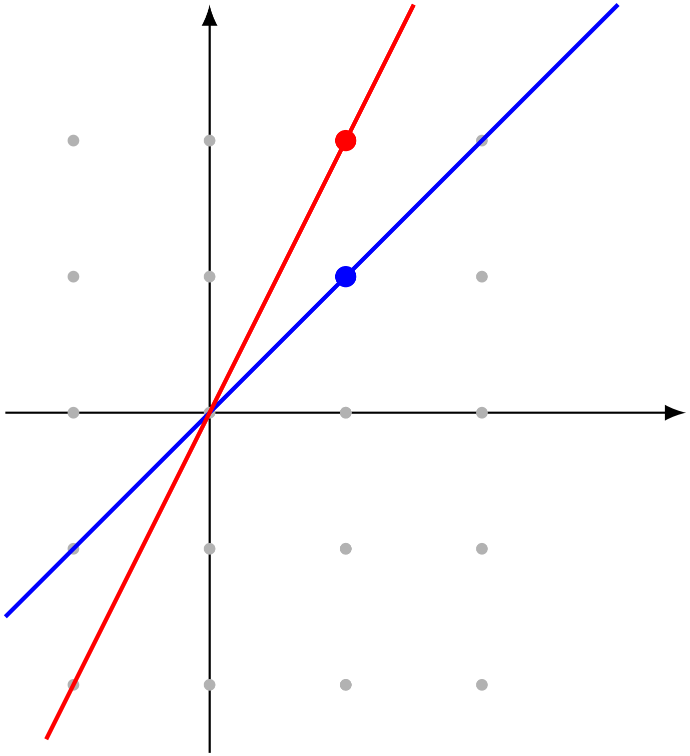

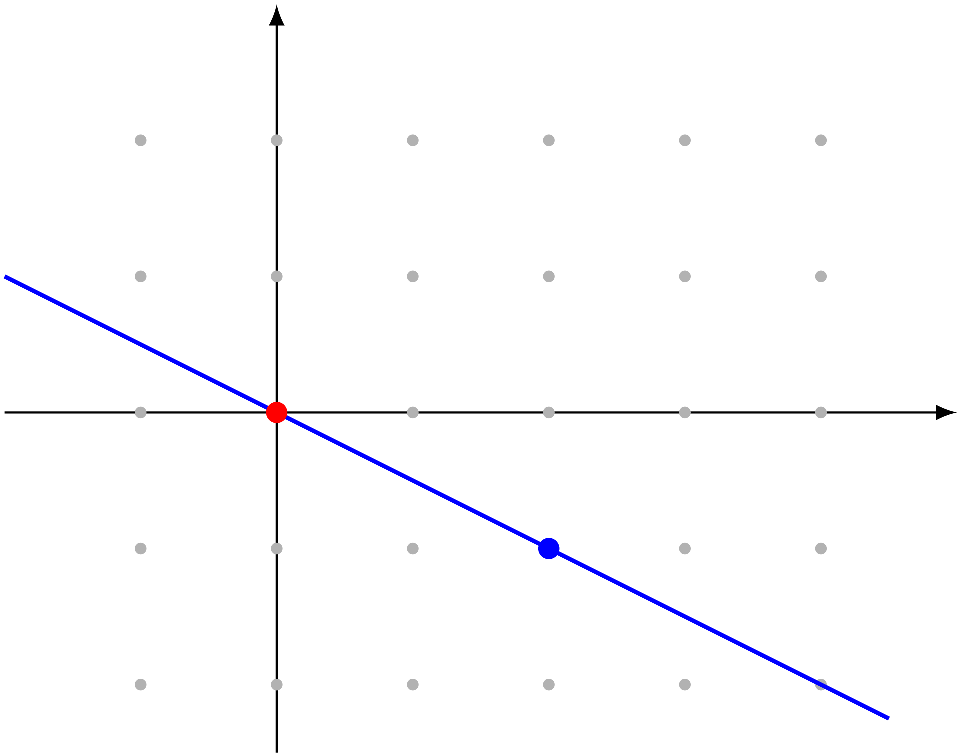

Take $(1,2)$. It's image is $T(1,2)=(0,0)$. $T(\operatorname{span}(1,2))=\{(0,0)\}$.

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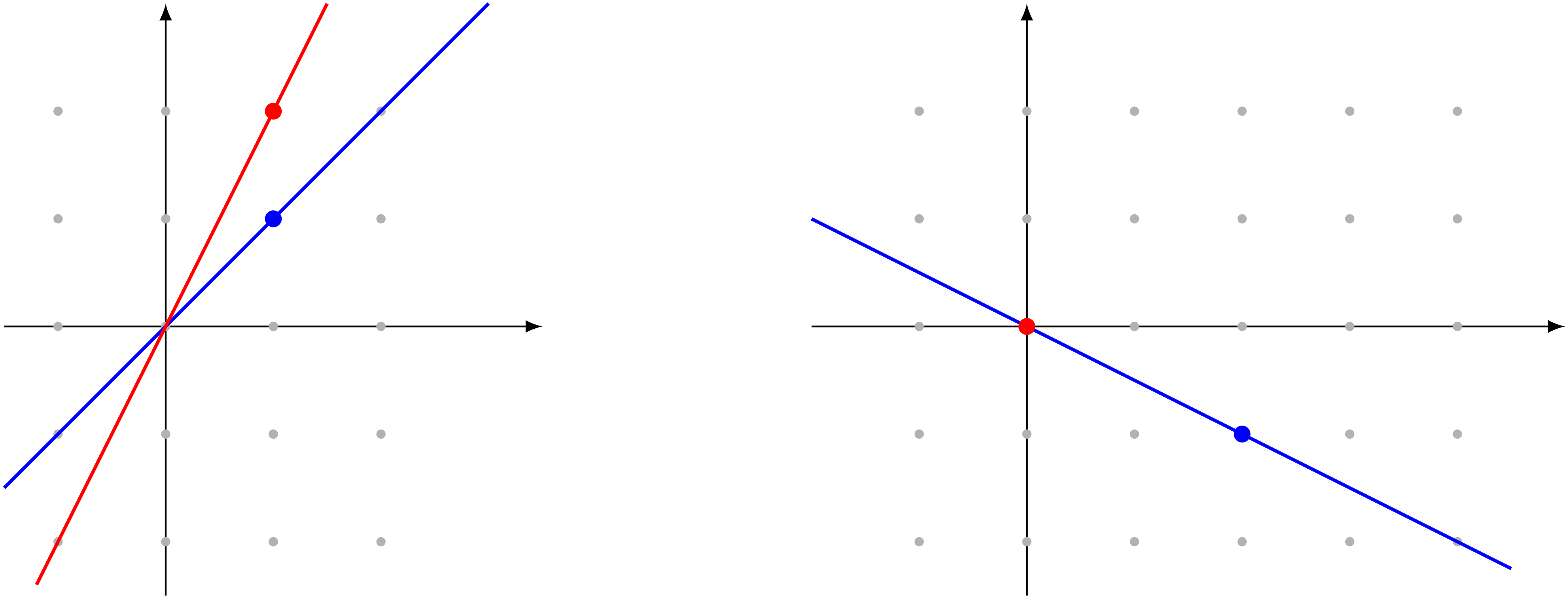

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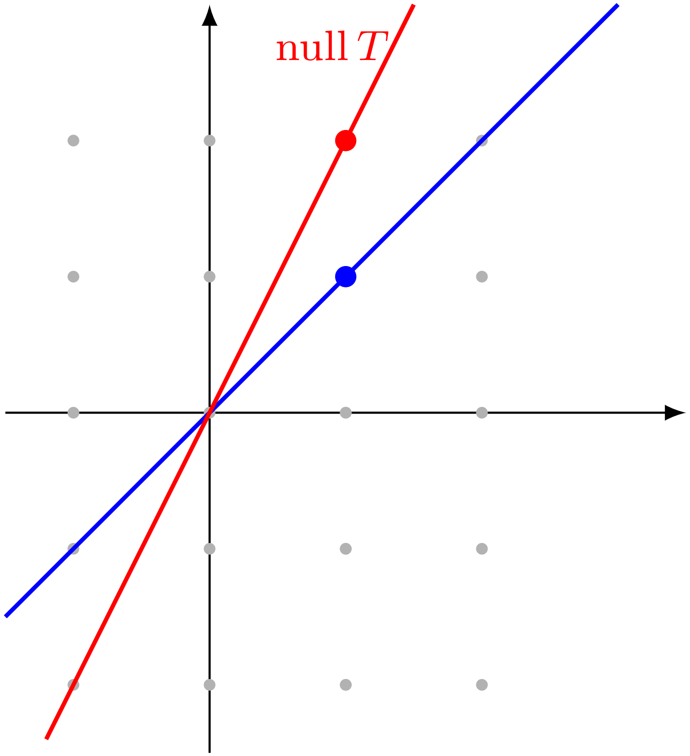


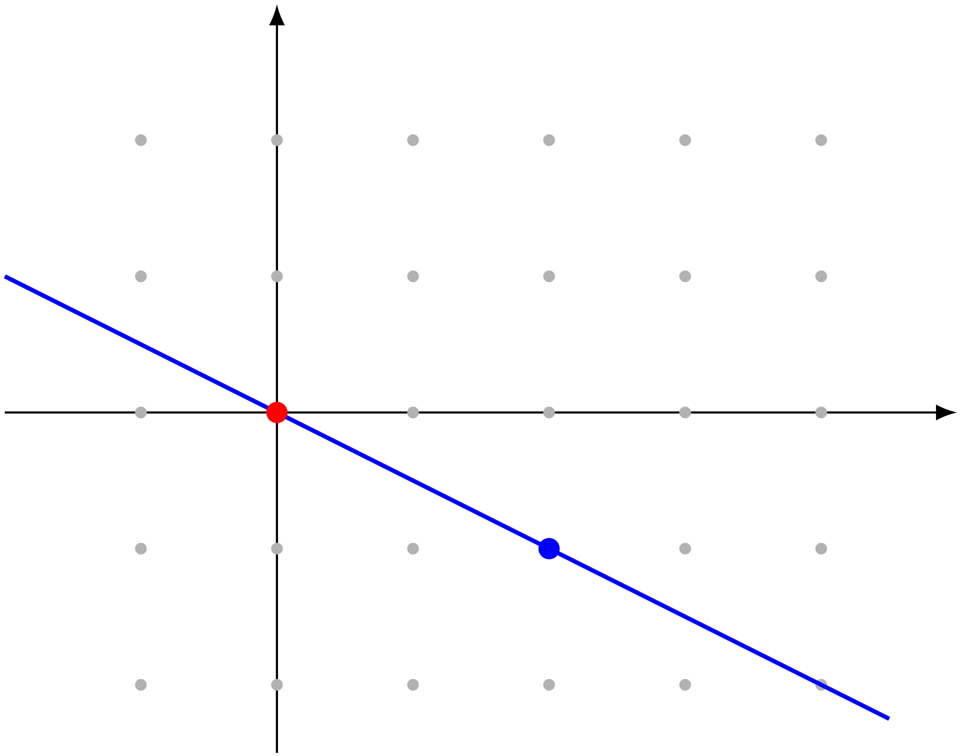

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$\mathrm{span}(1,2) \subset \mathrm{null}\, T$. In fact $\mathrm{span}(1,2) = \mathrm{null}\, T$. Why?

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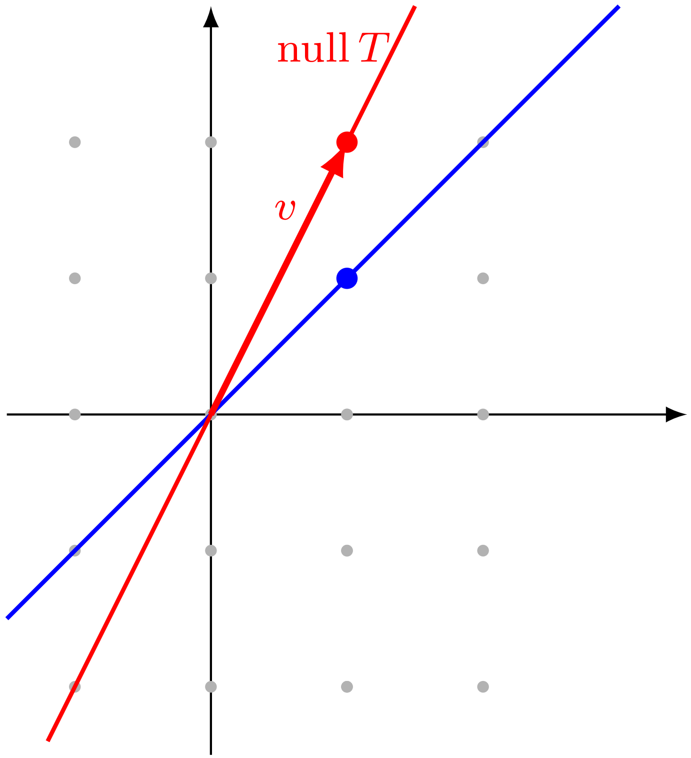


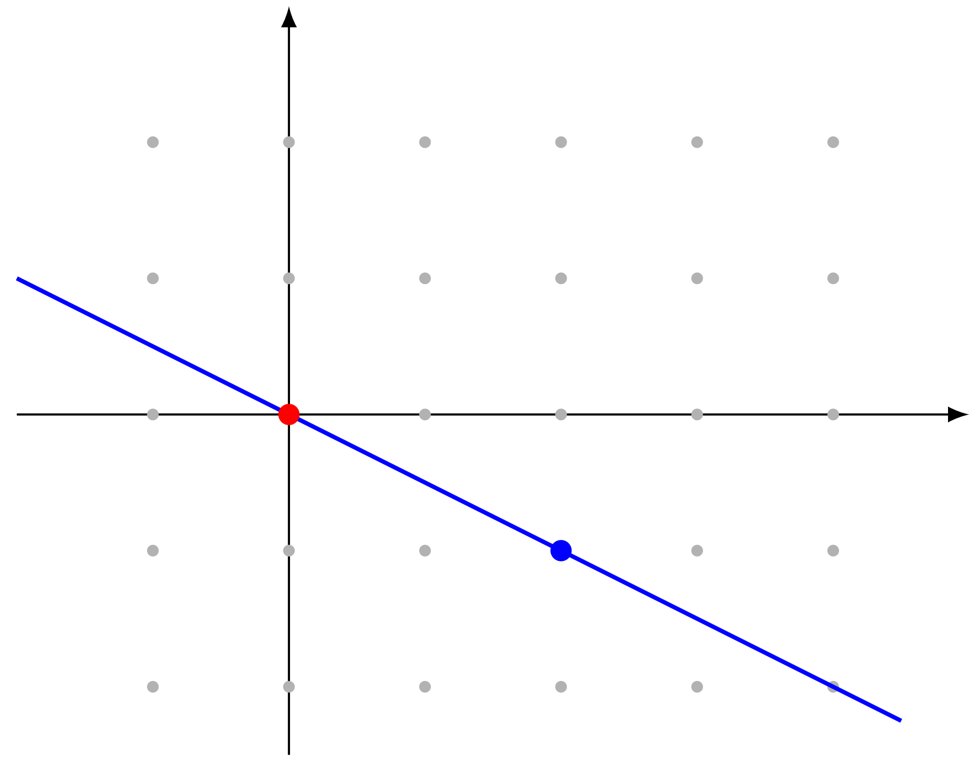

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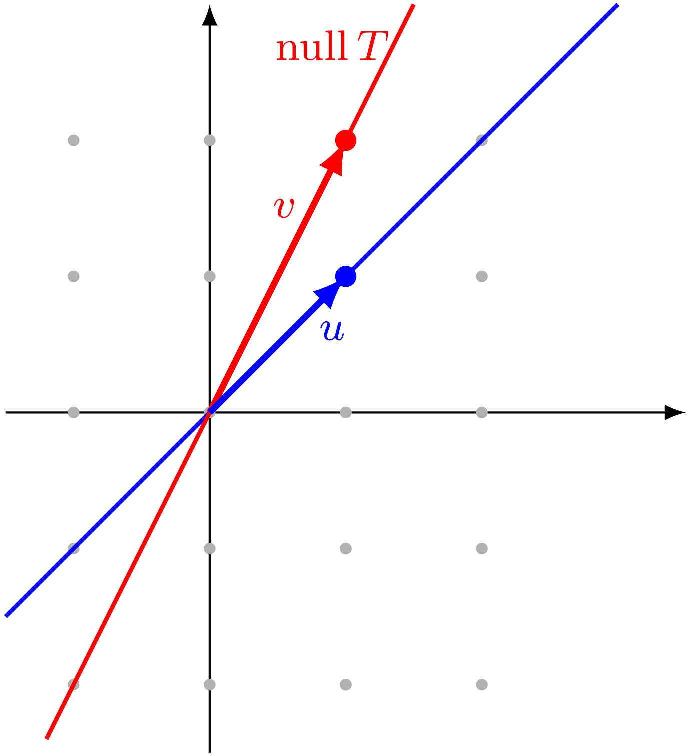


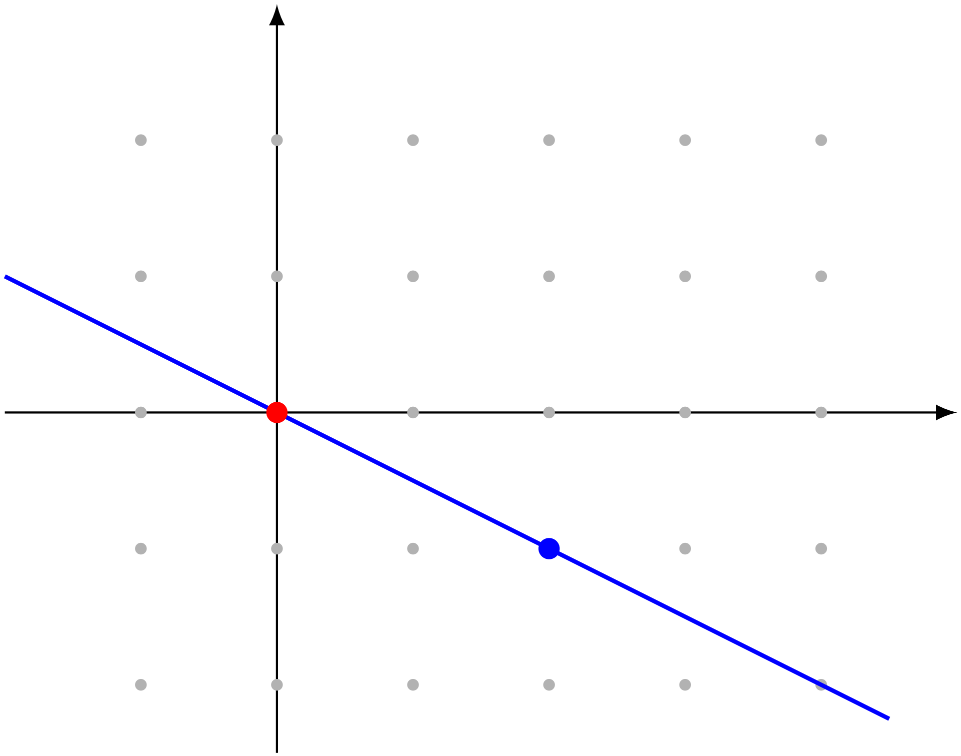

Choose a basis $v$ of $\operatorname{null} T$. Extend it to a basis $(v, u)$ of $\mathbb{R}^2$.

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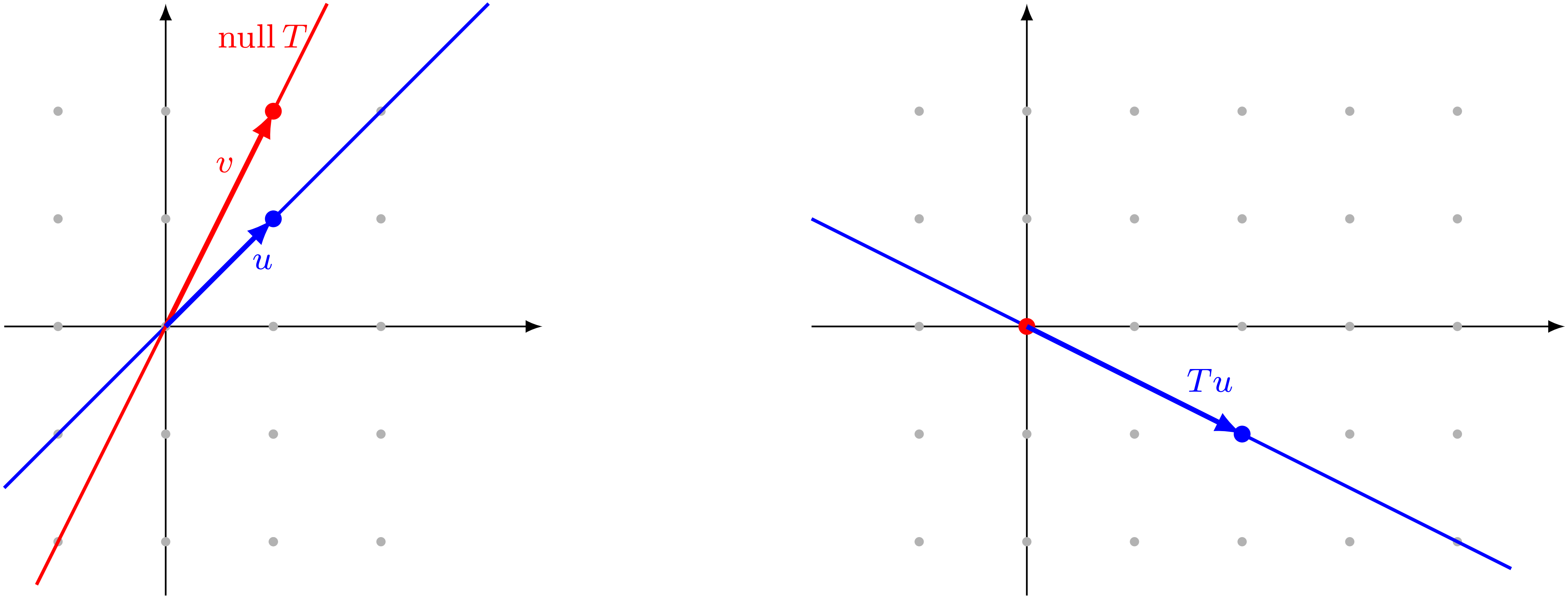


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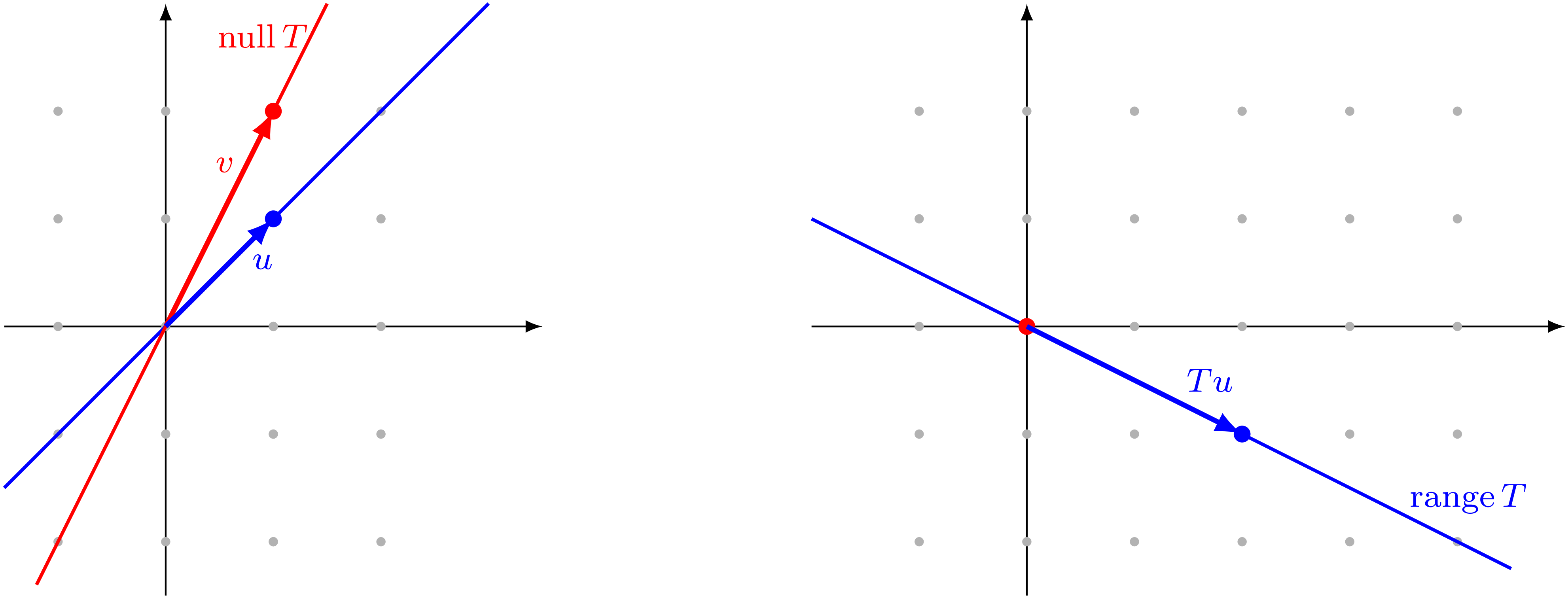


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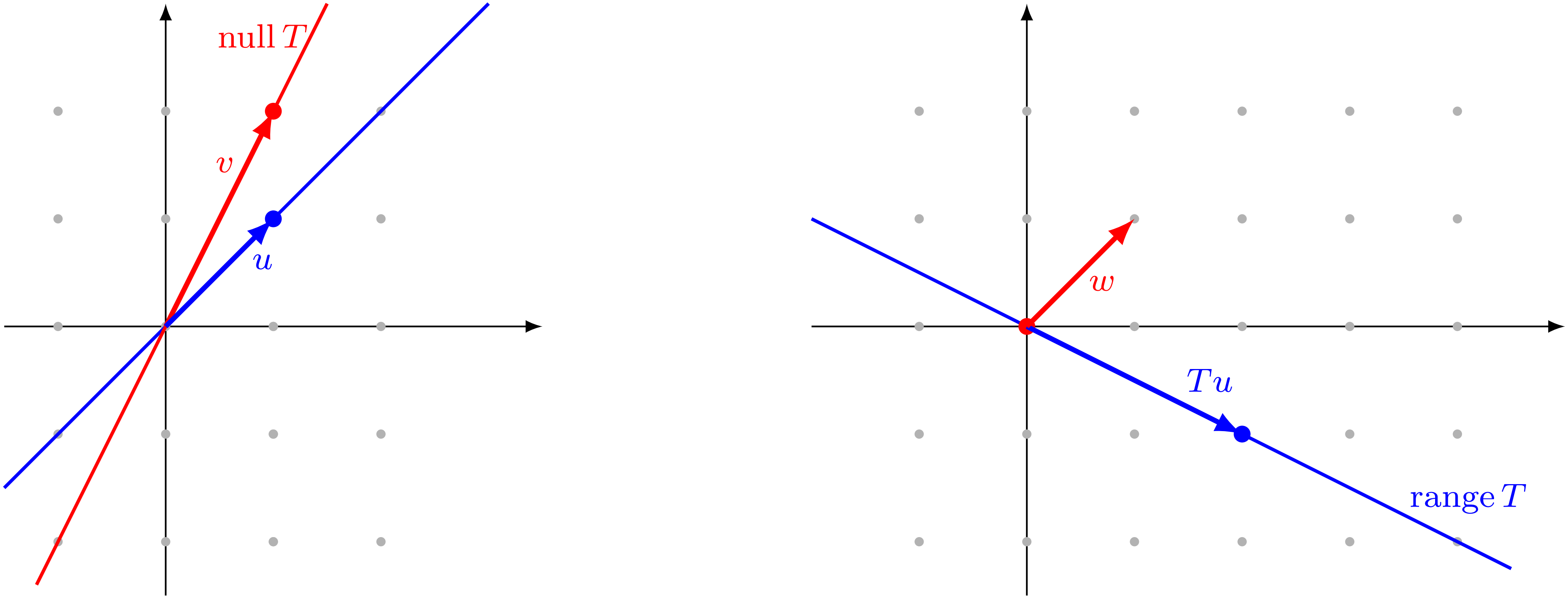


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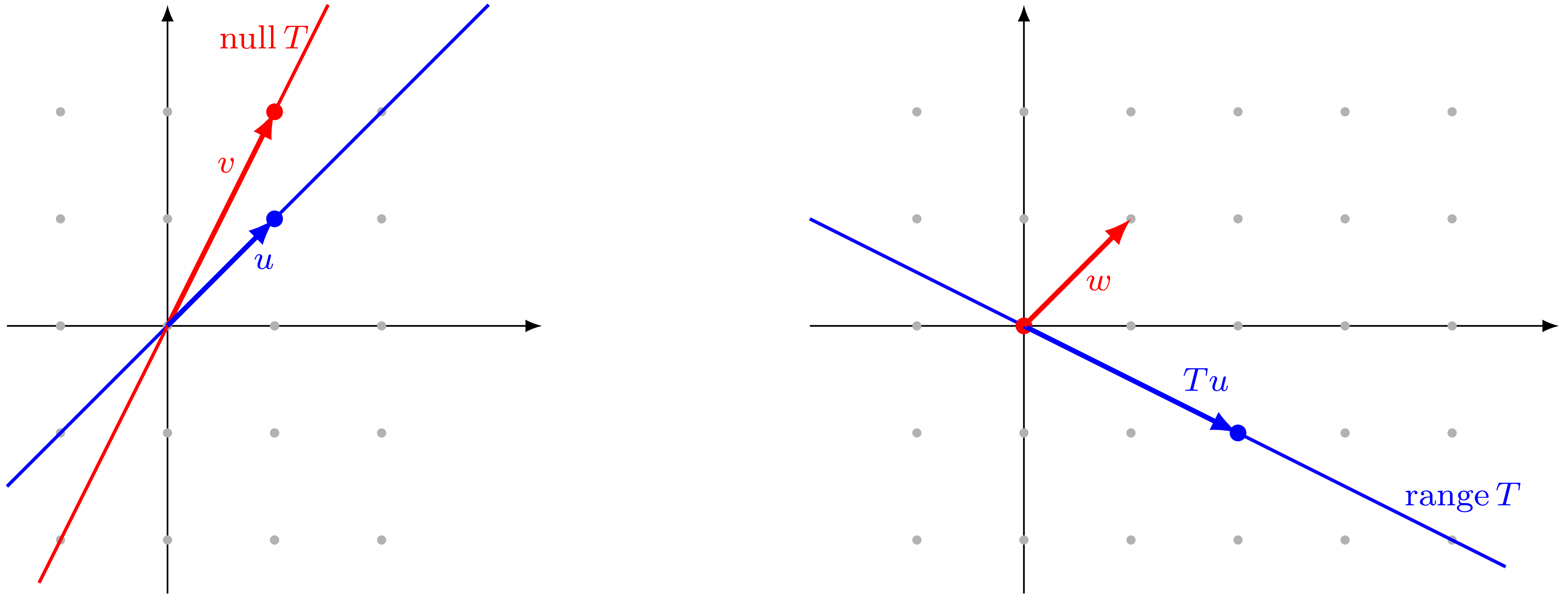


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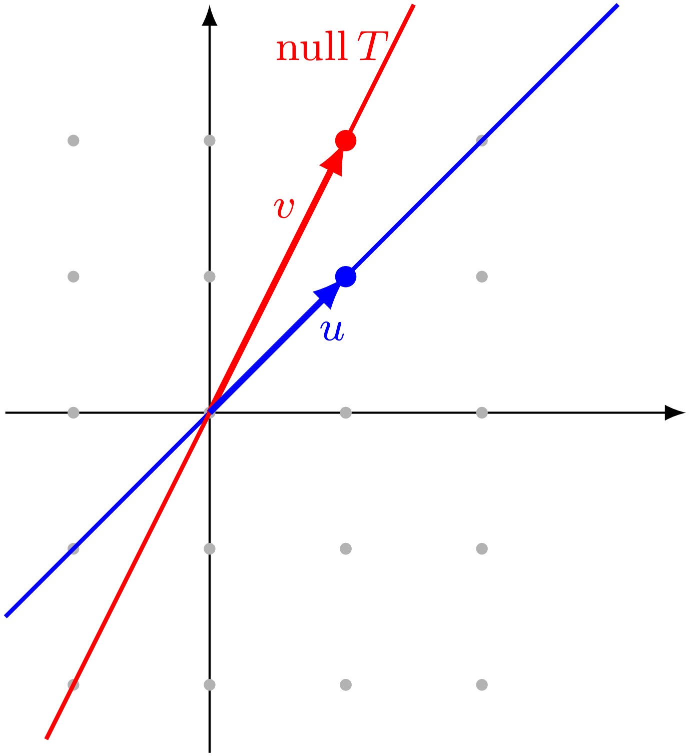
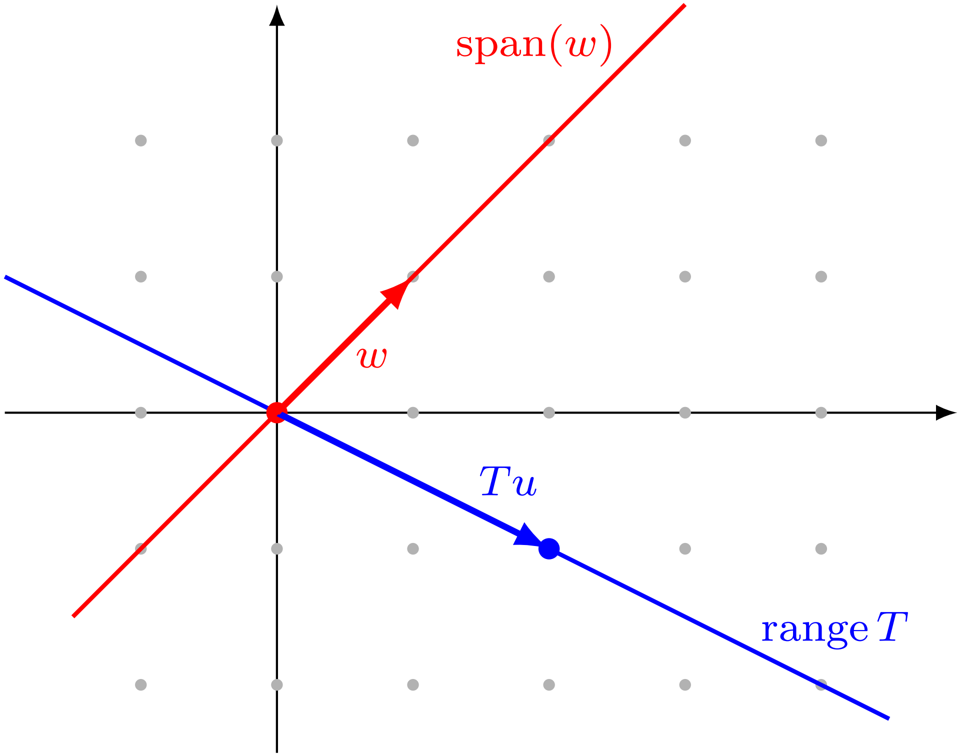
$$\mathbb{R}^2 = \operatorname{span}(v) \oplus \operatorname{span}(u) \qquad .$$

# Example of linear map



Let us recover a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by its values.




Choose a basis $v$ of $\operatorname{null} T$. Extend it to a basis $(v, u)$ of $\mathbb{R}^2$. $Tu$ is basis of $\operatorname{range} T$. Extend $Tu$ to a basis $(Tu, w)$ of $\mathbb{R}^2$.

$$\mathbb{R}^2 = \operatorname{span}(v) \oplus \operatorname{span}(u) \xrightarrow{0 \oplus T} \operatorname{span}(w) \oplus \operatorname{span}(Tu) = \mathbb{R}^2 .$$

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Isomorphisms $\phi$ and $\psi$ form an isomorphism $(0 \oplus \mathrm{id}) \to T$:

$$\begin{tikzcd}
\mathbb{F}^n \oplus \mathbb{F}^r \arrow[r, "\phi"] \arrow[d, "0\oplus\mathrm{id}"] & \operatorname{null} T \oplus U \arrow[r, "="] \arrow[d, "0\oplus T"] & V \arrow[d, "T"] \\
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Linear maps $T : V \to W$ and $T' : V' \to W'$ are isomorphic
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# Back to matrices

# Encoding a linear map by its values on a basis

**3.5 Linear maps and basis of domain**

Let $v_1, \ldots, v_n$ be a basis of $V$ and $w_1, \ldots, w_n \in W$.

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**Reformulation.** Any map $\{v_1, \dots, v_n\} \to W$ of a basis of $V$ to a vector space $W$
extends uniquely to a linear map $V \to W$.

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# From a linear map $\mathbb{F}^p \to \mathbb{F}^q$ to its matrix

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$$

**Conclusion:** any linear map $\mathbb{F}^p \to \mathbb{F}^q$ is a multiplication by a $q \times p$-matrix.

# Matrices

3.30 **Definition.** Let $q$ and $p$ denote positive integers.
A $q \times p$ **matrix** $A$ is a rectangular array of elements of $\mathbb{F}$ with $q$ rows and $p$ columns:

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The $k$th column of $\mathcal{M}(T)$ is formed of the coordinates of the $k$th basis vector $v_k$.

# The matrix of composition

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**3.43 The matrix of a composition of linear maps**

If $T : U \to V$ and $S : V \to W$ are linear maps, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

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&= B_{1,k}(A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{m,1}w_m) \\
&+ B_{2,k}(A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{m,2}w_m) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
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&= B_{1,k}(A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{m,1}w_m) \\
&\quad + B_{2,k}(A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{m,2}w_m) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + B_{n,k}(A_{1,n}w_1 + A_{2,n}w_2 + \cdots + A_{m,n}w_m) \\
&= \sum_{r=1}^{n} A_{1,r}B_{r,k}w_1 + \sum_{r=1}^{n} A_{2,r}B_{r,k}w_2 + \cdots + \sum_{r=1}^{n} A_{m,r}B_{r,k}w_m
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$$

$$
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Hence $(STu_1 \ldots STu_p) = (w_1 \ldots w_m) \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \ldots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,1} & \ldots & B_{1,p} \\ \vdots & & \vdots \\ B_{n,1} & \ldots & B_{n,p} \end{pmatrix}$

# Systems of linear equations vs. linear maps

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Any system of $q$ linear equations with $p$ unknowns looks as follows:

$$\begin{cases} A_{1,1}x_1 + \cdots + A_{1,p}x_p = b_1 \\ A_{2,1}x_1 + \cdots + A_{2,p}x_p = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{q,1}x_1 + \cdots + A_{q,p}x_p = b_q \end{cases}$$

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It can be re-written as a matrix equation $AX = B$, where

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & \vdots & \vdots \\ A_{q,1} & \dots & A_{q,p} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}.$$

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# Systems of linear equations vs. linear maps

Any system of $q$ linear equations with $p$ unknowns looks as follows:

$$\begin{cases} A_{1,1}x_1 + \dots + A_{1,p}x_p = b_1 \\ A_{2,1}x_1 + \dots + A_{2,p}x_p = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{q,1}x_1 + \dots + A_{q,p}x_p = b_q \end{cases}$$

It can be re-written as a matrix equation $AX = B$, where

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & \vdots & \vdots \\ A_{q,1} & \dots & A_{q,p} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}.$$

Each solution of $AX = B$ is a vector from $T^{-1}(B)$,
where $T : \mathbb{F}^p \to \mathbb{F}^q$ defined by matrix $A$, namely $T : X \mapsto AX$.

This allows to convert results about linear maps
into results about systems of linear equations.

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For a system
$$\begin{aligned}&\textstyle\sum_{k=1}^{p} A_{1,k}x_k = 0\\ &\vdots\\ &\textstyle\sum_{k=1}^{p} A_{q,k}x_k = 0\end{aligned}$$

Define $T : \mathbb{F}^p \to \mathbb{F}^q$ by
$T(x_1, \ldots, x_p) =$
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An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.