Advanced Linear Algebra MAT 315

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Range

3.17 Definition
For a map $T : V \to W$, the **range** of T is $\operatorname{range} T = T(V) = \{Tv \mid v \in V\}$.

Another name: **image**. Notation: $\operatorname{Im} T$.

3.18 Examples

- For $T : V \to W : v \mapsto 0$, $\operatorname{range} T = \{0\}$.
- For differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $\operatorname{range} D = \mathcal{P}(\mathbb{R})$.
- For multiplication by x^3 $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F}) : Tp = x^3 p(x)$,
  $\operatorname{range} T =$ polynomials without monomials of degree < 3.

Surjectivity and range

3.15 Definition (reminder)
A map $T : V \to W$ is called **surjective** if $\operatorname{range} T = W$.

3.14 The range of a linear map is a subspace.
For $T \in \mathcal{L}(V, W)$, $\operatorname{range} T$ is a subspace of W.

Proof $0 \in \operatorname{range} T$, since $T(0) = 0$.

If $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$, then $\exists v \in V : w = Tv$, $T(\lambda v) = \lambda T v = \lambda w \in \operatorname{range} T$.

$$w_1, w_2 \in \operatorname{range} T \implies \exists v_1, v_2 \in V : w_1 = Tv_1, w_2 = Tv_2$$
$$\implies w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2) \in \operatorname{range} T.$$

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Linear maps $\mathbb{F}^n \to V$ vs. lists of vectors

Let V be a vector space and let $u = (u_1, \ldots, u_n)$ be a list of vectors of V.

Theorem. The map $T_u : \mathbb{F}^n \to V : (x_1, \ldots, x_n) \mapsto x_1 u_1 + \cdots + x_n u_n$ is linear.

Proof

Additivity: Let $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$

$$T_u(x + y) = (x_1 + y_1)u_1 + \cdots + (x_n + y_n)u_n$$
$$= x_1 u_1 + y_1 u_1 + \ldots x_n u_n + y_n u_n$$
$$= x_1 u_1 + \cdots + x_n u_n + y_1 u_1 + \cdots + y_n u_n = T_u(x) + T_u(y)$$

Homogeneity: $T_u(\lambda x) = \lambda x_1 u_1 + \cdots + \lambda x_n u_n = \lambda(x_1 u_1 + \cdots + x_n u_n) = \lambda T_u(x)$. ∎

Denote $e_1 = (1, 0 \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, …, $e_n = (0, 0, \ldots, 0, 1)$.
Clearly, $(x_1, x_2, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n$ for any $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Theorem. Any linear $T : \mathbb{F}^n \to V$ is $T_{(u_1,\ldots,u_n)}$, where $u_i = T(e_i)$ for $\forall i$.

Proof $T(x_1, \ldots, x_n) = T(x_1 e_1 + \cdots + x_n e_n) = T(x_1 e_1) + \cdots + T(x_n e_n)$
$= x_1 T(e_1) + \cdots + x_n T(e_n) = x_1 u_1 + \cdots + x_n u_n = T_u(x)$. ∎

Properties of linear maps $\mathbb{F}^n \to V$

A linear map $T_u : \mathbb{F}^n \to V$ $\rightsquigarrow$ a list u of n vectors in V

$$T_u(x_1, \ldots, x_n) = x_1 u_1 + \cdots + x_n u_n$$

T_u is surjective $\iff$ u spans V

For any $v \in V$, $\exists (x_1, \ldots, x_n) \in \mathbb{F}^n$ $v = x_1 u_1 + \cdots + x_n u_n = T_u(x_1, \ldots, x_n)$.

T_u is injective $\iff$ u is a linear independent list

$$\operatorname{null}(T_u) = 0 \iff (x_1 u_1 + \cdots + x_n u_n = 0 \implies \forall i \; x_i = 0)$$

T_u is bijective $\iff$ u is a basis of V

Inverse to a linear map is linear

Theorem If V and W are vector spaces and a linear map $T : V \to W$ is invertible, then T^{-1} is linear.

Proof. Additivity. Let $w_1, w_2 \in W$. Then

$$T^{-1}(w_1 + w_2) = T^{-1}(\mathrm{id}_W\, w_1 + \mathrm{id}_W\, w_2) = T^{-1}(TT^{-1}w_1 + TT^{-1}w_2)$$
$$= T^{-1}T(T^{-1}w_1 + T^{-1}w_2) = \mathrm{id}_V(T^{-1}w_1 + T^{-1}w_2) = T^{-1}w_1 + T^{-1}w_2\,.$$

Proof. Homogeneity.

$$T^{-1}(\lambda w) = T^{-1}(\lambda\, \mathrm{id}_W\, w) = T^{-1}(\lambda TT^{-1}w) = T^{-1}(\lambda T(T^{-1}w))$$
$$= T^{-1}T(\lambda T^{-1}w) = \mathrm{id}_V(\lambda T^{-1}w) = \lambda T^{-1}w\,. \qquad \blacksquare$$

Corollary 1 A linear map $T : V \to W$ is an isomorphism in the category of vector spaces, if and only if it is bijective. $\blacksquare$

Corollary 2 A linear map $T : V \to W$ is an isomorphism in the category of vector spaces, if and only if $\operatorname{null} T = 0$ and $\operatorname{range} T = W$. $\blacksquare$

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Isomorphism classifications of vector spaces and linear maps

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Isomorphism classification of vector spaces

Theorem Each finite-dimensional vector space V over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^{\dim V}$.

Proof. Let $u = (u_1, \dots, u_{\dim V})$ be a basis of V.
Then the linear map $T_u : \mathbb{F}^{\dim V} \to V$ is bijective.
By Corollary 2 above, T_u is an isomorphism. $\blacksquare$

Corollary Finite-dimensional vector spaces V and W over a field $\mathbb{F}$ are isomorphic iff $\dim V = \dim W$.

Proof. $\Longleftarrow$ If $\dim V = \dim W = n$, then by Theorem above V are isomorphic to $\mathbb{F}^n$.
$\Longrightarrow$ If $T : \mathbb{F}^p \to \mathbb{F}^q$ is an isomorphism, $T = T_u$,
where $u = (u_1, \dots, u_p)$, $u_i = Te_i$ is a basis in F^q.
In $\mathbb{F}^q$ we got a basis of length p. Hence $p = q$. $\blacksquare$

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Direct sum of vector spaces

We have studied direct sums of **subspaces**.
In particular, if U and W are subspaces of V, $U \cap W = \{0\}$ and $V = U + W$,
then $V = U \oplus W$.

There is a construction, which starts with vector spaces U and W and produces
$V = U' \oplus W'$, where U' is isomorphic to U and W is isomorphic to W .

Let $V = U \times W = \{(u, w) \mid u \in U, w \in W\}$. Define:

Addition: $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$,
Multiplication: $\lambda(u, w) = (\lambda u, \lambda w)$.

This V with these operations is a required vector space over the same field.
$U' = U \times \{0\}$, $W' = \{0\} \times W$.

In particular, $\mathbb{F}^p \oplus \mathbb{F}^q$ is naturally isomorphic to $\mathbb{F}^{p+q}$.

If $f : A \to C$ and $g : B \to D$ are linear maps, then
define $f \oplus g : A \oplus B \to C \oplus D$ as $(a, c) \mapsto (f(a), g(c))$.
This is a linear map, the direct sum of f and g .

Isomorphism classification of linear maps

Theorem Any linear map $T : V \to W$ between finite-dimensional vector spaces over a field $\mathbb{F}$ is isomorphic to

$$0 \oplus \mathrm{id} : \mathbb{F}^{\dim \operatorname{null} T} \oplus \mathbb{F}^{\dim \operatorname{range} T} \to \mathbb{F}^{\dim W - \dim \operatorname{range} W} \oplus \mathbb{F}^{\dim \operatorname{range} T}$$

Proof. Let $u = (v_1, \dots, v_p)$ be a basis of $\operatorname{null} T$. Extend it to a basis $v_1, \dots, v_p, u_1, \dots, u_q$ of V . Notice that $\dim \operatorname{null} T = p$ and $\dim V = p + q$.
Denote $\operatorname{span}(u_1, \dots, u_q)$ by U . Clearly, $V = \operatorname{span}(v_1, \dots, v_p) \oplus \operatorname{span}(u_1, \dots, u_q)$ $= \operatorname{null} T \oplus U$. The restriction $T|_U$ is injective, because $U \cap \operatorname{null} T = 0$, and
$\phi = T_{(v_1,\dots,v_p)} \oplus T_{(u_1,\dots,u_q)} : \mathbb{F}^p \oplus \mathbb{F}^q \to \operatorname{null} T \oplus U$ is an isomorphism.
$(Tu_1, \dots, Tu_q)$ is a basis of $\operatorname{range} T$.
Extend it to a basis $w_1, \dots, w_r, Tu_1, \dots, Tu_q$ of W . Denote $\operatorname{span}(w_1, \dots, w_r)$ by C .
$\psi = T_{w_1,\dots,w_r} \oplus T_{Tu_1,\dots,Tu_q} : \mathbb{F}^r \oplus \mathbb{F}^q \to C \oplus \operatorname{range} T$ is an isomorphism.
Isomorphisms ϕ and ψ form an isomorphism $(0 \oplus \mathrm{id}) \to T$:

$$\begin{tikzcd}
\mathbb{F}^p \oplus \mathbb{F}^q \arrow[r, "\phi"] \arrow[d, "0\oplus\mathrm{id}"] & \operatorname{null} T \oplus U \arrow[r, "="] \arrow[d, "0\oplus T'"] & V \arrow[d, "T"] \\
\mathbb{F}^r \oplus \mathbb{F}^q \arrow[r, "\psi"] & C \oplus \operatorname{range} T \arrow[r, "="] & W
\end{tikzcd}$$ ∎

Numerical invariants of a linear map

3.22 **Corollary. Fundamental Theorem of Linear Maps.**
Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$.
Then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

Proof. By Isomorphism Classification of Linear Maps Theorem,
there exists an isomorphism $\mathbb{F}^{\dim \operatorname{null} T} \oplus \mathbb{F}^{\dim \operatorname{range} T} \to V$. ∎

$\dim \operatorname{range} T$ is called the **rank** of linear map T. It is denoted by $\operatorname{rk} T$.

$\operatorname{rk} T \leq \dim W$ for any linear map $T : V \to W$.

Proof. By Isomorphism Classification of Linear Maps Theorem,
there exists an isomorphism $\mathbb{F}^{\dim W - \dim \operatorname{range} T} \oplus \mathbb{F}^{\dim \operatorname{range} T} \to W$. ∎

A linear map $T : V \to W$ with $\dim V = p$, $\dim W = q$ and $\operatorname{rk} T = r$ exists
$\iff$ $r \leq p$ and $r \leq q$.

Linear maps $T : V \to W$ and $T' : V' \to W'$ are isomorphic
$\iff$ $\dim V = \dim V'$, $\dim W = \dim W'$ and $\operatorname{rk} T = \operatorname{rk} T'$.