Advanced Linear Algebra MAT 315

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02/18/2020

Spaces associated to a linear map, continued

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Theorem. The map $T_u : \mathbb{F}^n \to V : (x_1, \ldots, x_n) \mapsto x_1 u_1 + \cdots + x_n u_n$ is linear.

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Linear maps $\mathbb{F}^n \to V$ vs. lists of vectors

Let V be a vector space and let $u = (u_1, \dots, u_n)$ be a list of vectors of V.

Theorem. The map $T_u : \mathbb{F}^n \to V : (x_1, \dots, x_n) \mapsto x_1 u_1 + \dots + x_n u_n$ is linear.

Proof

Additivity: Let $x = (x_1, \dots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{F}^n$

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Isomorphism classifications of vector spaces and linear maps

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Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$.
Then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

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