Advanced Linear Algebra MAT 315

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Other notations: $\mathrm{Hom}_{\mathbb{F}}(V, W)$ or $\mathrm{Hom}(V, W)$.

Examples of linear maps

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Inclusion $\mathrm{in} \in \mathcal{L}(V, W) : x \mapsto x$ if $V \subset W$

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Differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$

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distributivity $(S_1 + S_2)T = S_1T + S_2T$ and $(T_1 + T_2)S = T_1S + T_2S$.

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With any object X, the **identity morphism** $\mathrm{id}_X : X \to X$ is associated:

for $\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "f"'] & X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd}$ we have $\mathrm{id}_X \circ f = f$

and for $\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[rr, bend right, "g"'] & X \arrow[r, "g"] & B \end{tikzcd}$ we have $g \circ \mathrm{id}_X = g$.

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Which categories will be used in this course?

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Morphisms $T : V \to W$ and $S : W \to V$ are said to be **inverse** to each other if $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$.

A morphism $T : V \to W$ is called **invertible** if there exists a morphism inverse to T.

3.54 **Uniqueness of Inverse** If a morphism is invertible then its inverse is unique.

Proof Let S_1 and S_2 be inverse to $T : V \to W$. Then

$$S_1 = S_1 \,\mathrm{id}_W = S_1(TS_2) = (S_1T)S_2 = \mathrm{id}_V\, S_2 = S_2$$ ■

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Inverses and invertibles

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A category does not recognize any difference between its isomorphic objects,
although the objects may be not identically the same.

surjectivity, injectivity and bijectivity

Back to the category of sets and maps

surjectivity, injectivity and bijectivity

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surjectivity, injectivity and bijectivity

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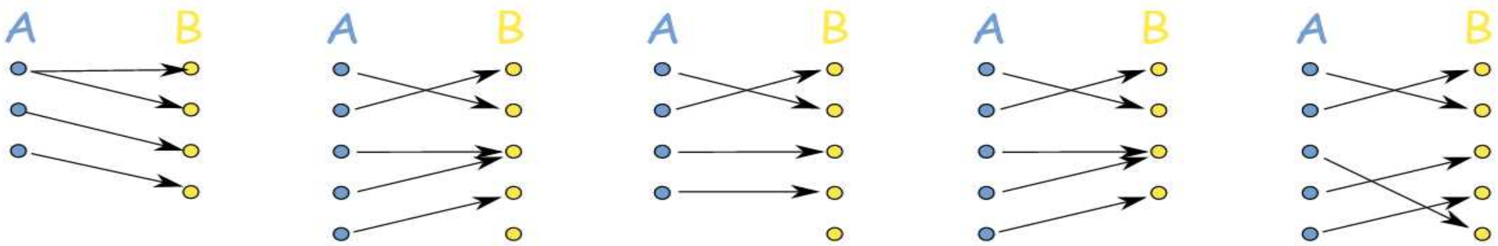

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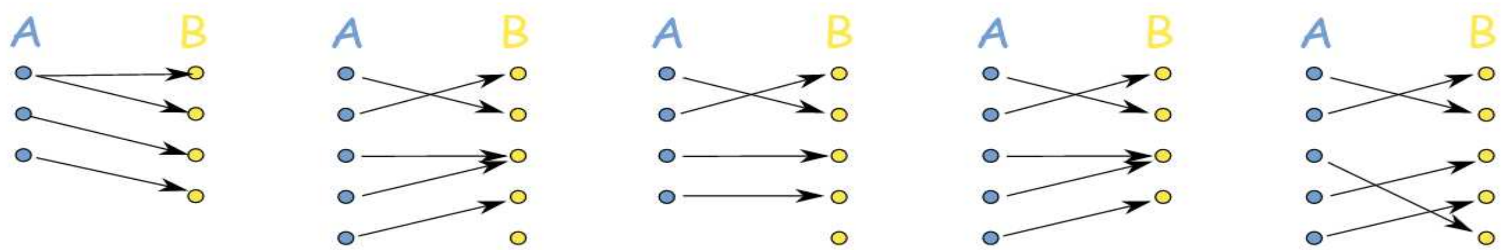

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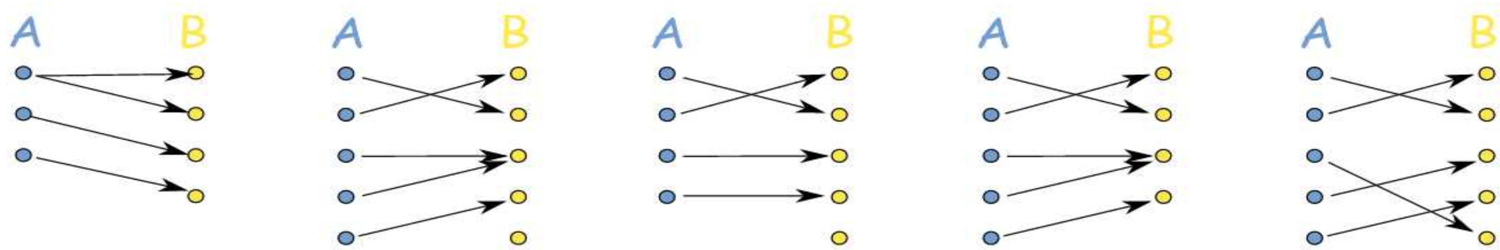

surjection,
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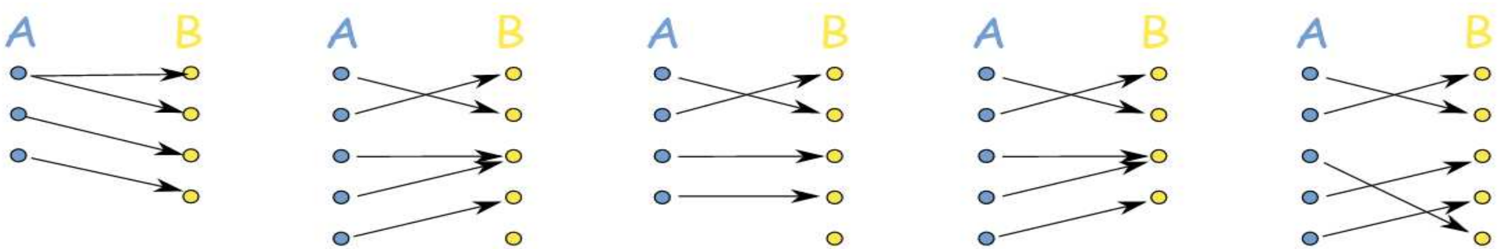

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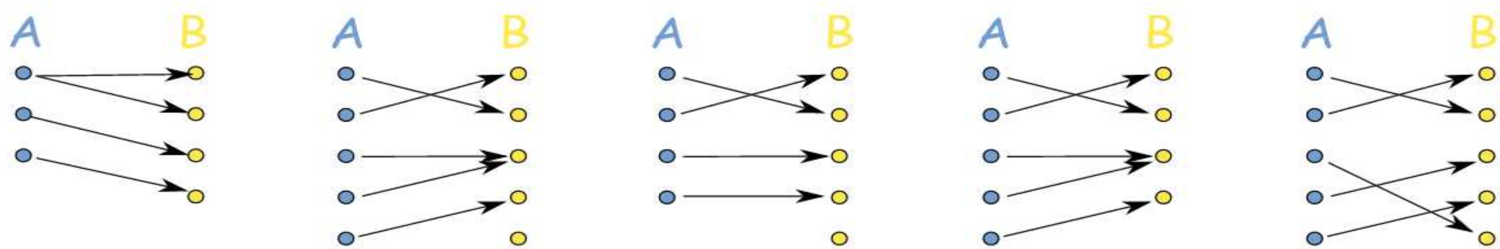

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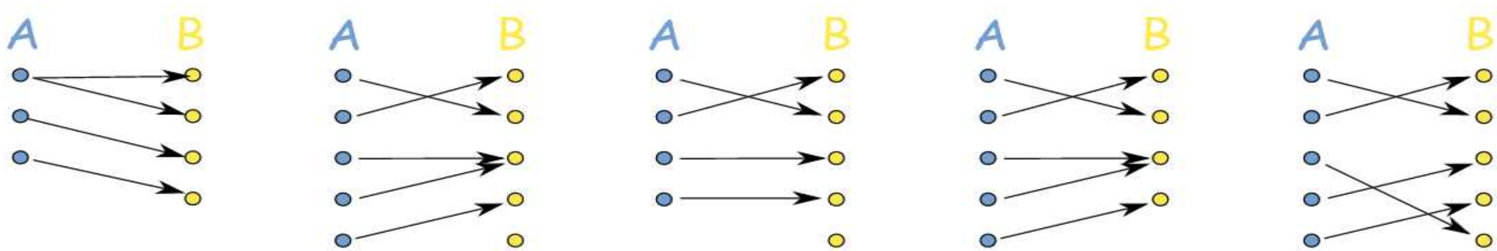

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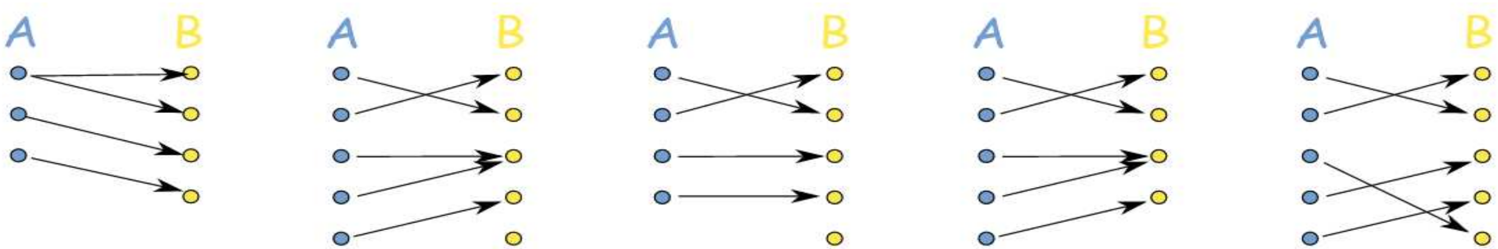

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1-to-1

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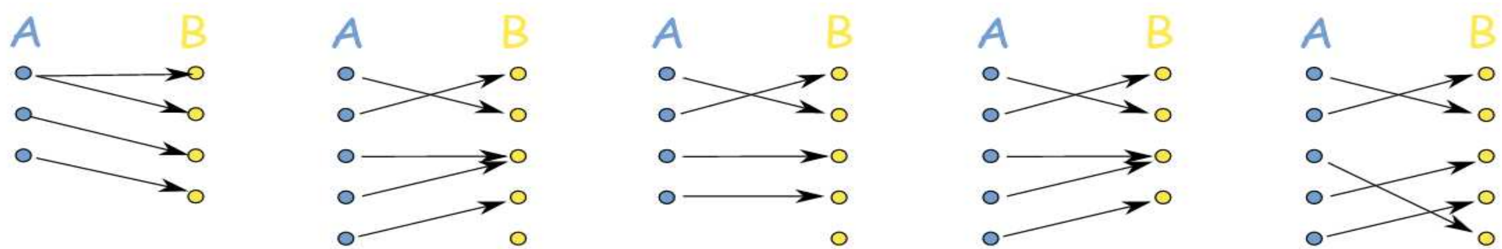

injection, but not surjection

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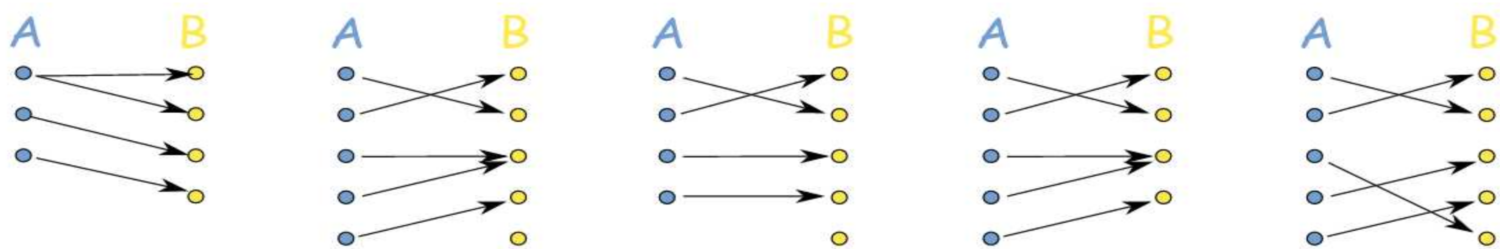

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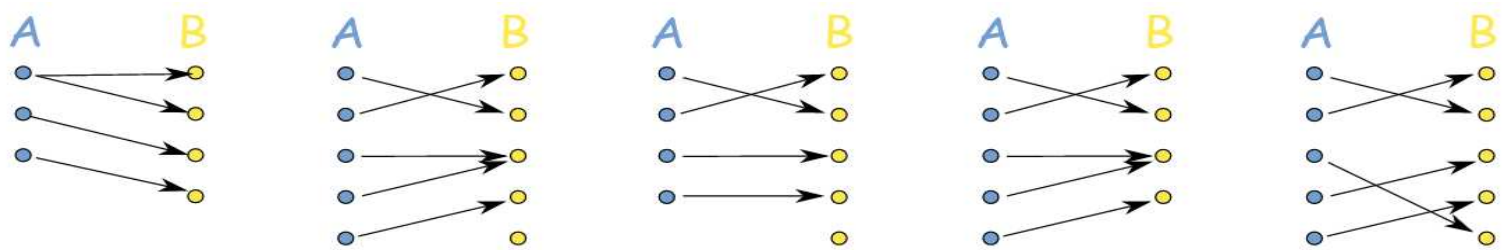

not a map		injection, but not surjection	surjection, but not injection	bijection
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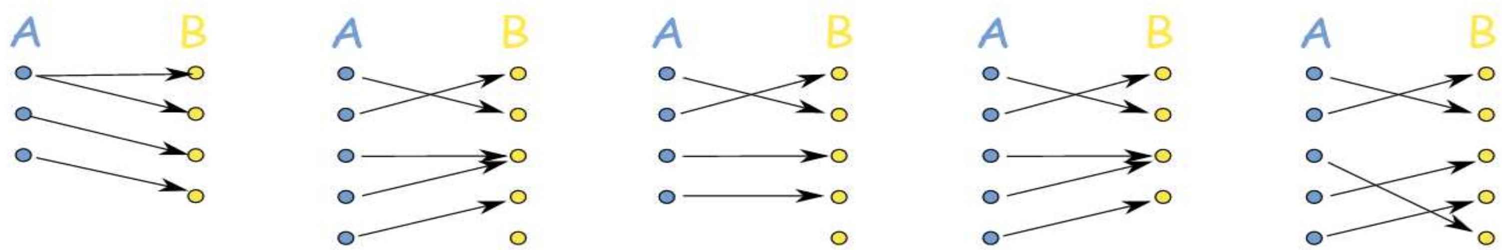

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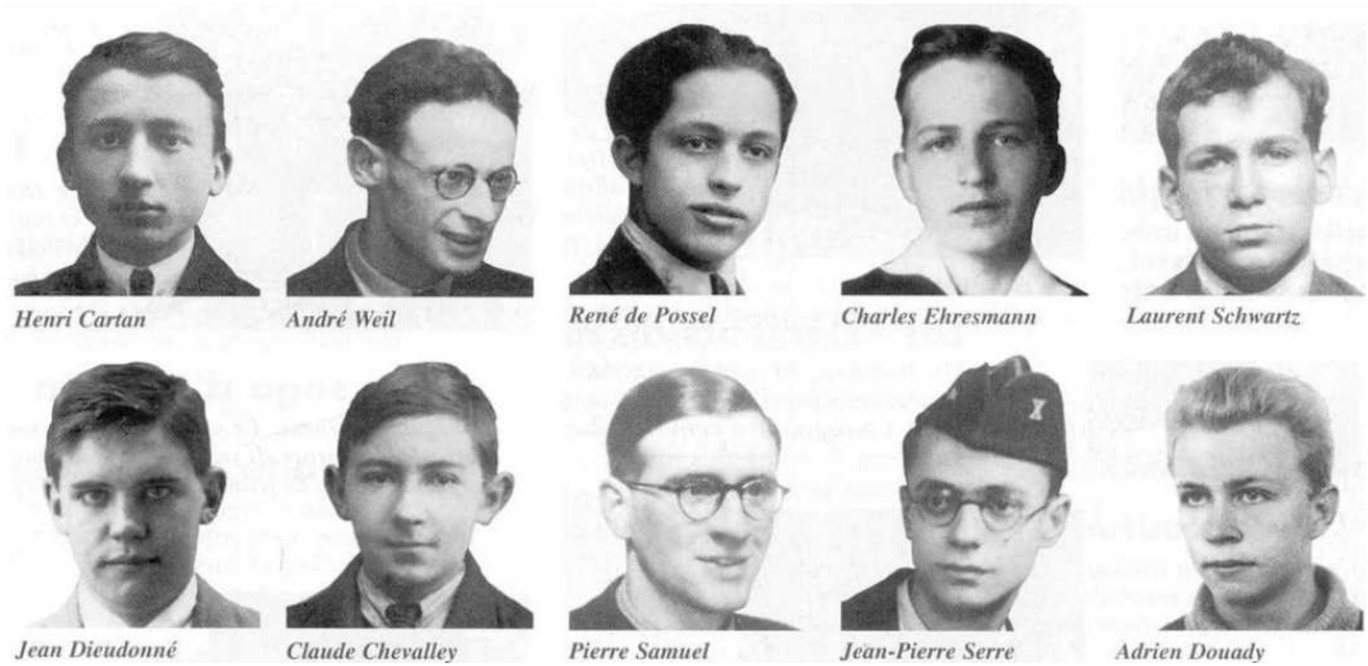
liberté, égalité et fraternité



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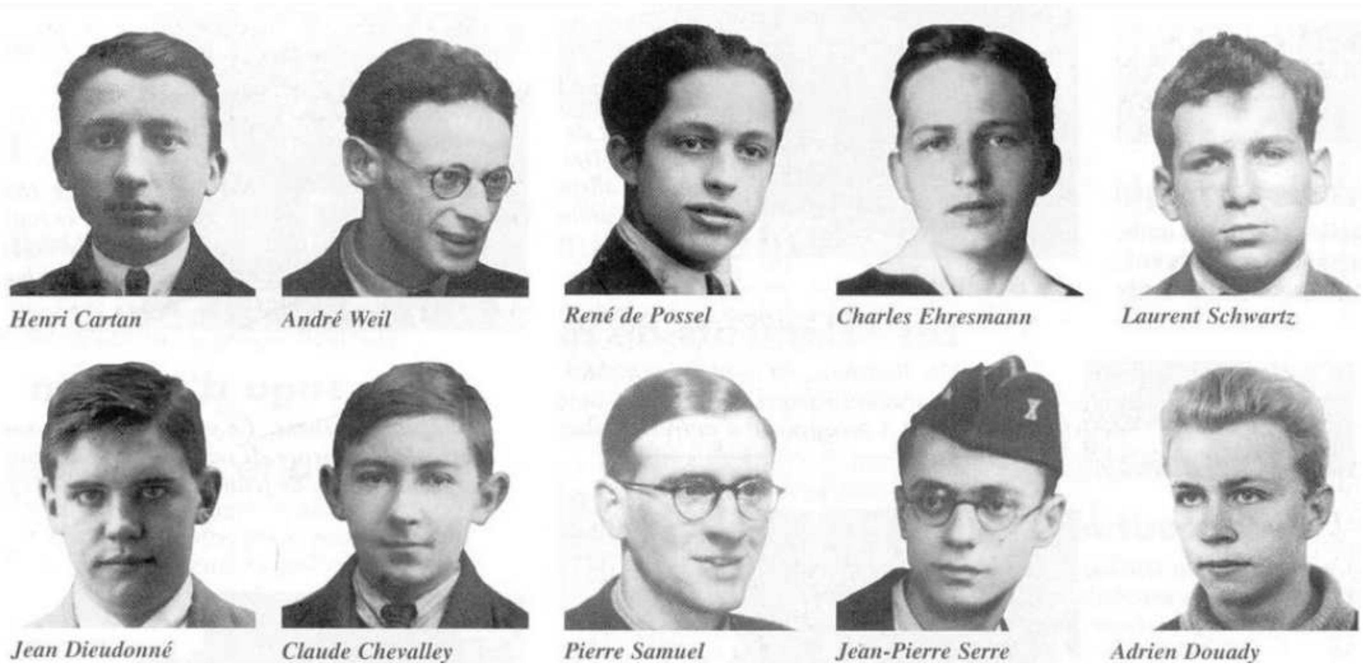

Nicolas Bourbaki

Invertible = bijection

Invertible $=$ bijection

Which sets are isomorphic in the category of sets and maps?

Invertible $=$ bijection

Which sets are isomorphic in the category of sets and maps?

3.56 **Theorem. Invertibility is equivalent to bijectivity.**

Invertible $=$ bijection

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You should know this.

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Spaces associated to a linear map

Null space

Null space

3.12 **Definition (reminder)** For $T \in \mathcal{L}(V, W)$, the null space of T is

$$\operatorname{null} T = T^{-1}\{0\}$$

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Another name: **kernel**.

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Another name: **kernel**. Notation: $\operatorname{Ker} T$.

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3.13 **Examples**

Null space

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3.13 Examples

- For $T : V \to W : v \mapsto 0$, $\operatorname{null} T =$

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3.13 **Examples**

- For $T : V \to W : v \mapsto 0$, $\operatorname{null} T = V$
- For differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $\operatorname{null} D =$

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Null space

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- For backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, F^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

  $\operatorname{null} T =$

Null space

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  $\operatorname{null} T = \{(a, 0, 0, \dots) \mid a \in \mathbb{F}\}$

Null space is a subspace

Null space is a subspace

3.14 **Theorem.**

Null space is a subspace

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Null space is a subspace

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